Tensor Renormalization Group (TRG)

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11th JICFuS Seminar on Non-perturbative Physics

Outline

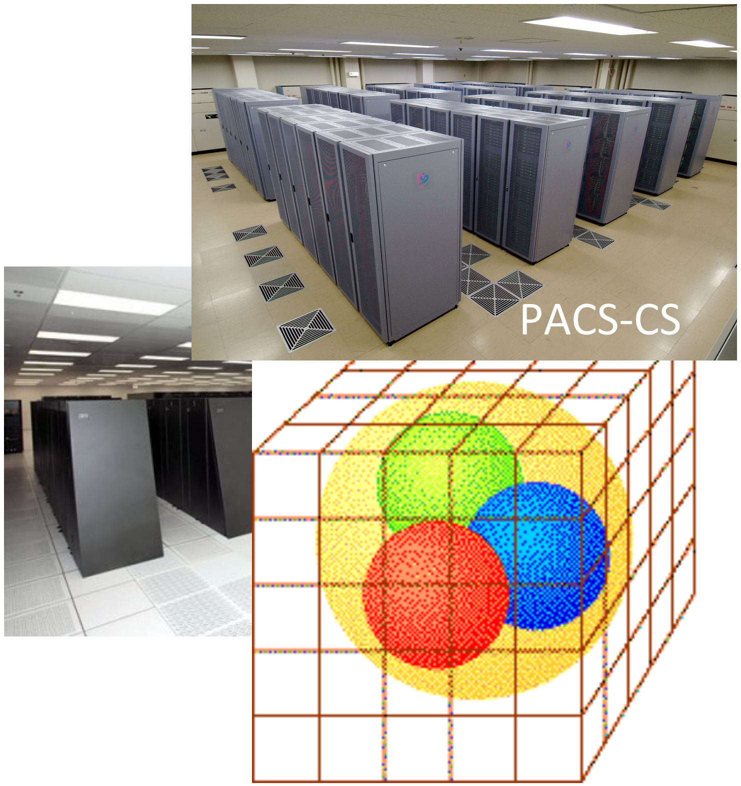
- Introduction
- TRG for 2D Ising model
- Summary & Future prospects
- Application to finite fermion density system:
  Gross-Neveu model
  = 2D four-fermion interaction

if time allows

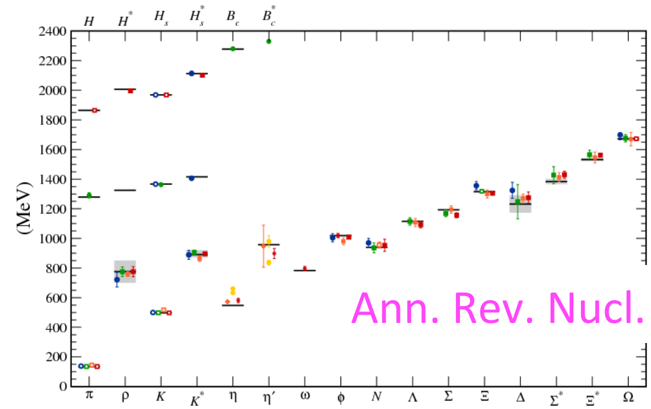
Introduction

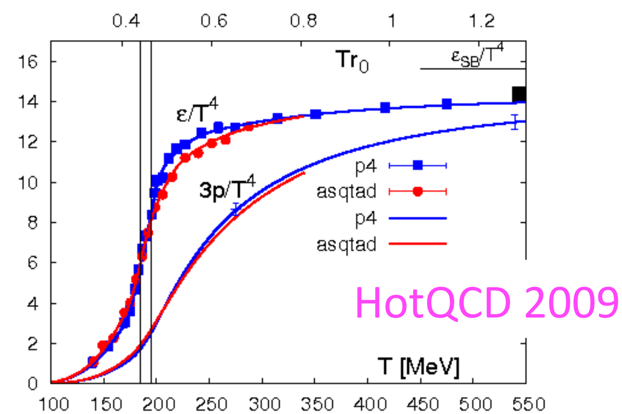
Why Tensor renormalization group instead of Monte Carlo method?

Lattice QCD by MC simulations



筑波大学素粒子論研究室





play crucial role in Hadron spectrum and T≠0 QCD

Limitations and problems in MC

- Light quark mass simulation Solved!

  new algorithm & improvement of action

- Critical slowing down

  continuum limit = critical point

- Sign problem (complex action problem)

  MC cannot be applied if Boltzmann weight is complex number

$$\frac{1}{Z} \det D[U] e^{-S_{\mathrm{G}}[U]} \in \mathbb{C}$$

Examples facing the sign problem

- QCD with finite quark density
  - EOS in core of compact stars
- Lattice chiral gauge theory
  - Simulation of weak interaction (SM)
- Lattice SUSY
  - not sure it exists but may be interesting
  - Dynamics of SUSY breaking
- θ term
  - Strong CP problem : Dynamics in the presence of θ-term is important

Approaches within MC framework

- Taylor expansion

  Can capture phase transition (non-analytic phenomena)?
- Phase-reweighting

  harder for larger volume
- Pure imaginary parameter (imaginary μ, θ)

  applicable range of analytic continuation
- Complex Langevin

  Convergence ?
- Lefschetz thimble

  Difficult! (at least for me)

Approaches within MC framework

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Go beyond Monte Carlo method!

Tensor Renormalization Group (TRG)

Levin & Nave 2007

Algorithm to compute partition function of lattice model approximately w/o relying on probability

for classical statistical system or quantum system in path-integral representation

- Good : No sign problem
- Bad : Higher dimensional system is still hard

see later

TRG for 2D Ising model

Procedures of TRG

① Rewrite Z in tensor network representation

$$Z \equiv \sum_{\{s\}} e^{-\beta H[s]} = \sum_{i,j,k,l,\dots} \dots T_{ijkl} T_{mnio} \dots$$

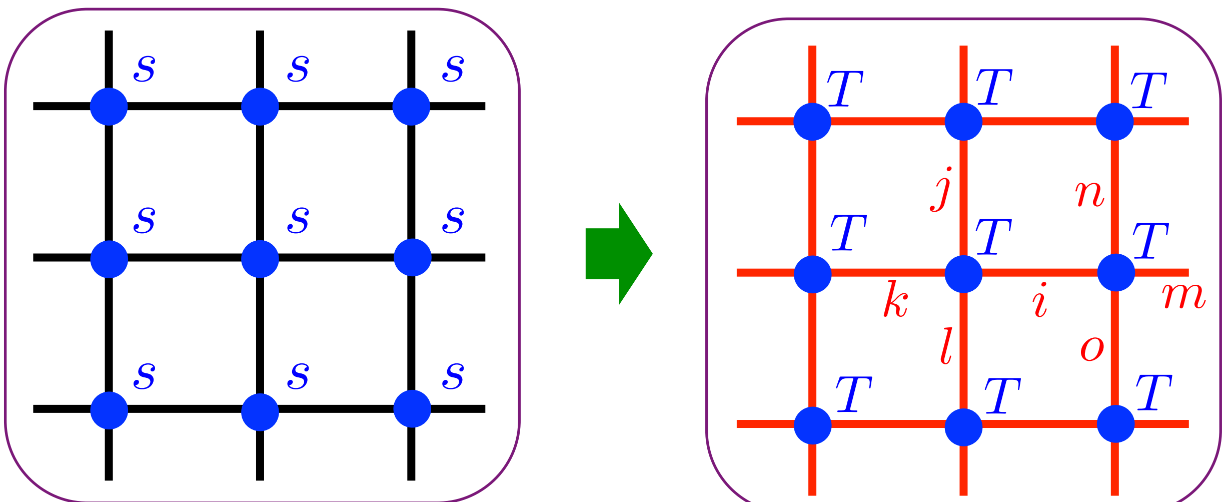
Tensor

Spins

will be explained soon

New degrees of freedom

- Analytic
- exact



site : Tensor T

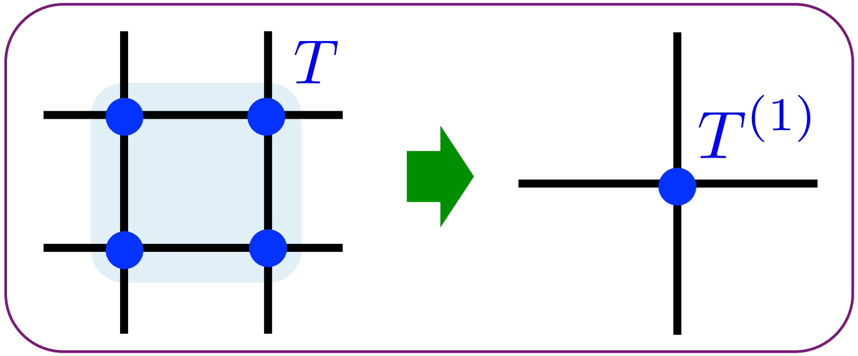
bond : index $i,j,k,l,\dots$

Procedures of TRG

① Rewrite Z in tensor network representation

② Coarse graining Tensor

Blocking of Tensor (like spin-blocking)



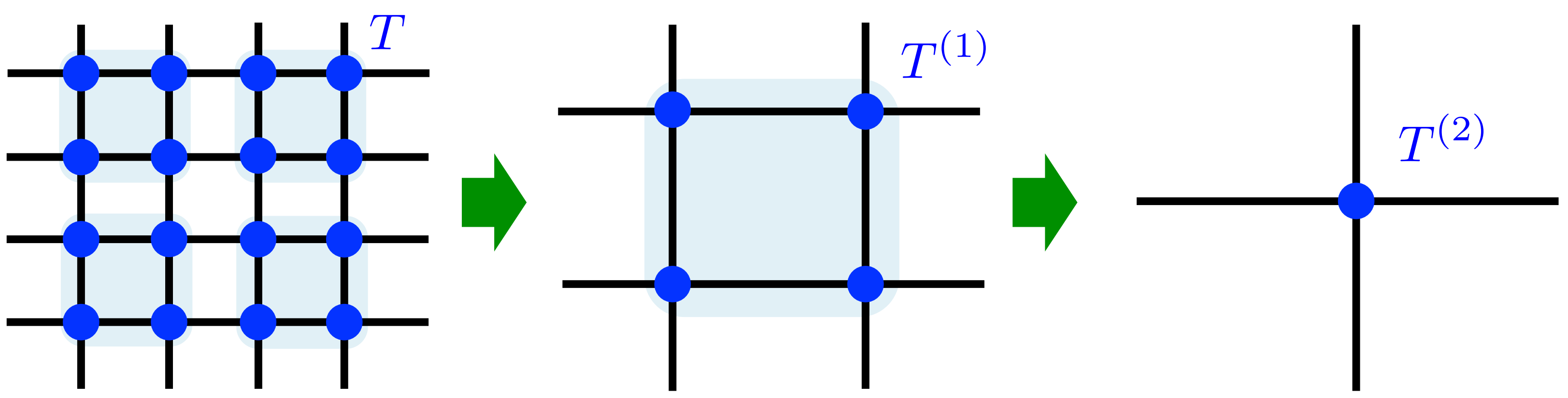
- extracting important information numerically
- selection of information introduces approximation

Procedures of TRG

① Rewrite Z in tensor network representation

② Coarse graining Tensor

③ Repeat the coarse graining and then reduce the number of tensors, finally compute Z by contraction



