Some topics in 2-flavor QCD at zero and finite temperature

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Workshop Lattice QCD at finite temperature and density,
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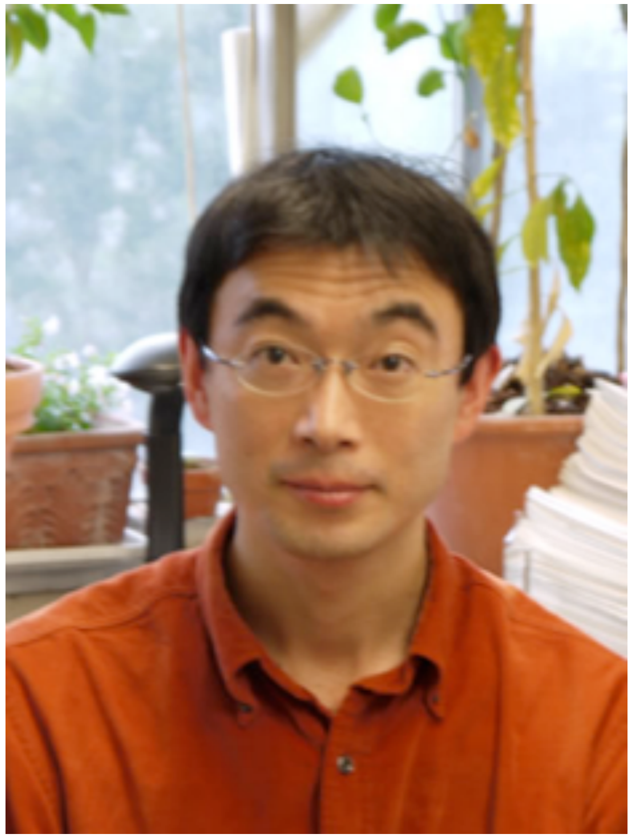
Part I

Chiral Symmetry Restoration and Eigenvalue Density of Dirac Operator at Finite Temperature

with H. Fukaya and Y. Taniguchi for JLQCD Collaboration



Also related to Taniguchi-san's talk after this.

1. Introduction

Chiral symmetry of QCD

phase transition

low T $U(1)_B \otimes SU(N_f)_V$ $\Rightarrow$ high T $U(1)_B \otimes SU(N_f)_L \otimes SU(N_f)_R$

restoration of chiral symmetry

Some questions

1. Eigenvalue distribution of Dirac operator

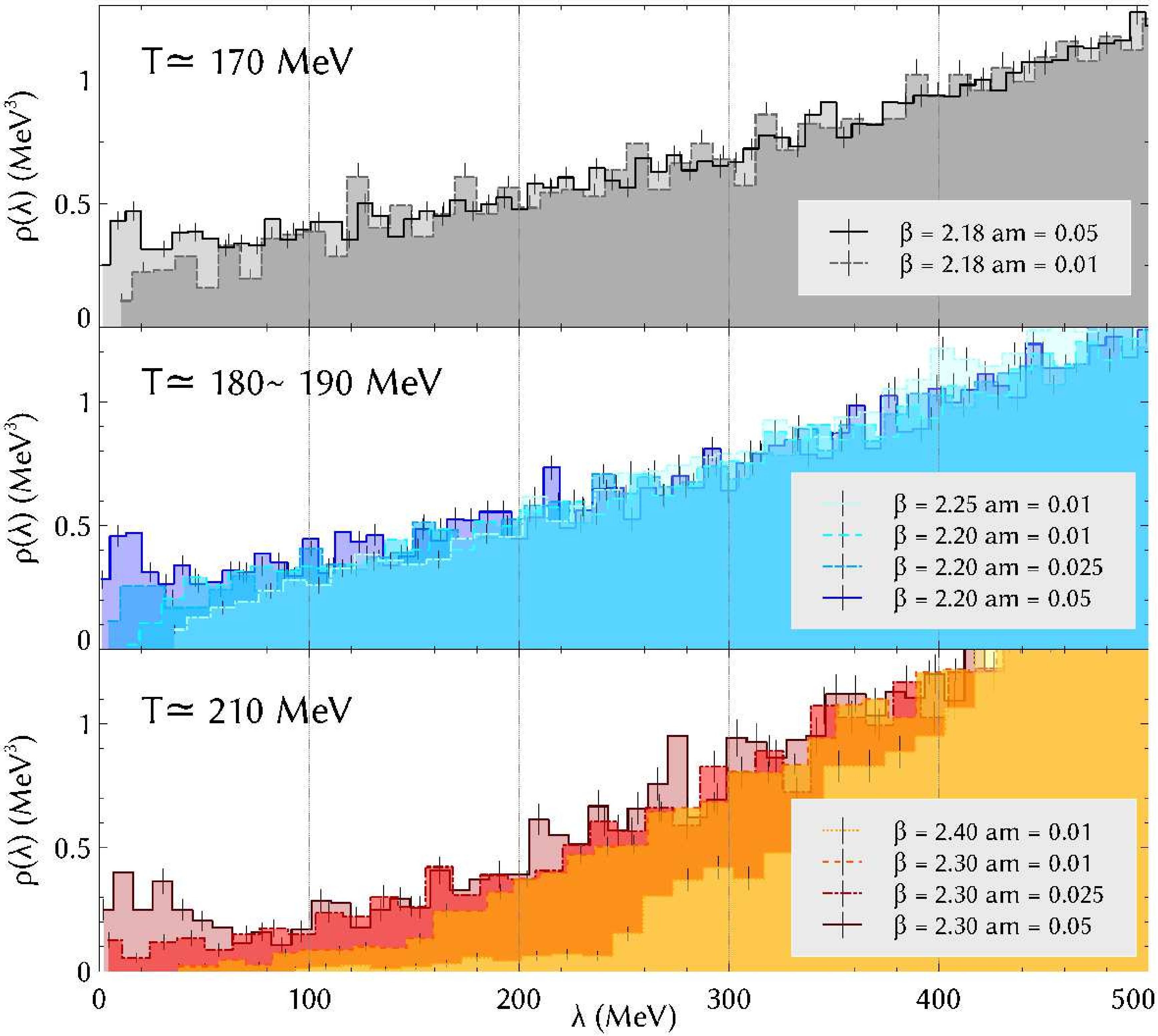
$\Updownarrow$ related ?