Procedures of TRG

① Rewrite Z in tensor network representation

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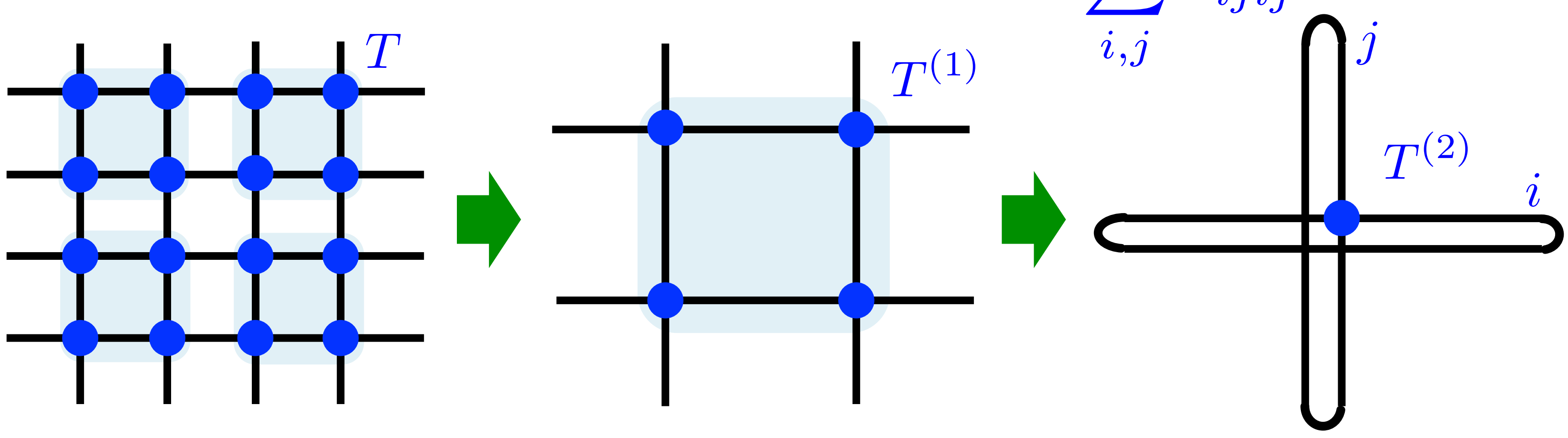
③ Repeat the coarse graining and then reduce the number of tensors, finally compute Z by contraction

$$Z \approx \sum_{i,j} T^{(n)}_{ijij}$$ for Periodic BC

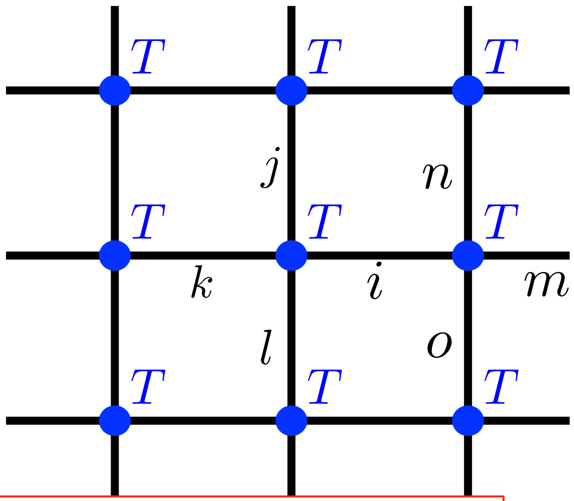


① Tensor network rep.

Direction

$$Z = \sum_{\{s\}} e^{-\beta H[s]} = \sum_{i,j,k,l,\ldots} \ldots T_{ijkl} T_{mnio} \ldots$$

1) Expand Boltzmann weight as in High-T expansion
2) Identify integer, which appears in the expansion, as new d.o.f. → index of tensor
3) Integrate out old d.o.f. (spin variable s)
4) Get tensor network rep. !

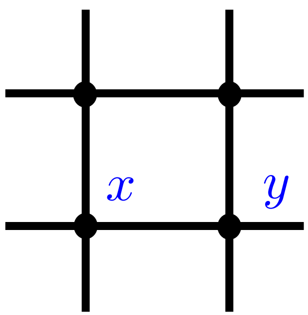


Basic procedure is common to fermion/gauge system

① Tensor network rep.

$$\mathcal{Z} = \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y \right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y)$$

nearest neighbor pair



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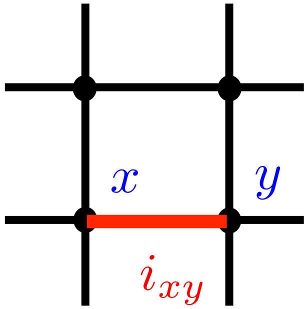
$$= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}}$$

V = # of lattice sites

$$\begin{aligned}
\exp(\beta s_x s_y) &= \cosh(\beta s_x s_y) + \sinh(\beta s_x s_y) \\
&= \cosh\beta + s_x s_y \sinh\beta \\
&= \cosh\beta(1 + s_x s_y \tanh\beta) \\
&= \cosh\beta \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}}
\end{aligned}$$

New d.o.f.

$$s_x = \pm 1$$



① Tensor network rep.

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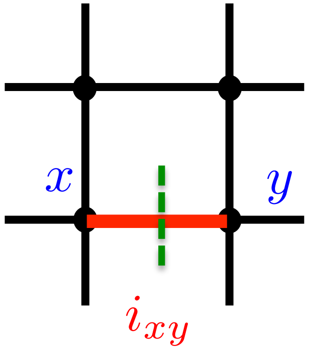
&= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{<x,y>} (s_x \sqrt{\tanh\beta} \cdot s_y \sqrt{\tanh\beta})^{i_{xy}}
\end{aligned}$$



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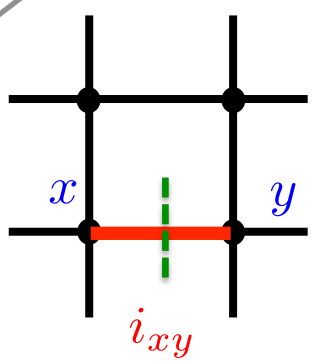
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{x} (s_x \sqrt{\tanh\beta})^{i_{xy}} (s_x \sqrt{\tanh\beta})^{i_{xz}} (s_x \sqrt{\tanh\beta})^{i_{xw}} (s_x \sqrt{\tanh\beta})^{i_{xv}}
\end{aligned}
$$

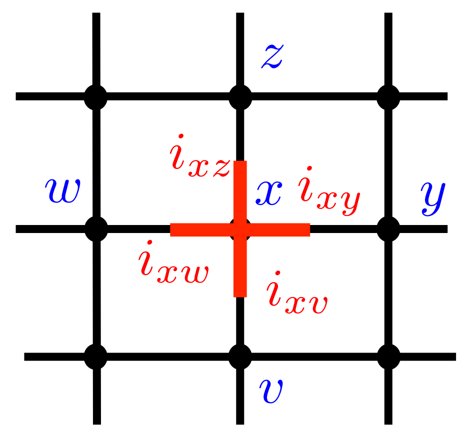
$$\prod_{<x,y>}$$

Bond



$$\prod_{x}$$

Site



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&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{x} (\sqrt{\tanh\beta})^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} s_x^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \prod_{x} (\sqrt{\tanh\beta})^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} \sum_{s_x=\pm1} s_x^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \prod_{x} \boxed{(\sqrt{\tanh\beta})^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} 2\delta(\mathrm{mod}(i_{xy}+i_{xz}+i_{xw}+i_{xv}, 2))}
\end{aligned}
$$

Spin sum can be done

$$= T_{i_{xy} i_{xz} i_{xw} i_{xv}}$$

New d.o.f. : index of tensor

① Tensor network rep.

$$\mathcal{Z} = 2^V (\cosh\beta)^{2V} \sum_{..,i,j,k,l,m,n,o,..} \cdots T_{ijkl} T_{mnio} \cdots$$

$$T_{ijkl} = (\sqrt{\tanh\beta})^{i+j+k+l} \delta(\mathrm{mod}(i+j+k+l), 2)$$

$$\begin{bmatrix} T_{0000} & T_{0001} & T_{0010} & T_{0011} \\ T_{0100} & T_{0101} & T_{0110} & T_{0111} \\ T_{1000} & T_{1001} & T_{1010} & T_{1011} \\ T_{1100} & T_{1101} & T_{1110} & T_{1111} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \tanh\beta \\ 0 & \tanh\beta & \tanh\beta & 0 \\ 0 & \tanh\beta & \tanh\beta & 0 \\ \tanh\beta & 0 & 0 & (\tanh\beta)^2 \end{bmatrix}$$

Contents of tensor depend on model

① Tensor network rep.

$$\mathcal{Z} = 2^V (\cosh\beta)^{2V} \sum_{..,i,j,k,l,m,n,o,..} \cdots T_{ijkl} T_{mnio} \cdots$$

Key points

translational invariance

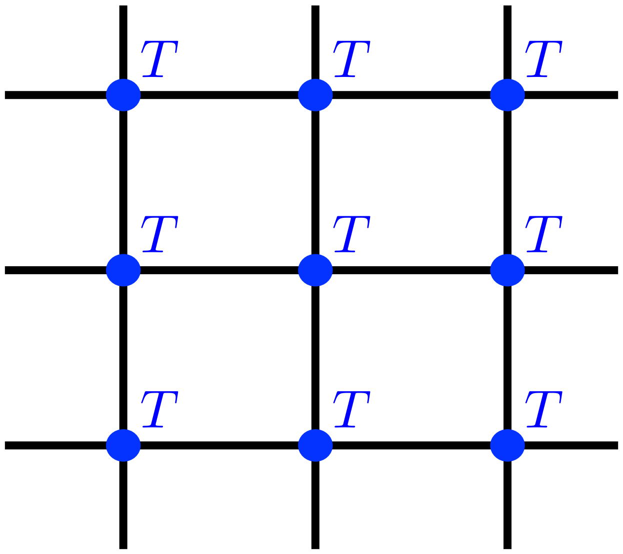
⇕

all tensors are common

local interaction
(nearest neighbor)

⇕

network (nearest neighbor)



① Tensor network rep.

$$\mathcal{Z} = \sum_{..,i,j,k,l,m,n,o,..} \cdots T_{ijkl} T_{mnio} \cdots$$

- So far, we have just rewritten Z
- Next step is to carry out the summation
- But, naïve approach costs ∝ 2^{2V}
- One has to reduce the cost and introduce approximation but wants to keep an efficiency by summing important part in Z

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Coarse graining (renormalization, blocking)

② Coarse graining

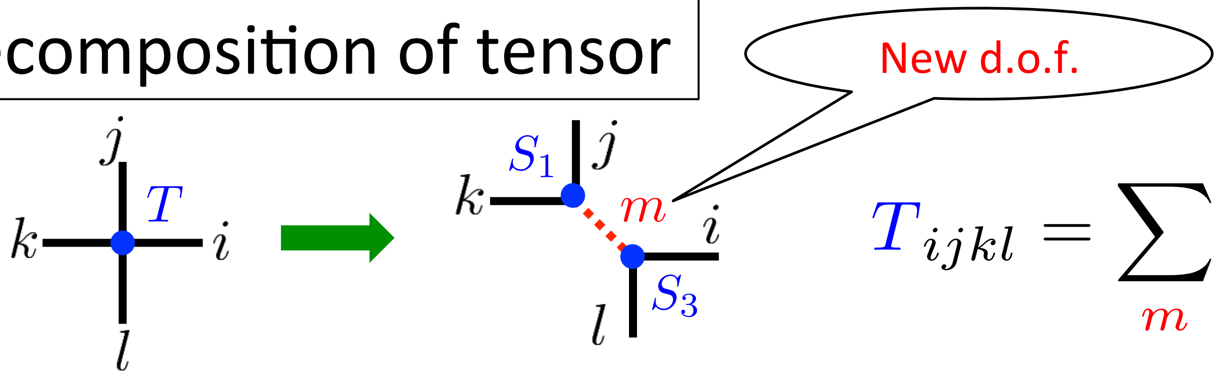
Direction

assuming translational invariance & local interaction

- Decompose tensor & extract important part = compression of information & emergence of new d.o.f.
- Making new tensor by combining the compact tensors = contracting old d.o.f.
- By repeating the decomposition and contraction, # tensors can be reduced
- After decreasing # of tensors, Z can be computed easily = ③

②Coarse graining

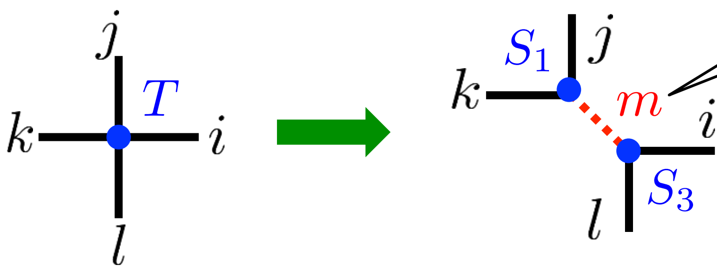
Decomposition of tensor



$$T_{ijkl} = \sum_{m} (S_1)_{jkm} (S_3)_{lim}$$

② Coarse graining

Decomposition of tensor



$$T_{ijkl} = \sum_{m} (S_1)_{jkm} (S_3)_{lim}$$

Singular value decomposition (SVD)

u, v : orthogonal matrix

$$M_{ab} = \sum_{m} u_{am} \sigma_m (v^T)_{mb}$$

$\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$: singular value

approx.