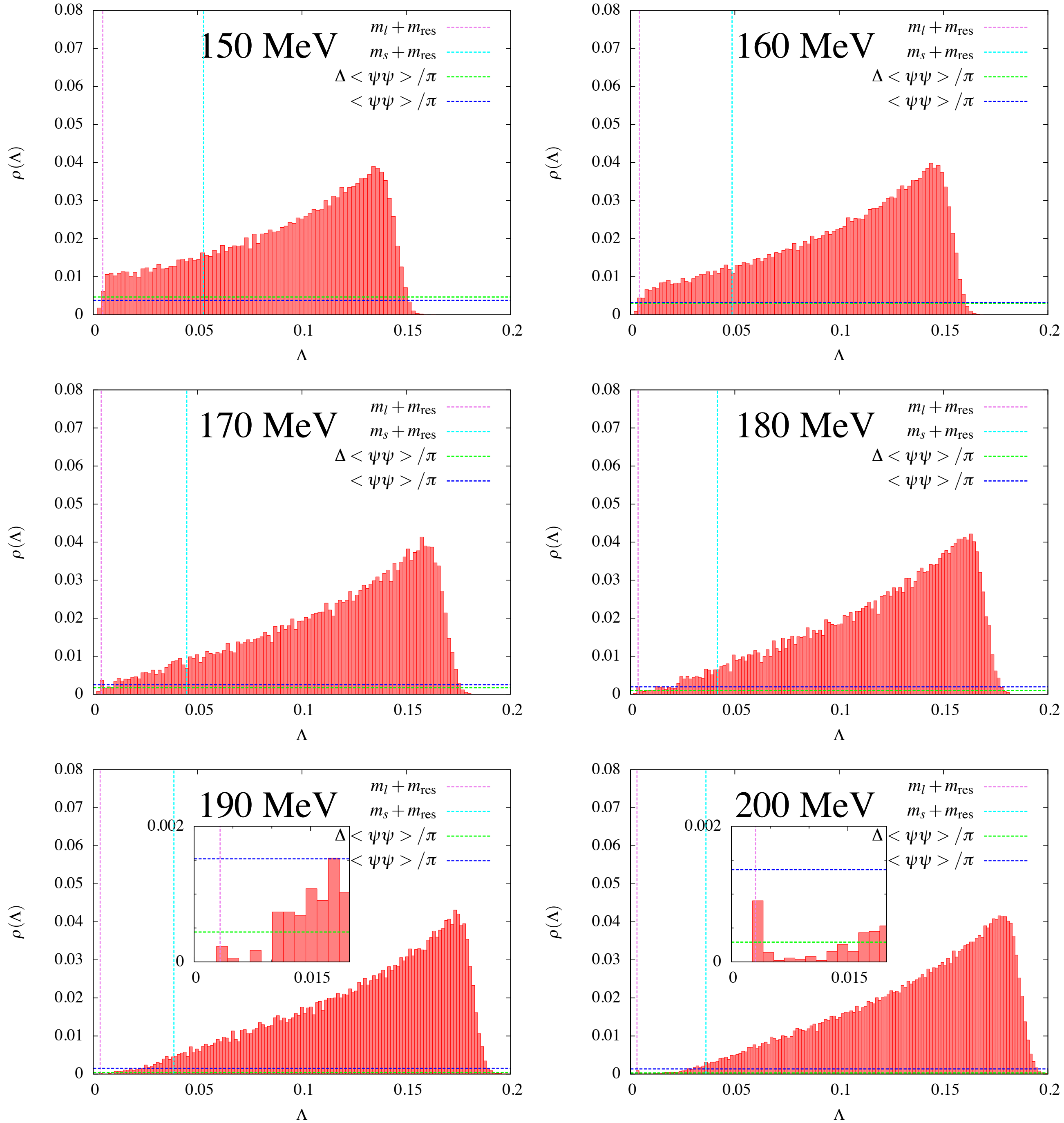
2. Recovery of U(1)_A symmetry at high T ?

Previous studies on 1

$$\rho(\lambda) = \lim_{V\to\infty} \frac{1}{V} \sum_n \delta(\lambda - \lambda_n)$$

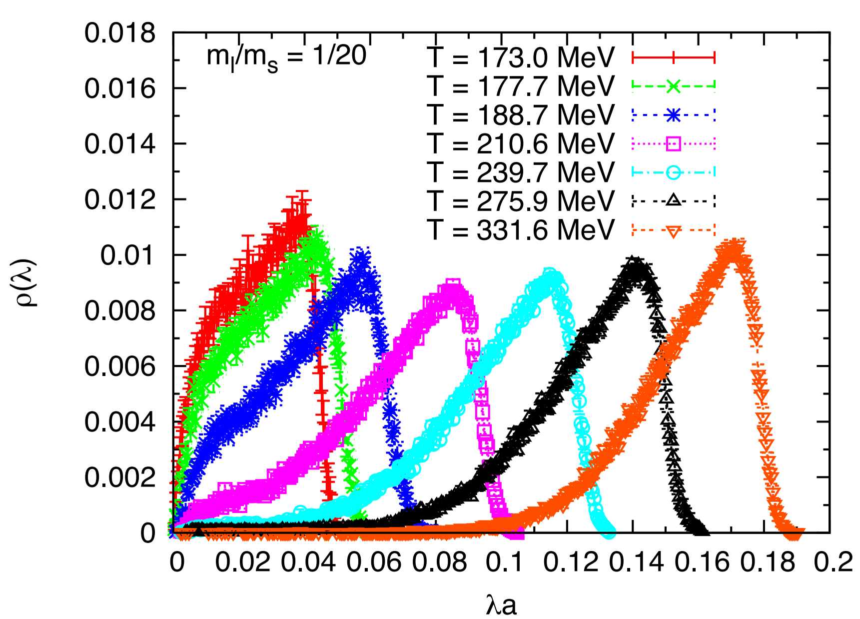
Cossu *et al.* (JLQCD 2013), Overlap
Phys. Rev. D87 (2013) 114514

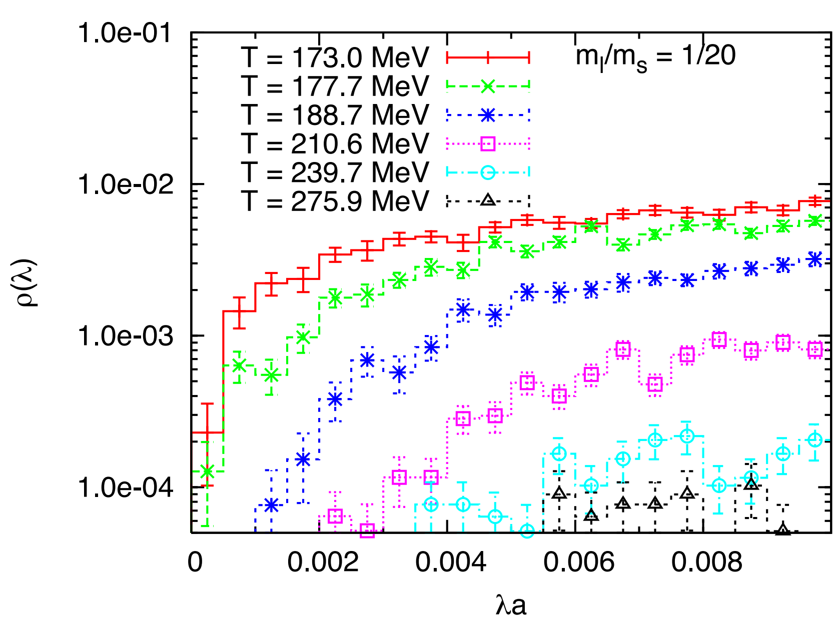




Lin (HotQCD11), DW

Is small λ suppressed ?





Ohno *et al.* (11), HISQ

Previous studies on 2

$$\chi_{U(1)_A} = \int d^4x \, \langle \sigma(x)\sigma(0) - \delta(x)\delta(0) \rangle$$

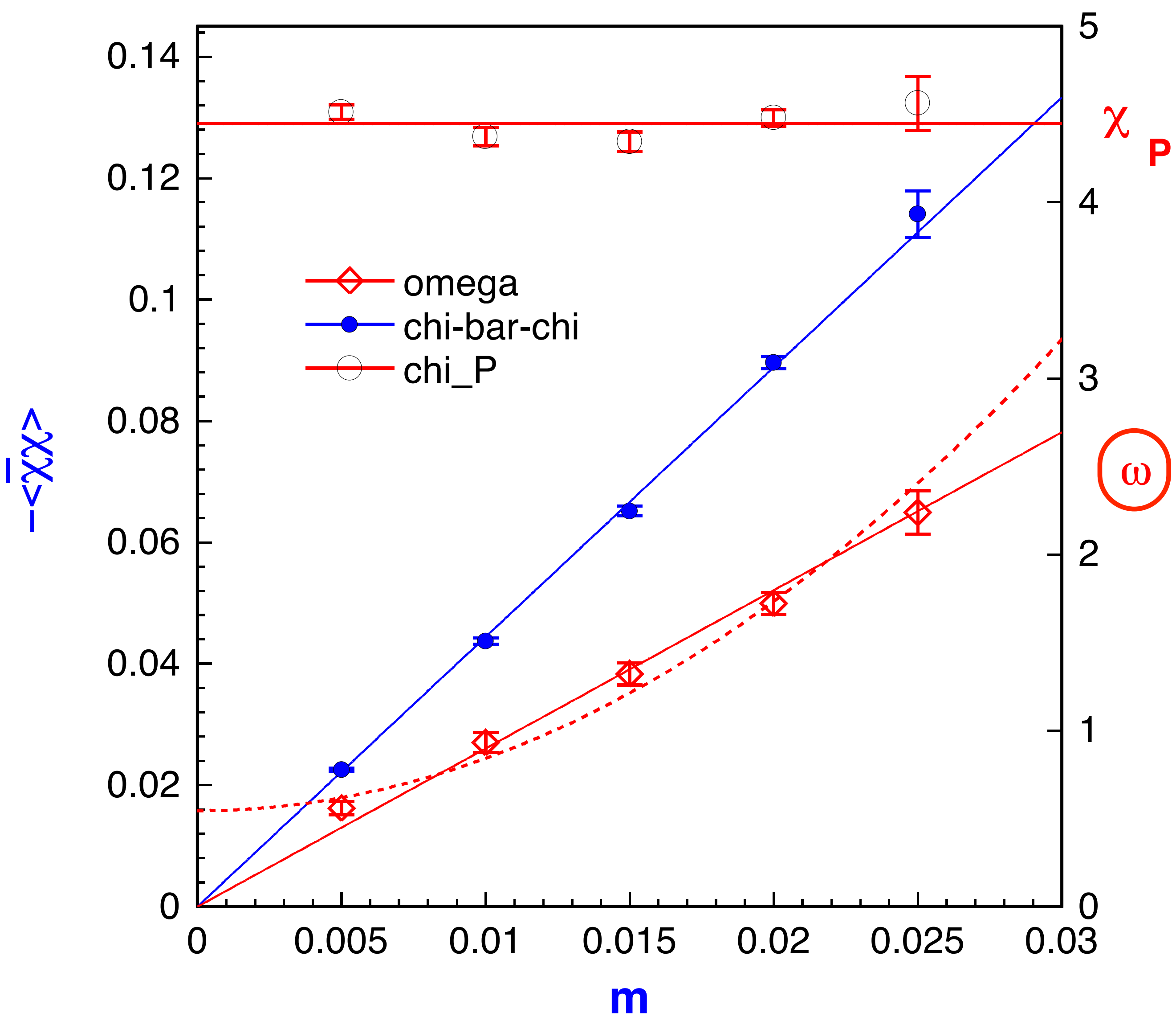
Cohen(96), Theory Yes ! $\chi_{U(1)_A} = 0, \quad (m \to 0)$