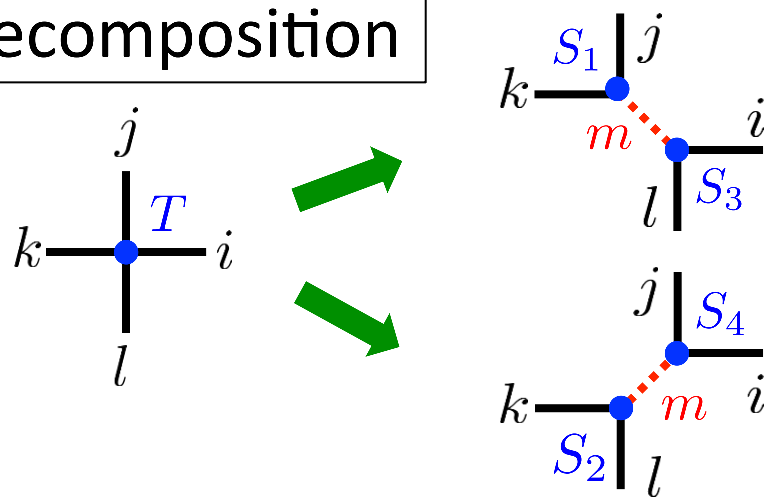
$$T_{ijkl} = M_{(kj),(il)} = \sum_{m}^{\text{all}} u_{(kj),m} \sqrt{\sigma_m} \cdot \sqrt{\sigma_m} v^{\mathrm{T}}_{m,(il)} \approx \sum_{m=1}^{D_{\text{cut}}} (S_1)_{jkm} (S_3)_{lim}$$

D_{cut}-truncated matrix is the best approximation among all rank-D_{cut} matrices

Tensor (matrix) is approximated by low-rank tensor = information compression

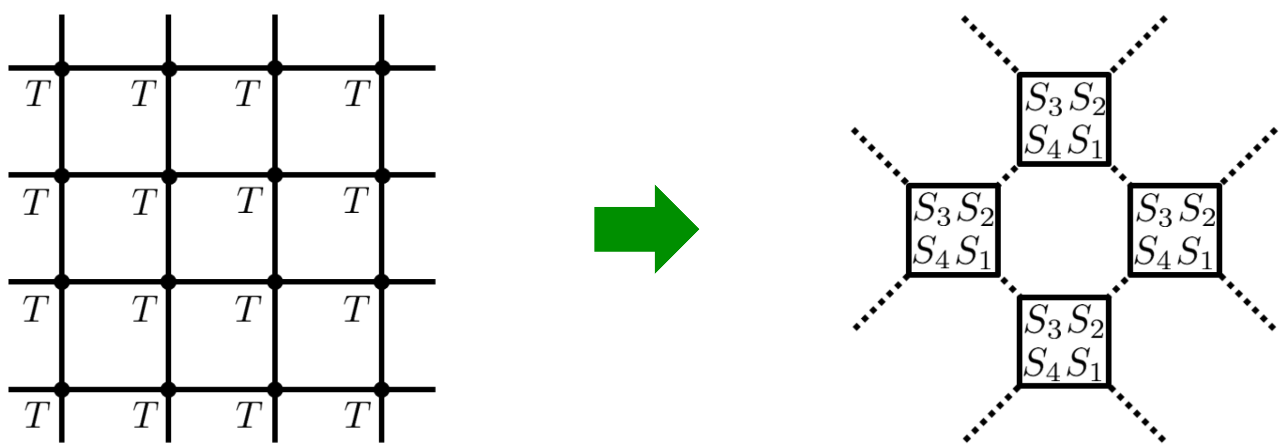
②Coarse graining

Decomposition



$$T_{ijkl} \approx \sum_{m=1}^{D_{\rm cut}} (S_1)_{jkm} (S_3)_{lim}$$

$$T_{ijkl} \approx \sum_{m=1}^{D_{\rm cut}} (S_2)_{klm} (S_4)_{ijm}$$



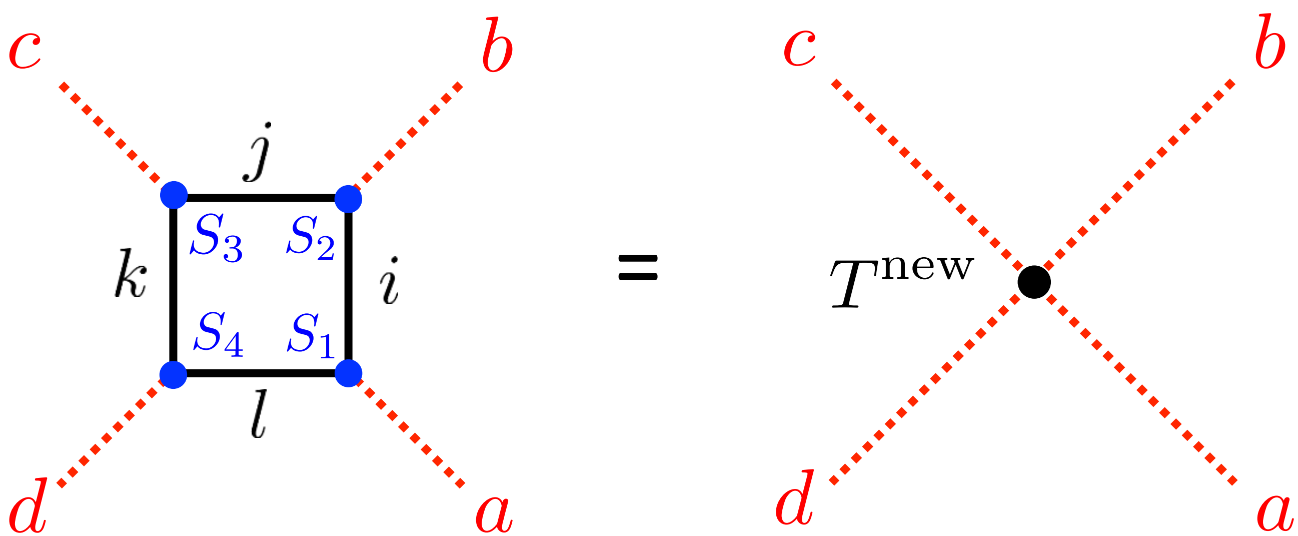
②Coarse graining

Making new tensor by contraction

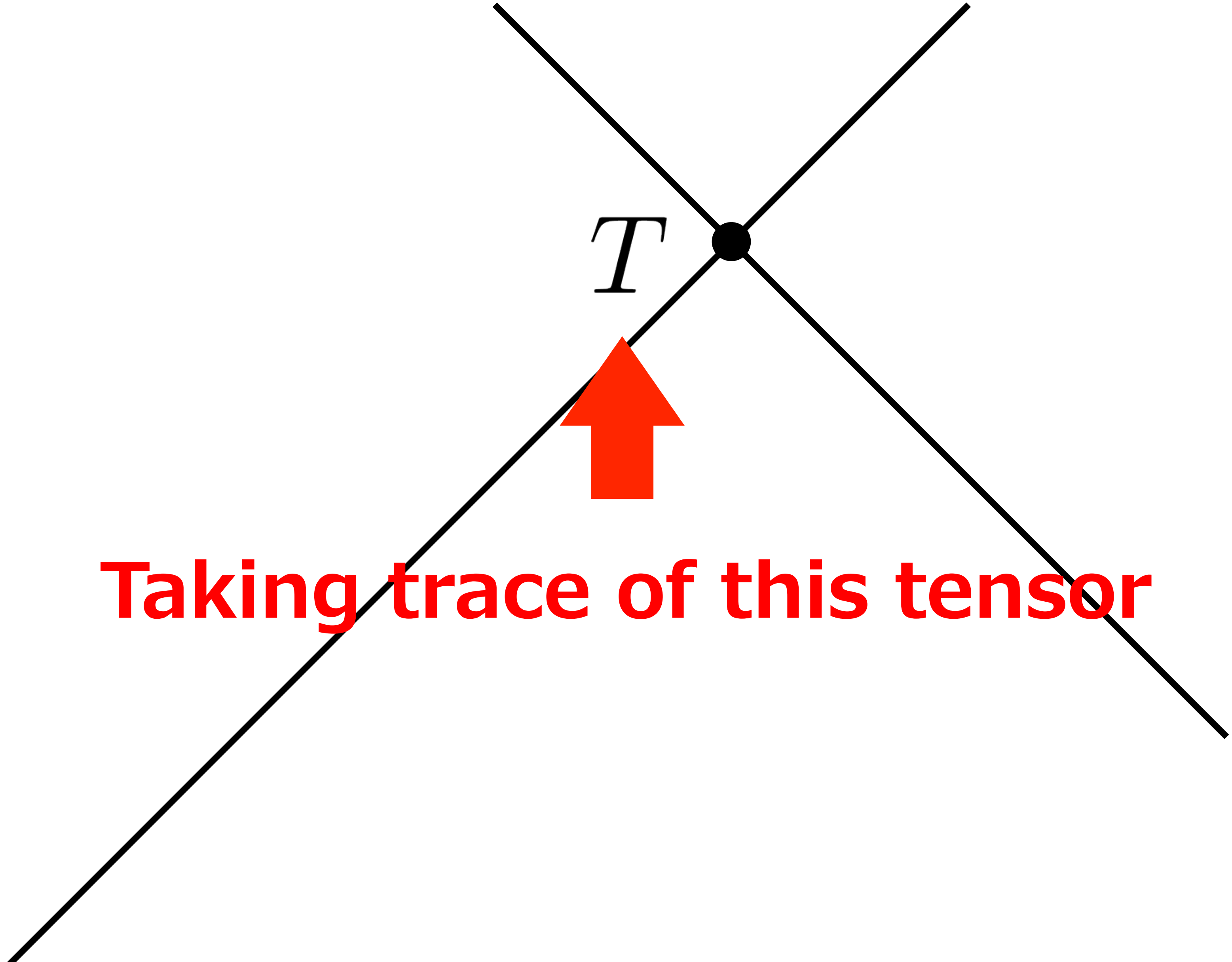
$$\sum_{i,j,k,l}^{\text{all}} (S_1)_{ila}(S_2)_{jib}(S_3)_{kjc}(S_4)_{lkd} = T^{\text{new}}_{abcd}$$

integrate out old d.o.f.