Lee-Hatsuda(96), Theory No ! zero mode contributions are important.

$\chi_{U(1)_A} = O(m^2) + \Delta$ $\Delta = O(1)$ at $N_f = 2$: contributions from $Q = \pm 1$

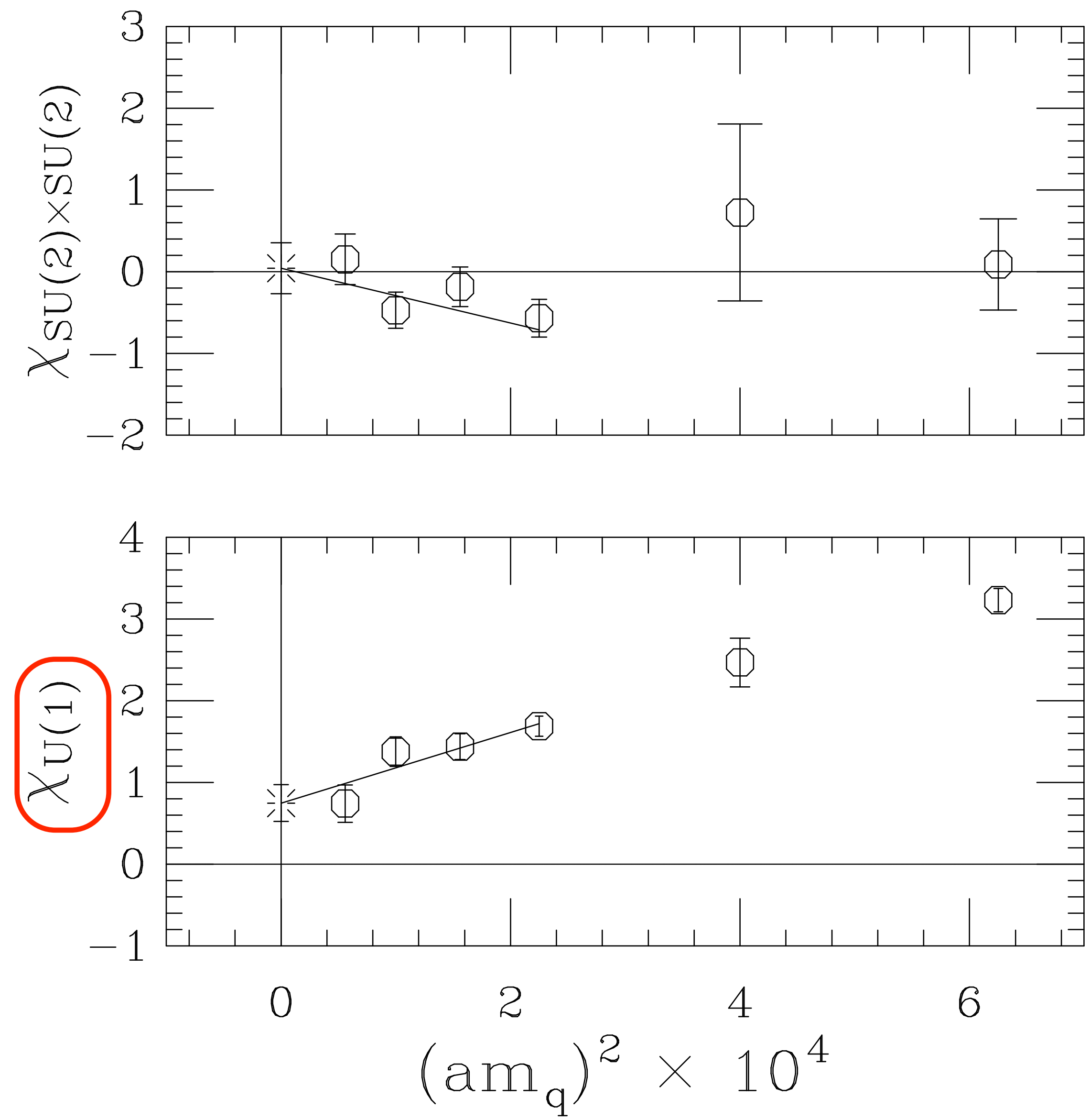
Old Lattice results

Chandrasekharan *et al.*, (98), KS No !



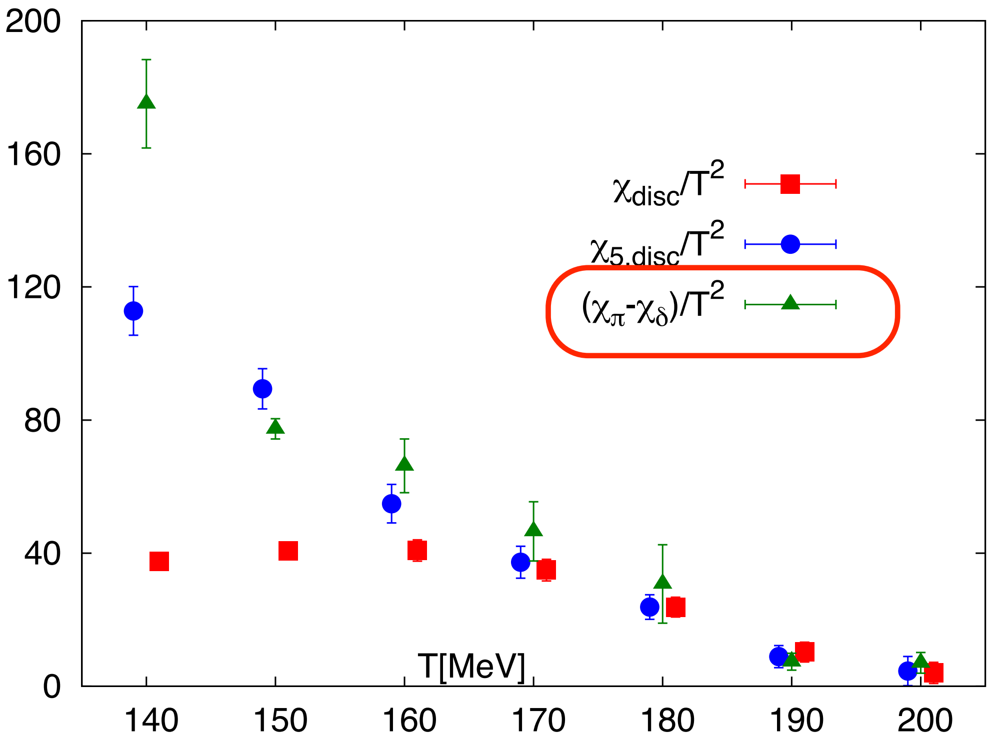
Chiral symmetry is restored.

Bernard, *et al.* (96), KS No !



U(1)$_A$ is NOT.

Recent lattice results

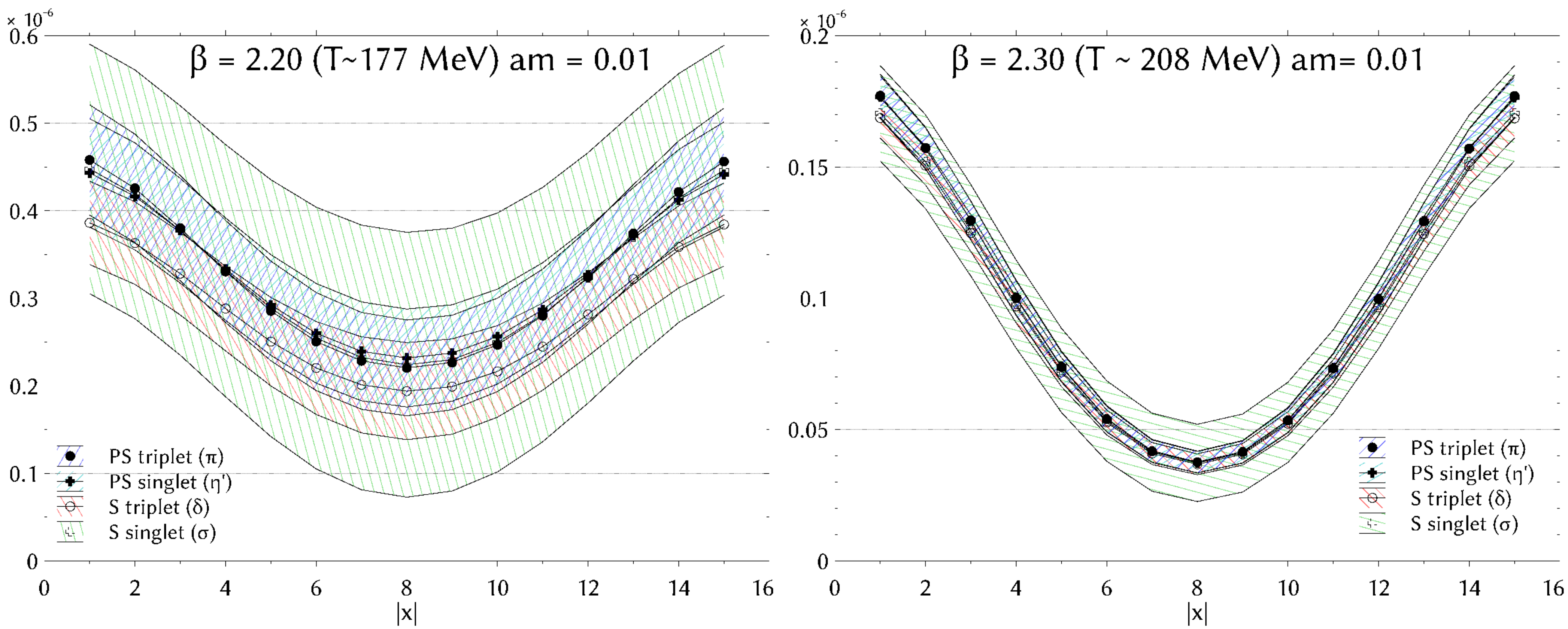


Hegde (HotQCD11), DW   No ?!