Renormalization!

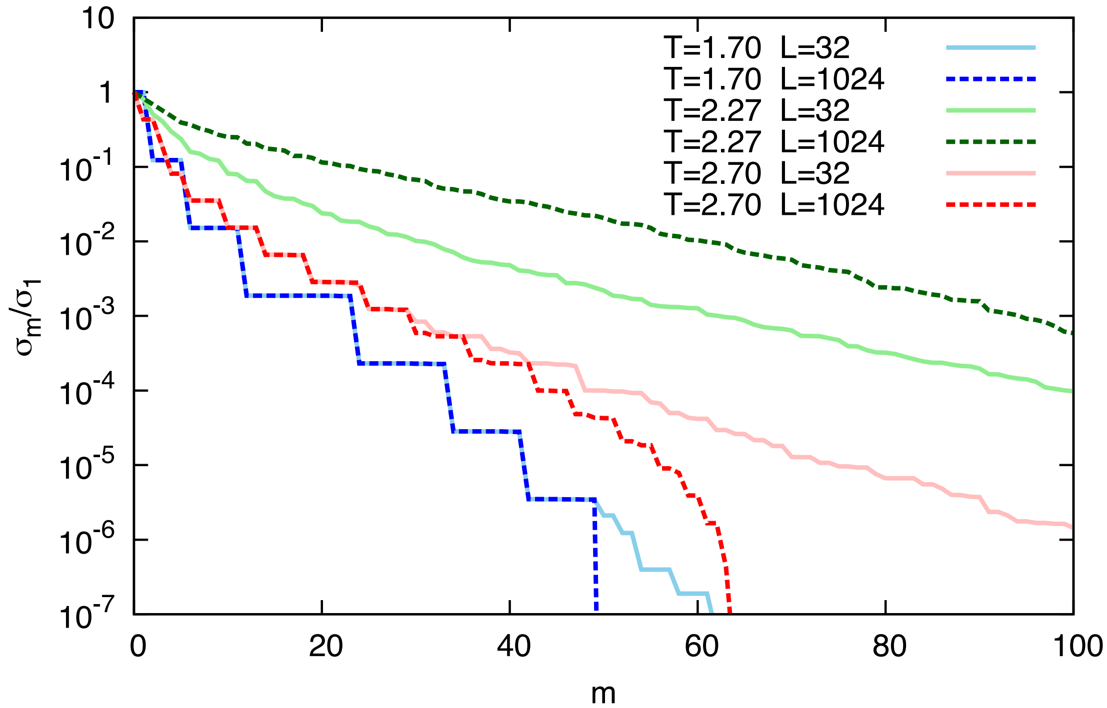


by H. Kawauchi (Kanazawa U.,D1)



Hierarchy of singular value

2D Ising model



D_{cut}=32

$$T_c = 2/[\ln(1+\sqrt{2})] = 2.269...$$

Entanglement entropy

$$S = -\sum_i \tilde{\sigma}_i \ln \tilde{\sigma}_i$$

normalized singular value

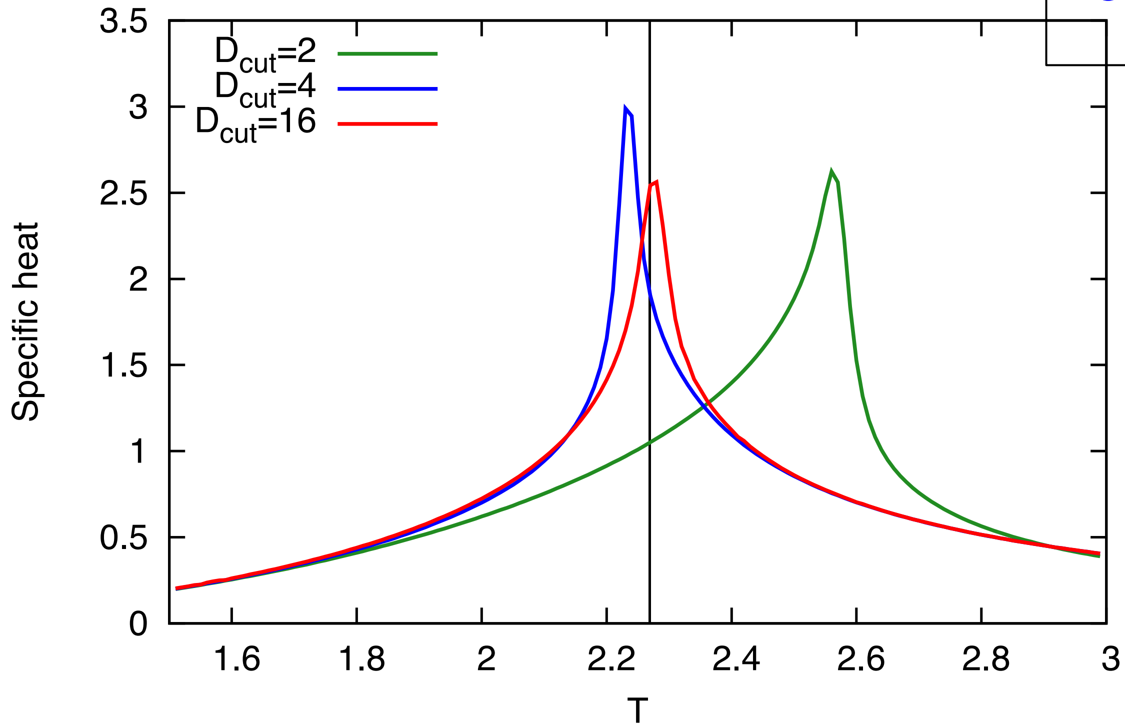
- Off criticality: good hierarchy (small S)
- Near criticality: hierarchy gets worse (large S)

like critical slowing down in MC

Tensor network renormalization (TNR) Evenbly&Vidal 2014 can cure the situation

D_{cut}-dependence of Specific heat

T_c=2.269...



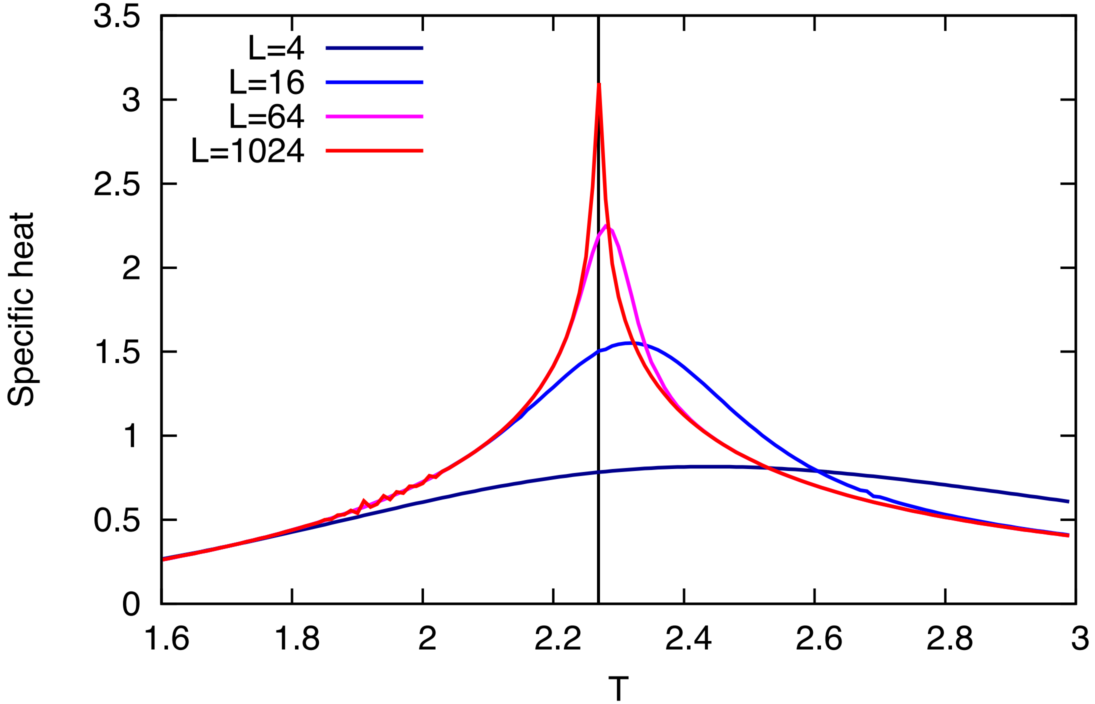
$$C = \frac{1}{L^2}\frac{\partial}{\partial T}\left(T^2 \frac{\partial \ln \mathcal{Z}}{\partial T}\right)$$

numerical derivative

L=128

For larger D_{cut}, transition T is closer to the exact T_c

Large volume



D_{cut}=32

$$T_c = 2/[\ln(1+\sqrt{2})] = 2.269\ldots$$

one-day work by using this MacBook Air

Cost ∝ log(Lattice size) × $(D_{cut})^6$ × [# temperature mesh]

Status of numerical study of TRG

- 2D system
  - Spin : Ising model Levin & Nave 2007, X-Y model Yu et al. 2013, O(3) Judah et al. 2014
  - Scalar : ϕ^4 theory Shimizu 2012
  - Gauge + Fermion : QED_2 Shimizu & Kuramashi 2013
  - $QED_2 + \theta$: Shimizu & Kuramashi 2014
  - Finite density : Gross-Neveu model ST & Yoshimura 2014
- Higher dimensional system
  - Higher order TRG(HOTRG): new coarse graining method applicable for any dimensional system
  - 3D Ising : Xie et al. 2012, 4D Ising : Yoshimura et al., 2015
- Specialized to Gauge theory
  - Decorated tensor network renormalization: Wittrich et al. 2014

Historical background

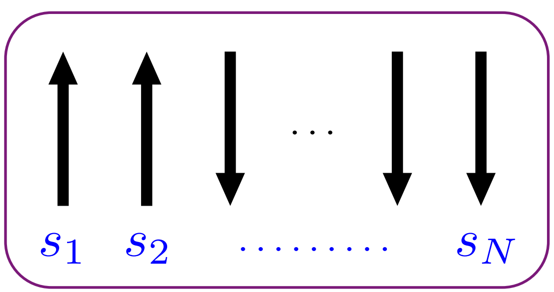
- Density matrix renormalization group (DMRG) White 1992
  - Variational method to obtain ground state in 1D quantum sys.
  - By selecting GOOD basis using SVD, one can drastically reduce the # of data $O(2^N) \rightarrow O(N)$, N:# sites (information compression)
  - Target: Wave function (in Tensor network representation)
  - Before this appears, limited to N=30. But DMRG enables N=100

2^N elements

$$|\psi\rangle = \sum_{s_1, s_2, \ldots, s_N} \psi_{s_1, s_2, \ldots, s_N} |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_N\rangle$$

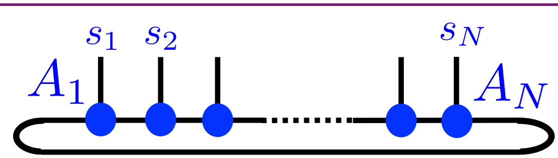
$$\mathrm{tr}\left[A_1^{s_1} A_2^{s_2} \cdots A_N^{s_N}\right]$$

$A^s : d \times d$ matrix



Matrix product state (MPS)

$2Nd^2$ elements



Tensor network rep. of wave function

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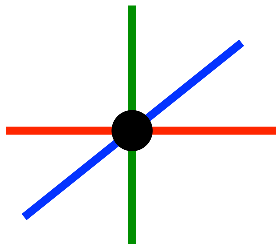
  - Before this appears, limited to N=30. But DMRG enables N=100
- Tensor renormalization group (TRG) Levin & Nave 2007
  - Target : Partition function of classical Stat. system
  - Express partition function in terms of tensor network rep., compress tensor by using SVD and coarse graining tensor
  - Very powerful in 2D system. Comparable to MC or more