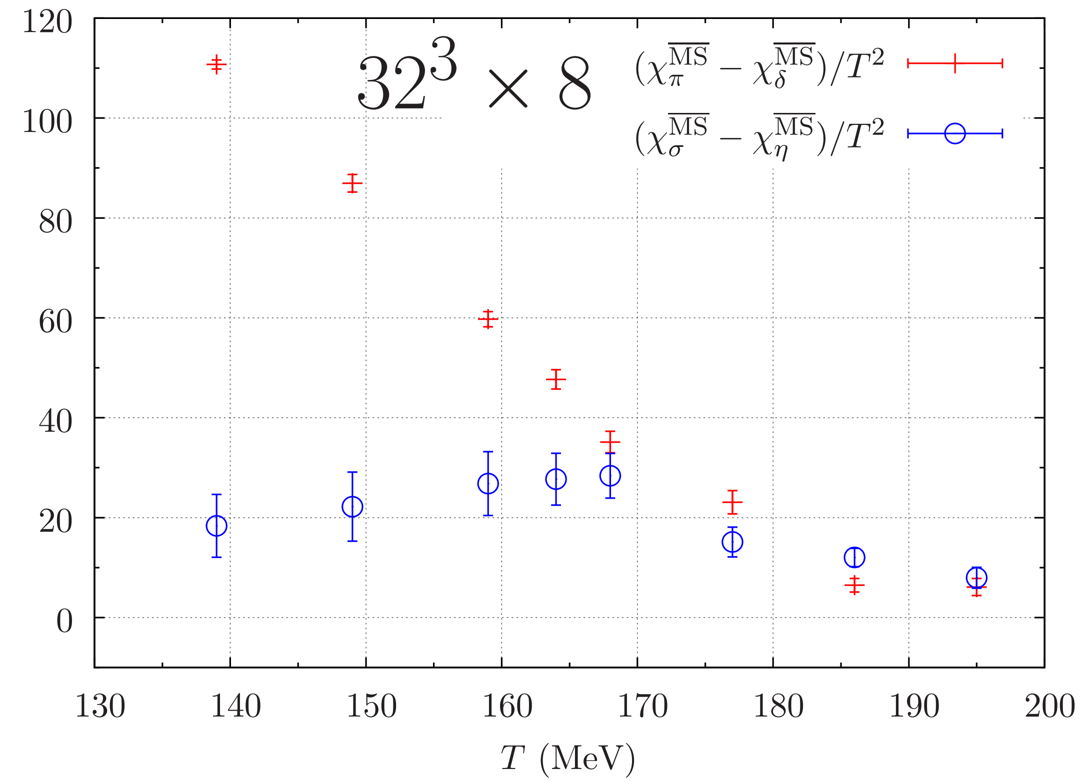
$$\chi_{U(1)_A} = 0 \text{ or not ?}$$

Cossu *et al.* (JLQCD 2013), Overlap
Phys. Rev. D87 (2013) 114514

meson correlators   Yes ?!



Buchoff *et al.* (LLNL/RBC 2013), DWF
arXiv:1309.4149[hep-lat]



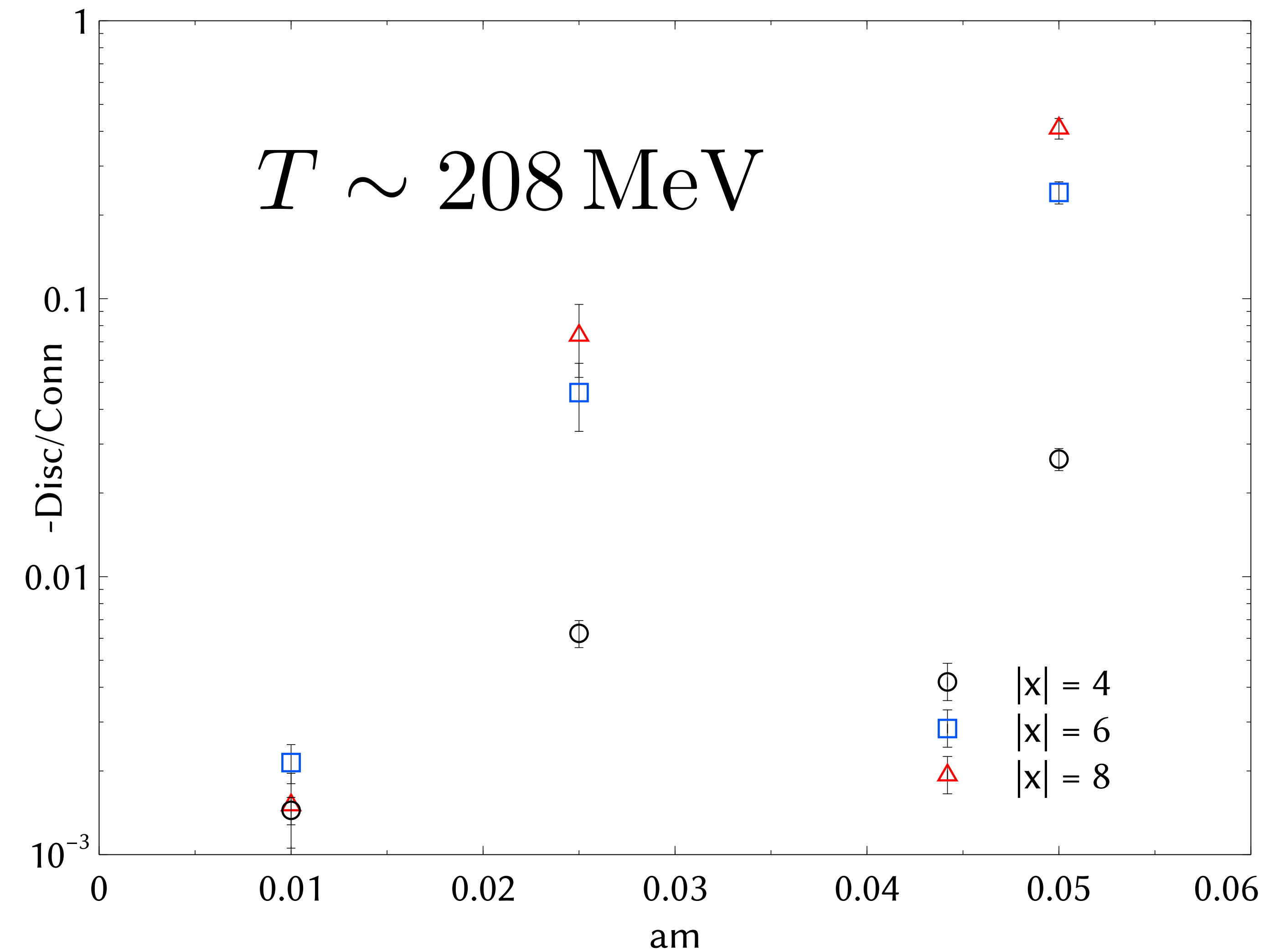
U(1) violating susceptibilities

No ?!

Cossu *et al.* (JLQCD 2013), Overlap
Phys. Rev. D87 (2013) 114514

$$\frac{\langle \eta(x)\eta(0) - \pi(x)\pi(0)\rangle}{\langle \pi(x)\pi(0)\rangle}$$

Yes ?



Our work

give constraints on eigenvalue densities of 2-flavor overlap fermions, if chiral symmetry in QCD is restored at finite temperature.
discuss a behavior of singlet susceptibility using the constraints.

Content

1. Introduction
2. Overlap fermions
3. Constraints on eigenvalue densities
4. Discussions: singlet susceptibility

2. Overlap fermions

Action

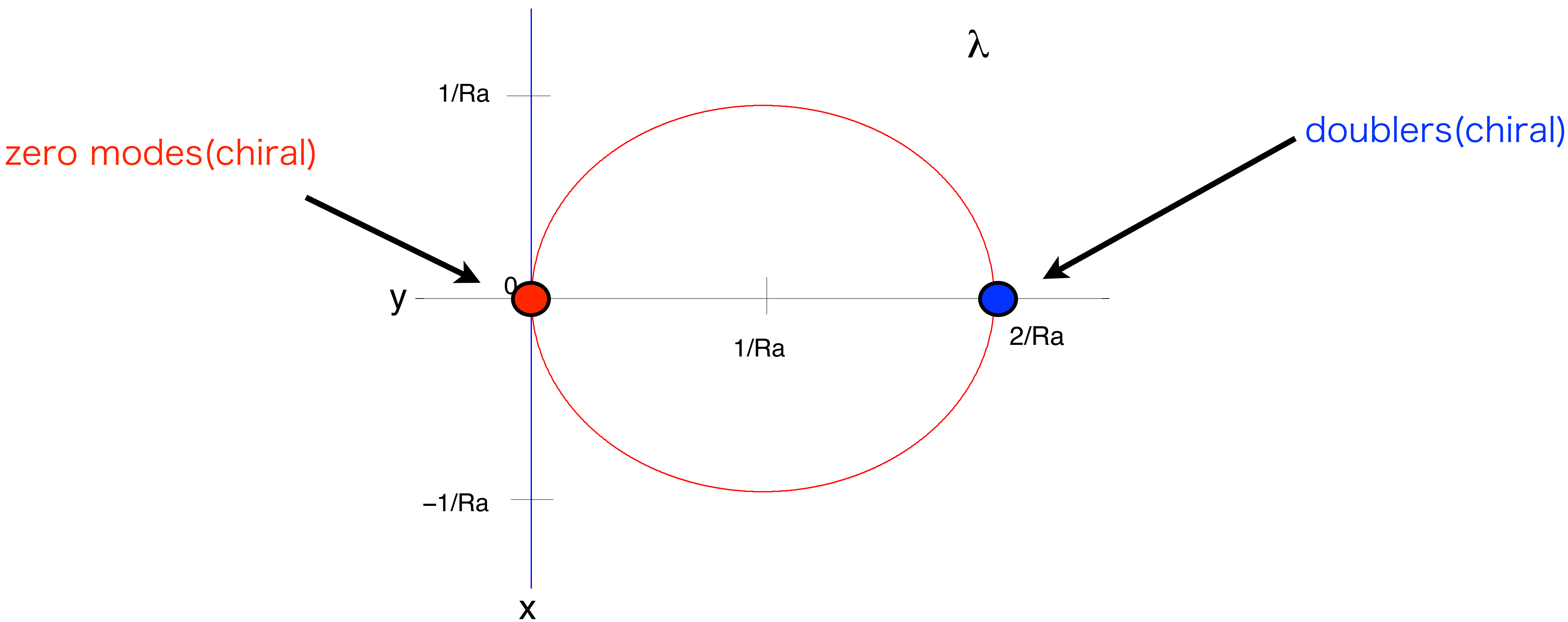
$$S = \bar{\psi}[D - mF(D)]\psi, \quad F(D) = 1 - \frac{Ra}{2}D$$