MC	TRG
Boltzmann weight is interpreted as probability	Tensor network rep. of partition function (no probability interpretation)
Importance sampling	Compression of tensor by SVD
Statistical errors	Systematic errors
Sign problem may appear	No sign problem ∵ no probability
Critical slowing down	Efficiency of compression gets worse around criticality

Numerical aspect of TRG and Task

- Main computation（For HOTRG, n-dim system）
  - Decomposition ⇒ SVD(EVD) : $O(D_{cut}^{6})$
  - Contraction ⇒ matrix-matrix product : $O(D_{cut}^{4n-1})$ Hot spot
- Memory
  - # elements of tensor : $O(D_{cut}^{2n})$
  - internal d.o.f. ⇒ more memory



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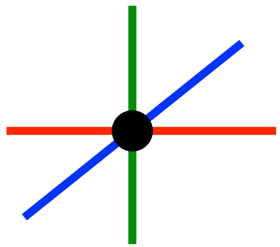


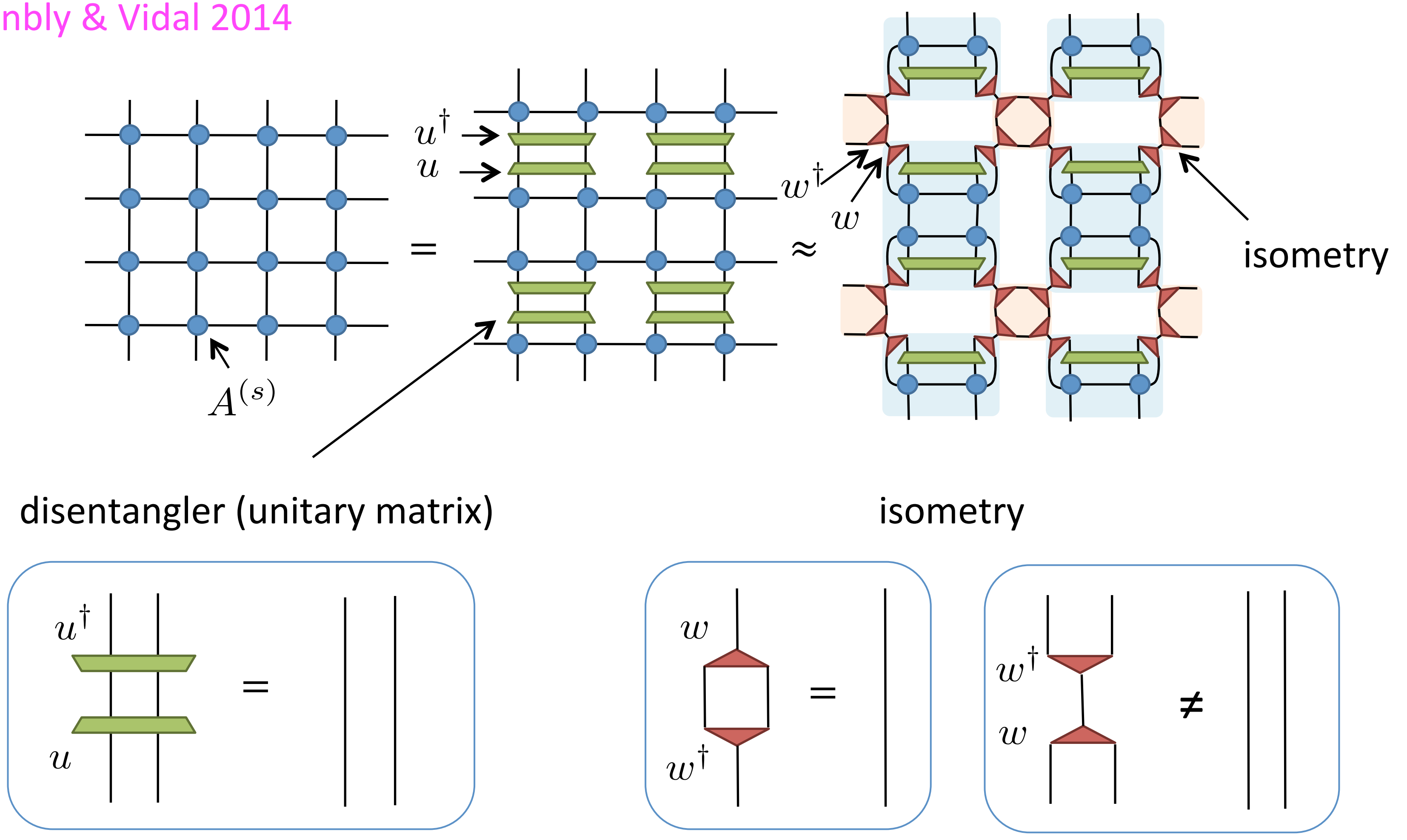
- matrix-matrix product : Level 3 BLAS ≫ SVD
- Better coarse graining with small D_{cut} (highly compression)?

Improvement of Coarse graining

- Tensor Entanglement Filtering Renormalization
  Gu et al. 2009
  - Removing short range correlation (partially)
  - works in off-criticality but not near criticality
- Second TRG Xie et al. 2009
  - Optimization including environment (TRG: locally optimal)
  - works in off-criticality but not near criticality
- Tensor Network Renormalization (TNR)
  Evenbly & Vidal 2014
  - First remove short correlation (entanglement) by using disentangler, and then coarse graining is performed
  - Even around criticality, sustainable coarse graining is realized

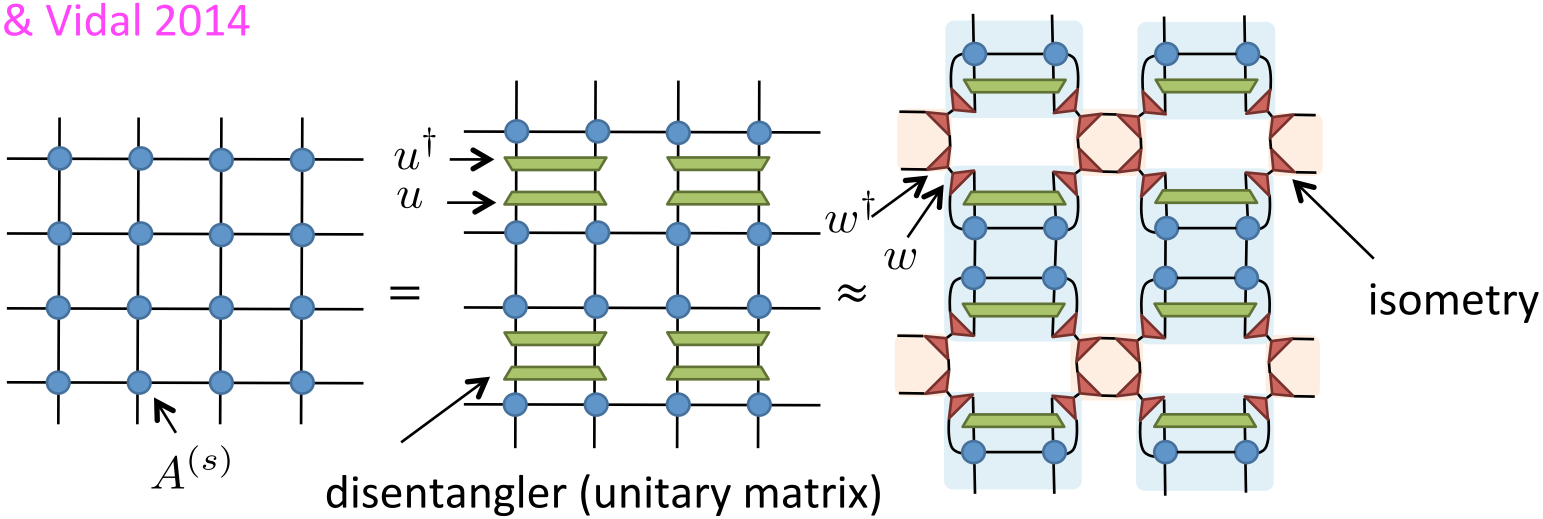
Tensor Network Renormalization

Evenbly & Vidal 2014

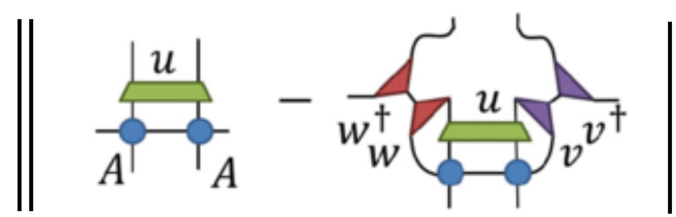


Tensor Network Renormalization

Evenbly & Vidal 2014



u & *w* are determined such that δ is minimized

$\delta =$ 

Cost: $O(D_{cut}^7)$ for TNR
$O(D_{cut}^6)$ for TRG

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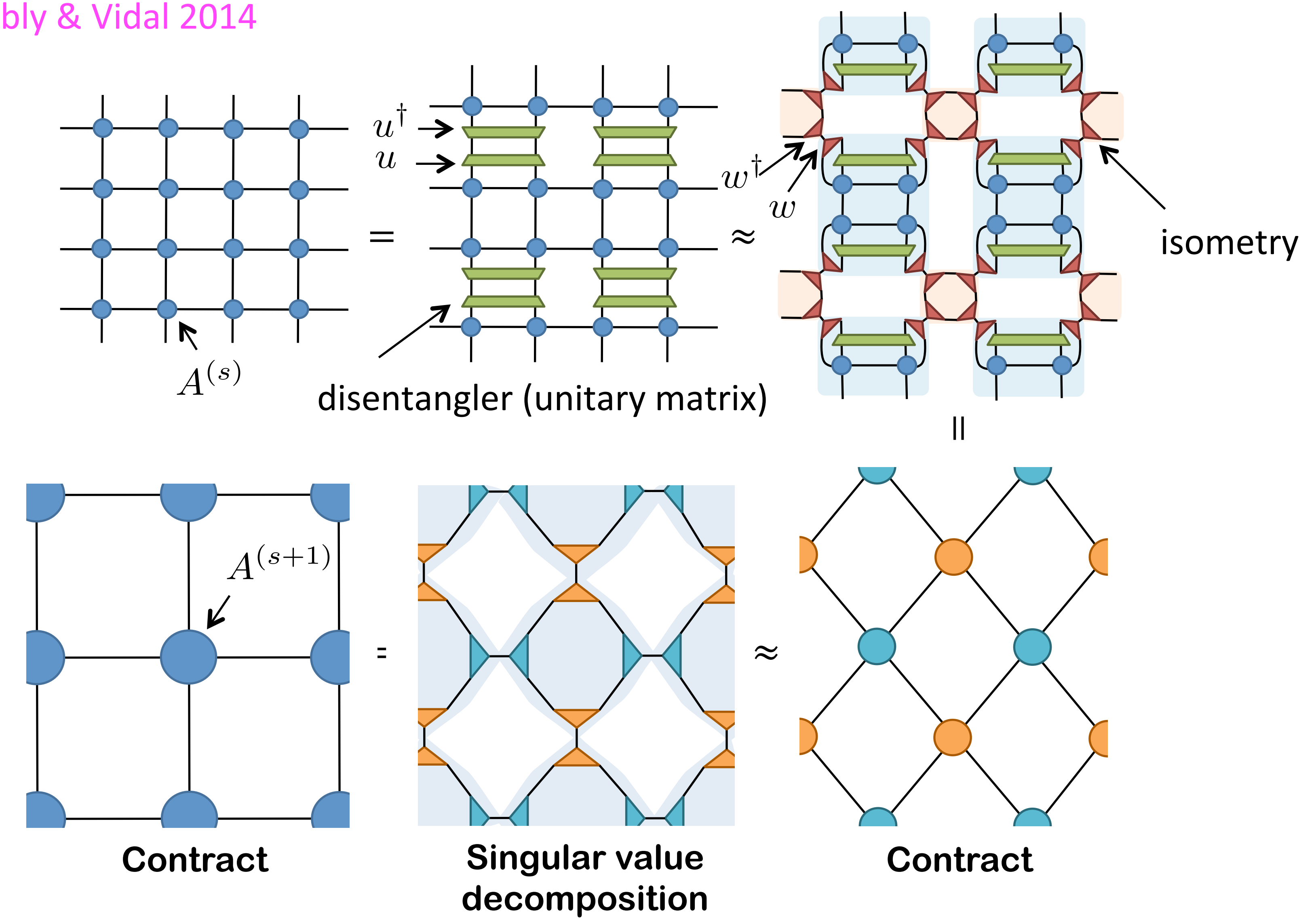

Tensor Network Renormalization

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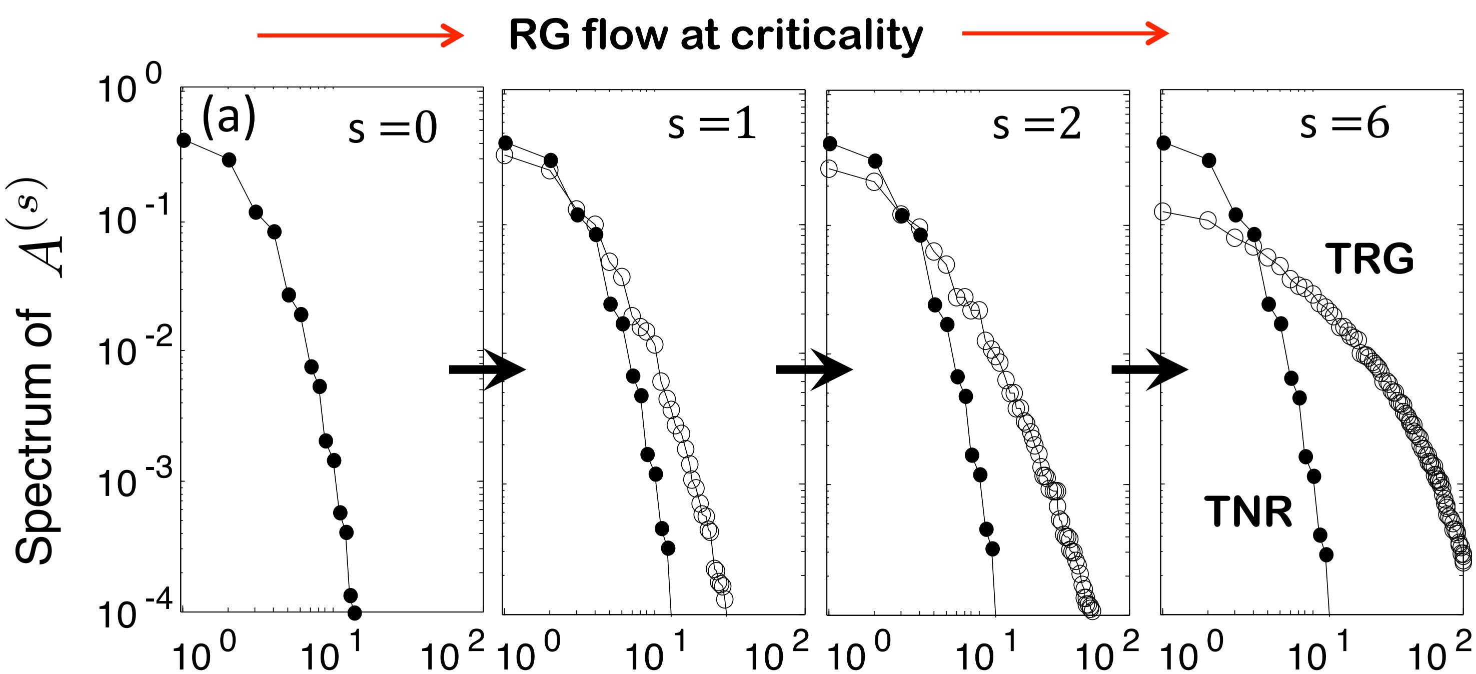
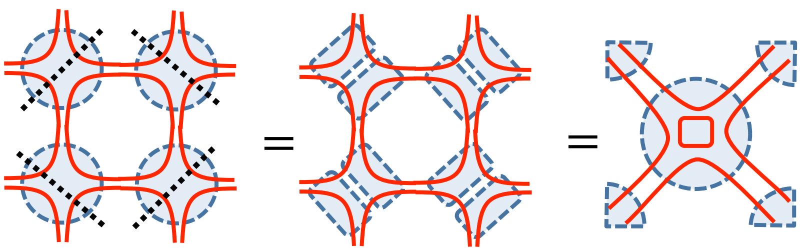


Hierarchy for TNR is robust even after several iterations:
TNR is a sustainable coarse graining

Why TNR works

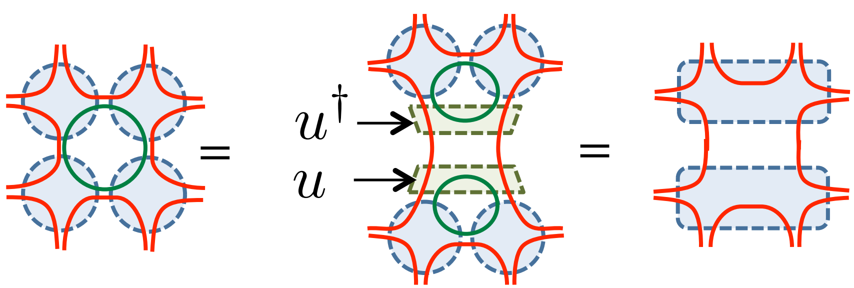
Evenbly & Vidal 2014

TRG with Corner Double Line tensor

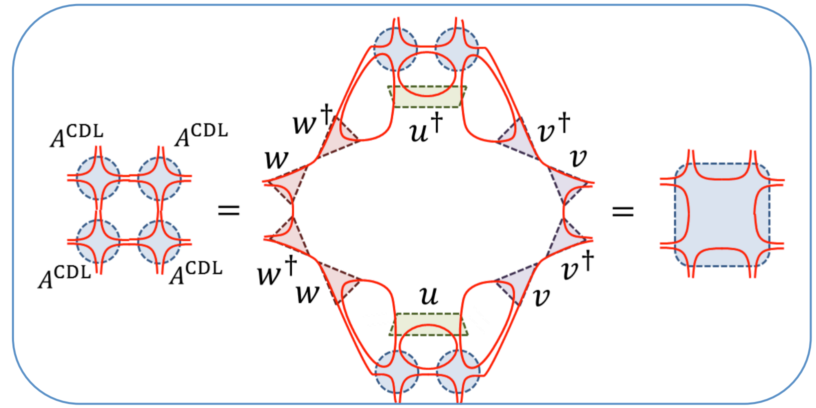


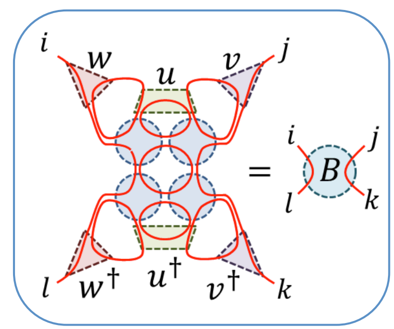
short-correlation represented by CDL remains

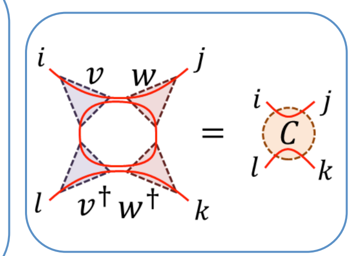
A role of disentangler

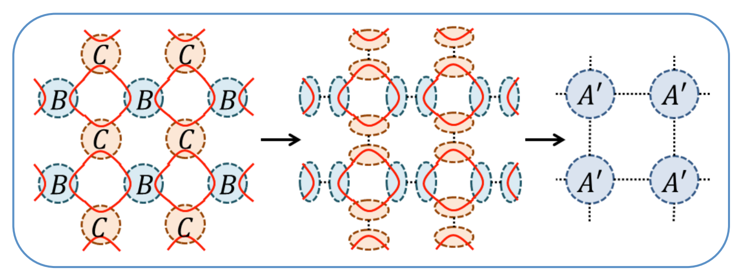


short-correlation can be removed









Summary

- No sign problem in TRG
- Key of TRG: information compression using SVD
- Inefficient around criticality
- But, Tensor Network Renormalization (TNR) can solve the problem. An extension to higher dimensional system is still missing
- In any case, TRG is very powerful in low (2D) dimensional system

Future prospects

- Long way to 4D QCD (+μ&θ)
  - Higher dimensional system (4D system)
    - Cost : $O(D_{cut}^{15})$, Memory : $O(D_{cut}^{8})$
    - efficient parallelization?
    - TNR?
  - Non-Abelian gauge theory
    - Character expansion ⇒ Tensor network rep. is OK but internal d.o.f. is huge
- Low dim. system suffering from the sign problem ?
  - 2D CP(N-1) + θ: Strong CP problem 2015 Kawauchi&ST
  - Lattice SUSY, Lattice chiral gauge theory

Application to finite fermion density system: Gross-Neveu model

2D Gross-Neveu model

Continuum

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + m)\psi - \frac{g^2}{2}\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\right]$$

$N = 1$

$\partial\!\!\!/ = \partial_\nu \gamma_\nu$ $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$ 2 components in spinor space

2D Gross-Neveu model

Continuum

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + m)\psi - \frac{g^2}{2}\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\right]$$

$N = 1$

$\partial\!\!\!/ = \partial_\nu \gamma_\nu$ $\quad \bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$ 2 components in spinor space

Lattice

site

chemical potential

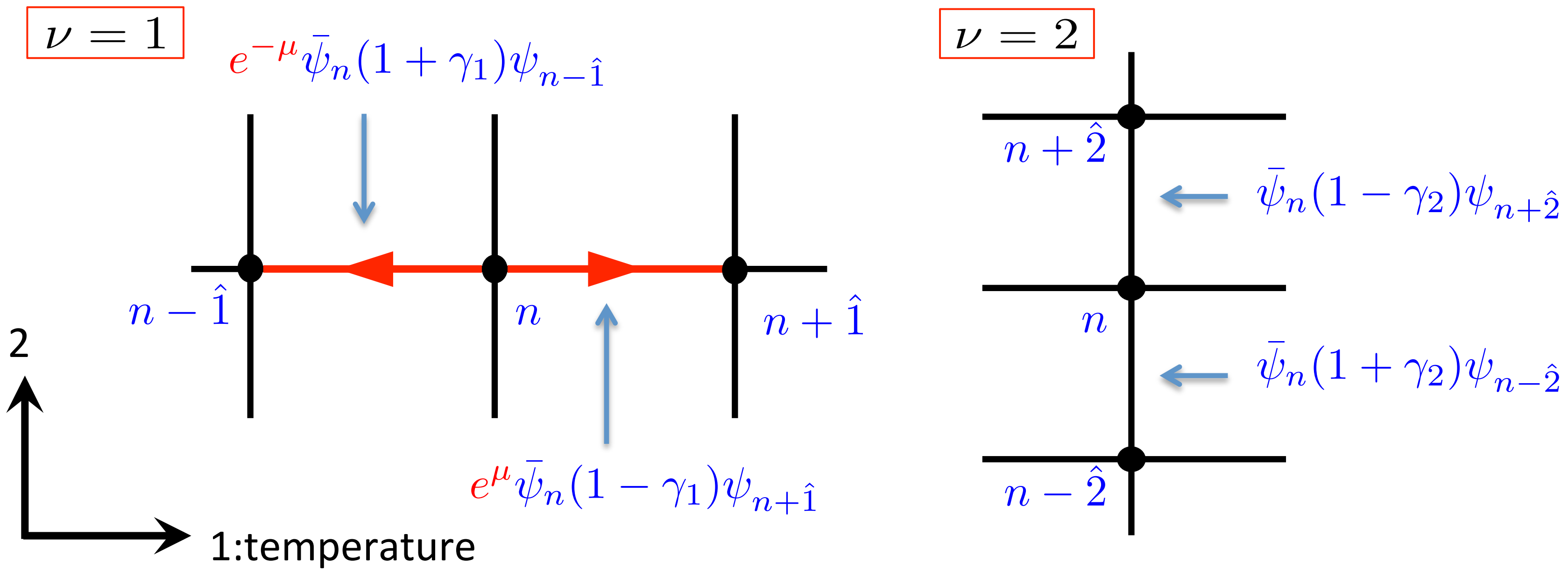
$$\partial\!\!\!/ \longrightarrow \frac{\Delta\!\!\!/^{\rm f} + \Delta\!\!\!/^{\rm b} - \Delta^{\rm b}_\nu \Delta^{\rm f}_\nu}{2} \qquad \Delta^{\rm f/b}_\nu \psi_n = \pm\left(e^{\pm\mu\delta_{\nu,1}}\psi_{n\pm\hat{\nu}} - \psi_n\right)$$

Wilson fermions

ν=1: temperature direction

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\sum_{\nu=1}^{2}\left[e^{\mu\delta_{\nu,1}}\bar{\psi}_n(1-\gamma_\nu)\psi_{n+\hat{\nu}} + e^{-\mu\delta_{\nu,1}}\bar{\psi}_n(1+\gamma_\nu)\psi_{n-\hat{\nu}}\right] \\ &+ (m+2)\bar{\psi}_n\psi_n - \frac{g^2}{2}\left[(\bar{\psi}_n\psi_n)^2 + (\bar{\psi}_n i\gamma_5\psi_n)^2\right]\end{aligned}$$

2D Gross-Neveu model



$$\mathcal{L} = -\frac{1}{2}\sum_{\nu=1}^{2}\left[e^{\mu\delta_{\nu,1}}\bar{\psi}_n(1-\gamma_\nu)\psi_{n+\hat{\nu}} + e^{-\mu\delta_{\nu,1}}\bar{\psi}_n(1+\gamma_\nu)\psi_{n-\hat{\nu}}\right]$$

Hopping term : Energy term in Ising model

$$+ \quad (m+2)\bar{\psi}_n\psi_n - \frac{g^2}{2}\left[(\bar{\psi}_n\psi_n)^2 + (\bar{\psi}_n i\gamma_5\psi_n)^2\right]$$

mass term and interaction term : magnetic term in Ising model

Grassmann TRG

- Formulation :

  Gu et al., 2010, Gu 2011

- 2D relativistic system

  Shimizu & Kuramashi 2013

For fermion system, procedure is similar to that of Ising model. But, one has to make tensor for the model by oneself.

How to make tensor

1) Expand BW, and then new d.o.f.（bosonic） appears

$$e^{-\sum_n \mathcal{L}} = \prod_n \cdots \exp\left[\frac{1}{2}e^{-\mu}\bar{\psi}_{n+\hat{1}}(1+\gamma_1)\psi_n\right]\cdots$$

Hopping term for 1st direction

How to make tensor

1) Expand BW, and then new d.o.f.（bosonic） appears

$$e^{-\sum_n \mathcal{L}} = \prod_n \cdots \exp\left[\frac{1}{2} e^{-\mu} \bar{\psi}_{n+\hat{1}} (1+\gamma_1) \psi_n \right] \cdots$$

$$= \prod_n \cdots \exp\left[e^{-\mu} \bar{\psi}_{n+\hat{1},1} \psi_{n,1} \right] \cdots$$

$$\gamma_1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

How to make tensor

1) Expand BW, and then new d.o.f.（bosonic）appears

$$e^{-\sum_n \mathcal{L}} = \prod_n \cdots \exp\left[\frac{1}{2} e^{-\mu} \bar{\psi}_{n+\hat{1}} (1+\gamma_1) \psi_n \right] \cdots$$

$$= \prod_n \cdots \exp\left[e^{-\mu} \bar{\psi}_{n+\hat{1},1} \psi_{n,1} \right] \cdots$$

$$= \prod_n \cdots \sum_{t_{n,1}=0}^{1} \left(e^{-\mu} \bar{\psi}_{n+\hat{1},1} \psi_{n,1} \right)^{t_{n,1}} \cdots$$

$$= \sum_{\{t\}} \prod_n \cdots \left(e^{-\mu} \bar{\psi}_{n+\hat{1},1} \psi_{n,1} \right)^{t_{n,1}} \cdots$$

Finite expansion due to Grassmann

new d.o.f. (bosonic)

the same goes for

- ✧ other hopping terms
- ✧ mass term
- ✧ interaction term

How to make tensor

1) Expand BW, and then new d.o.f. (bosonic) appears
2) Integrate out old d.o.f. (Grassmann) and then one obtains tensor network rep.

New d.o.f.

$$\mathcal{Z} = \sum_{\{t,x,s\}=0,1} \int \left(\prod_n d\psi_n d\bar{\psi}_n \right)$$

$$\times \prod_n \left[e^{-\mu} \bar{\psi}_{n+\hat{1},1} \psi_{n,1} \right]^{t_{n,1}} \left[e^{\mu} \bar{\psi}_{n,2} \psi_{n+\hat{1},2} \right]^{t_{n,2}}$$ hopping term for 1st direc.

$$\times \quad [\ldots]^{x_{n,1}} [\ldots]^{x_{n,2}}$$ hopping term for 2nd direc.

$$\times \left[-(m+2)\bar{\psi}_{n,1}\psi_{n,1} \right]^{s_{n,1}} \left[-(m+2)\bar{\psi}_{n,2}\psi_{n,2} \right]^{s_{n,2}}$$ mass term

$$\times \left[2g^2 \bar{\psi}_{n,1}\psi_{n,1}\bar{\psi}_{n,2}\psi_{n,2} \right]^{s_{n,3}}$$ 4-fermion interaction term

How to make tensor

1) Expand BW, and then new d.o.f.（bosonic） appears
2) Integrate out old d.o.f. (Grassmann) and then one obtains tensor network rep.
3) However, before/after the integration, sign factor originating from Grassmann nature may appear “Randomly” → “Sign problem”?

How to make tensor

1) Expand BW, and then new d.o.f.（bosonic） appears
2) Integrate out old d.o.f. (Grassmann) and then one obtains tensor network rep.
3) However, before/after the integration, sign factor originating from Grassmann nature may appear "Randomly" → "Sign problem"?
4) To deal with the sign factor better, introduce new Grassmann variables

New Grassmann variable

same exponent

Say

$$\left(e^{-\mu}\bar{\psi}_{n+\hat{1},1}\psi_{n,1}\right)^{t_{n,1}} = \int (d\xi_{n,1}\xi_{n,1})^{t_{n,1}} \left(e^{-\mu}\bar{\psi}_{n+\hat{1},1}\psi_{n,1}\right)^{t_{n,1}}$$

=1

New Grassmann variable

New Grassmann variable

Say

$$\left(e^{-\mu}\bar{\psi}_{n+\hat{1},1}\psi_{n,1}\right)^{t_{n,1}} = \int (d\xi_{n,1}\xi_{n,1})^{t_{n,1}}\left(e^{-\mu}\bar{\psi}_{n+\hat{1},1}\psi_{n,1}\right)^{t_{n,1}}$$

$$= \int \left(e^{-\mu}\psi_{n,1}d\xi_{n,1}\bar{\psi}_{n+\hat{1},1}\xi_{n,1}\right)^{t_{n,1}}$$

"shuffle"

New Grassmann variable

Say

$$\left(e^{-\mu}\underline{\bar{\psi}_{n+\hat{1},1}}\,\underline{\psi_{n,1}}\right)^{t_{n,1}} = \int (d\xi_{n,1}\xi_{n,1})^{t_{n,1}} \left(e^{-\mu}\bar{\psi}_{n+\hat{1},1}\psi_{n,1}\right)^{t_{n,1}}$$

$$= \int \left(e^{-\mu}\psi_{n,1}d\xi_{n,1}\bar{\psi}_{n+\hat{1},1}\xi_{n,1}\right)^{t_{n,1}}$$

$$= \int \left(e^{-\mu/2}\underline{\psi_{n,1}}d\xi_{n,1}\right)^{t_{n,1}} \left(e^{-\mu/2}\underline{\bar{\psi}_{n+\hat{1},1}}\xi_{n,1}\right)^{t_{n,1}}$$

separate $\underline{\psi_n}$ $\underline{\bar{\psi}_{n+\hat{1}}}$

By introducing new Grassmann variable and pairing it with old d.o.f., one can avoid an awkward manipulation of sign factors and can easily integrate out the old d.o.f.

Introduce new Grassmann variable for other hopping terms as well

Tensor network rep.

$$\mathcal{Z} = \sum_{\{t,x\}} \int \prod_n \mathcal{T}_{t_n x_n t_{n-\hat{1}} x_{n-\hat{2}}}$$

$$t_n = (t_{n,1}, t_{n,2})$$
$$x_n = (x_{n,1}, x_{n,2})$$

fermionic part

$$\mathcal{T}_{t_n x_n t_{n-\hat{1}} x_{n-\hat{2}}} = T_{t_n x_n t_{n-\hat{1}} x_{n-\hat{2}}} \times d\bar{\xi}_{n,2}^{t_{n,2}} d\xi_{n,1}^{t_{n,1}} d\bar{\eta}_{n,2}^{x_{n,2}} d\eta_{n,1}^{x_{n,1}} d\xi_{n,2}^{t_{n-\hat{1},2}} d\bar{\xi}_{n,1}^{t_{n-\hat{1},1}} d\eta_{n,2}^{x_{n-\hat{2},2}} d\bar{\eta}_{n,1}^{x_{n-\hat{2},1}} \times (\bar{\xi}_{n+\hat{1},1}\xi_{n,1})^{t_{n,1}} (\bar{\xi}_{n,2}\xi_{n+\hat{1},2})^{t_{n,2}} (\bar{\eta}_{n+\hat{2},1}\eta_{n,1})^{x_{n,1}} (\bar{\eta}_{n,2}\eta_{n+\hat{2},2})^{x_{n,2}}$$

bosonic part

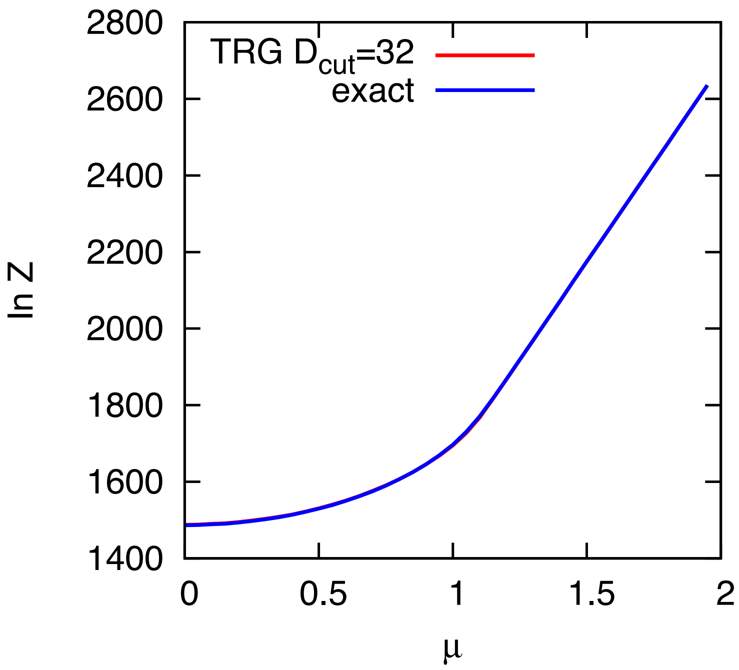
$$T_{t_n x_n t_{n-\hat{1}} x_{n-\hat{2}}} = \exp\left[\frac{\mu}{2}(-t_{n,1} - t_{n-\hat{1},1} + t_{n,2} + t_{n-\hat{1},2})\right] \times \sum_{s_{n,1}, s_{n,2}, s_{n,3}=0}^{1} \int d\psi_n d\bar{\psi}_n \left(2g^2 \bar{\psi}_{n,1}\psi_{n,1}\bar{\psi}_{n,2}\psi_{n,2}\right)^{s_{n,3}} \times \left(-(m+2)\bar{\psi}_{n,1}\psi_{n,1}\right)^{s_{n,1}} \left(-(m+2)\bar{\psi}_{n,2}\psi_{n,2}\right)^{s_{n,2}} \times \bar{\chi}_{n,1}^{x_{n-\hat{2},1}} \chi_{n,2}^{x_{n-\hat{2},2}} \bar{\psi}_{n,1}^{t_{n-\hat{1},1}} \psi_{n,2}^{t_{n-\hat{1},2}} \bar{\chi}_{n,2}^{x_{n,2}} \chi_{n,1}^{x_{n,1}} \psi_{n,1}^{t_{n,1}} \bar{\psi}_{n,2}^{t_{n,2}}$$

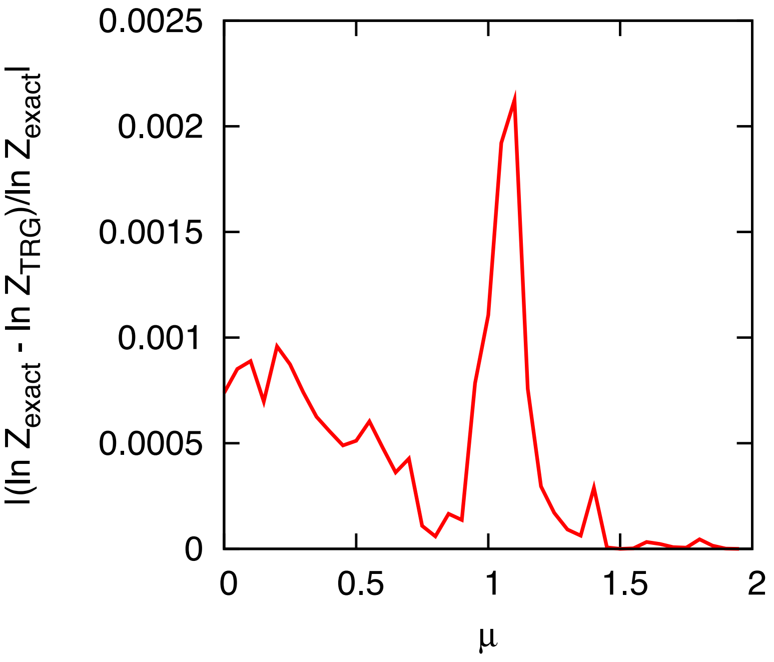
Other issues

- Coarse-graining can be done with some care about the fermionic part of tensor
- For finite temperature system, anti-periodic BC is imposed for fermions. This can be taken into account by inserting another tensor (matrix) in one-time slice
- If you want more flavors N, the number of index of tensor increases N times.

Numerical results: ln Z

$32^2 \qquad g = m = 0$





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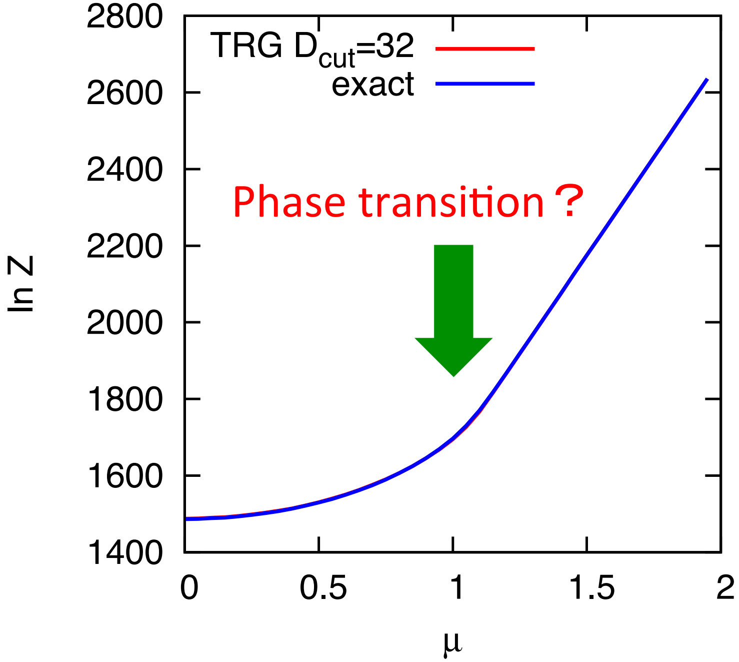
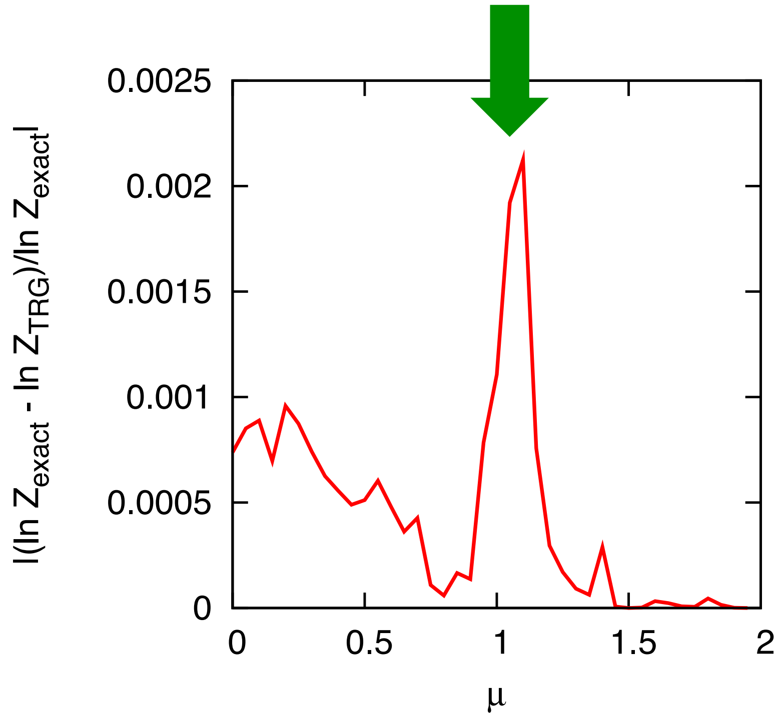
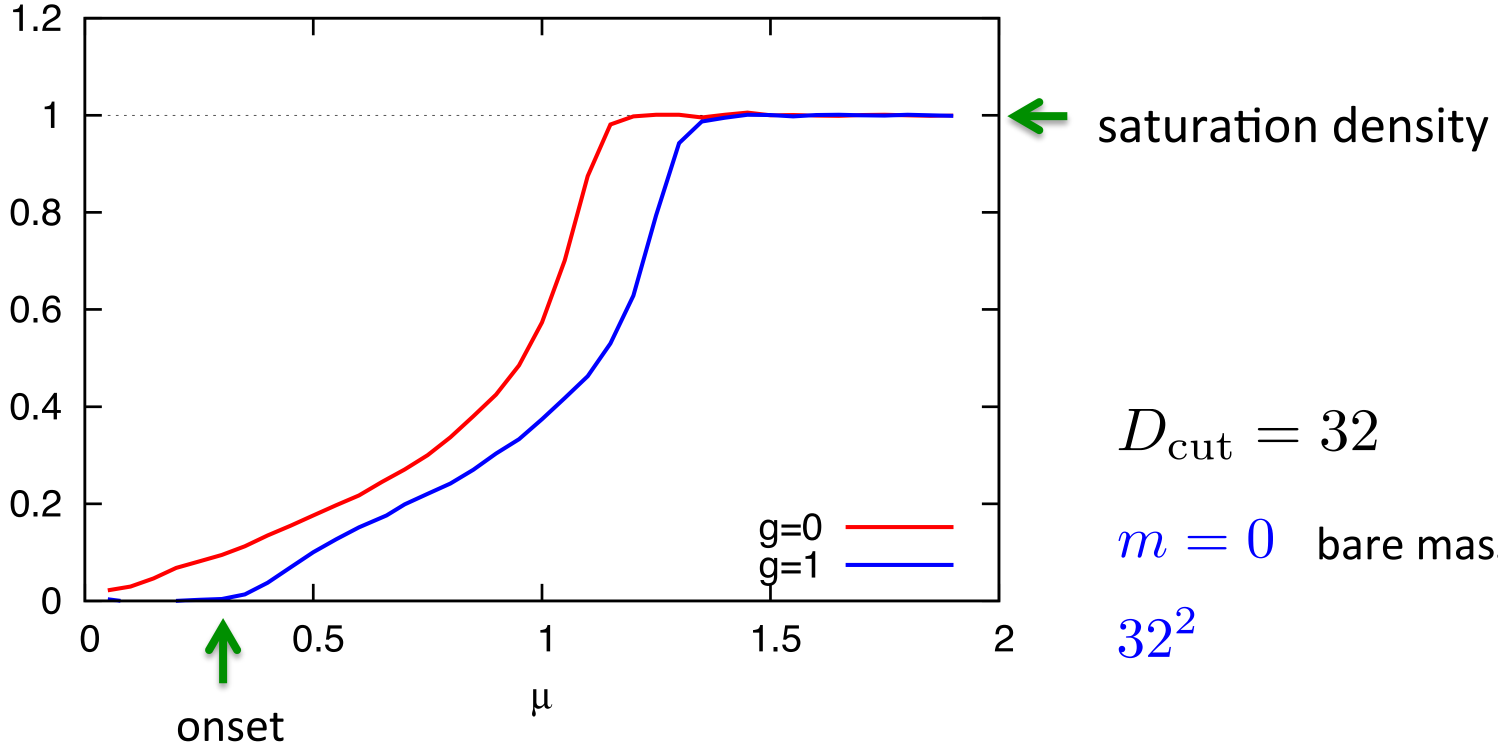




Truncation error gets larger around PT?

Fermion number density



$$n = \frac{T}{L}\frac{\partial \ln Z}{\partial \mu}$$

$D_{\mathrm{cut}} = 32$

$m = 0$ bare mass

32^2

Summary

- Grassmann TRG
  - Introduce new Grassmann variables to deal with "new sign problem"
  - Demonstrate an extension to finite density in GN model
- Outlook
  - Extension to Higher dimensional system
    - Higher order Grassmann TRG Sakai & ST in progress
    - 2+1D Wilson Fermions = Domain wall Fermions in 2D
  - Lattice SUSY, Lattice chiral gauge theory