Ginsparg-Wilson relation

$$D\gamma_5 + \gamma_5 D = aDR\gamma_5 D$$

Eigenvalue spectrum

$$\lambda_n^A + \bar{\lambda}_n^A = aR\bar{\lambda}_n^A\lambda_n^A$$



$$D(A)\phi_n^A = \lambda_n^A\phi_n^A \quad \Rightarrow \quad D(A)\gamma_5\phi_n^A = \bar{\lambda}_n^A\gamma_5\phi_n^A$$

Propagator

$$S(x,y)=\sum_n\left[\frac{\phi_n(x)\phi_n^\dagger(y)}{f_m\lambda_n-m}+\frac{\gamma_5\phi_n(x)\phi_n^\dagger(y)\gamma_5}{f_m\bar{\lambda}_n-m}\right]-\sum_{k=1}^{N_{R+L}}\frac{1}{m}\phi_k(x)\phi_k^\dagger(y)+\sum_{K=1}^{N_D}\frac{Ra}{2}\phi_K(x)\phi_K^\dagger(y)$$

bulk modes(non-chiral) zero modes(chiral) doublers(chiral)

$$f_m=1+\frac{Rma}{2}$$

Measure

$$P_m(A)=e^{-S_{YM}(A)}(-m)^{\overbrace{N_fN_{R+L}^A}^{}}\left(\frac{2}{Ra}\right)^{\overbrace{N_fN_D^A}^{\text{\# of doublers}}}\prod_{\Im\lambda_n^A>0}(Z_m^2\bar{\lambda}_n^A\lambda_n^A+m^2)$$

\# of zero modes

$$Z_m^2=1-(ma)^2\frac{R^2}{4}$$

positive definite and even function of $m\neq 0$ for even N_f

N_f=2 in this talk.

Ward-Takahashi identities under "chiral" rotation

$$\begin{aligned}\theta^a(x)\delta_x^a\psi(x) &= i\theta^a(x)T^a\gamma_5(1-RaD)\psi(x),\\ \theta^a(x)\delta_x^a\bar\psi(x) &= i\bar\psi(x)\theta^a(x)T^a\gamma_5,\end{aligned}$$

Integrated operators

$$S^a=\int d^4x\, S^a(x),\quad P^a=\int d^4x\, P^a(x)$$

$$\begin{aligned}S^a(x) &= \bar\psi(x)T^aF(D)\psi(x), && \text{scalar}\\ P^a(x) &= \bar\psi(x)T^a i\gamma_5F(D)\psi(x), && \text{pseudo-scalar}\end{aligned}$$

chiral rotation at N_f=2

$$\delta^aS^b=2\delta^{ab}P^0,\,\delta^aP^b=-2\delta^{ab}S^0$$
$$\delta^aS^0=2P^a,\,\delta^aP^0=-2S^a$$

If the chiral symmetry is restored,

$$\boxed{\lim_{m\to 0}\langle\delta^a\mathcal{O}_{n_1,n_2,n_3,n_4}\rangle_m=0}$$

WT identities

$$\mathcal{O}_{n_1,n_2,n_3,n_4}=(P^a)^{n_1}(S^a)^{n_2}(P^0)^{n_3}(S^0)^{n_4}$$

$$N=\sum_i n_i,\quad n_1+n_2=\mathrm{odd},\quad n_1+n_3=\mathrm{odd}$$

flavor parity

δ^a: flavor non-singlet, parity-odd

explicit form

$$\frac{\delta^a}{2}\mathcal{O}_{n_1,n_2,n_3,n_4}=-n_1\mathcal{O}_{n_1-1,n_2,n_3,n_4+1}+n_2\mathcal{O}_{n_1,n_2-1,n_3+1,n_4}-n_3\mathcal{O}_{n_1,n_2+1,n_3-1,n_4}+n_4\mathcal{O}_{n_1+1,n_2,n_3,n_4-1}$$

3. Constraints on eigenvalue densities

Assumption 1

non-singlet chiral symmetry is restored:

$$\lim_{m\to 0}\lim_{V\to\infty}\langle\delta_a\mathcal{O}\rangle_m = 0 \quad (\text{for } a\neq 0),$$

$$\langle\mathcal{O}(A)\rangle_m = \frac{1}{Z}\int\mathcal{D}A\,P_m(A)\,\mathcal{O}(A), \quad Z=\int\mathcal{D}A\,P_m(A).$$

$P_m(A)$: even in m

Assumption 2 if $\mathcal{O}(A)$ is m-independent

$$\langle\mathcal{O}(A)\rangle_m = f(m^2)$$

$f(x)$ is analytic at $x=0$

Note that this does not hold if the chiral symmetry is spontaneously broken.

Ex.

$$\lim_{V\to\infty}\frac{1}{V}\langle Q(A)^2\rangle_m = m\frac{\Sigma}{N_f}+O(m^2)$$

Assumption 3 if $\mathcal{O}(A)$ is m-independent and positive, and satisfies

$$\lim_{m\to 0}\frac{1}{m^{2k}}\langle\mathcal{O}(A)\rangle_m = 0$$

$$\Rightarrow\quad \langle\mathcal{O}(A)\rangle_m = m^{2(k+1)}\underbrace{\int\mathcal{D}A\,\hat{P}(m^2,A)\mathcal{O}(A)}_{\text{finite}}$$

$\hat{P}(0,A)\neq 0$ for ${}^{\exists}A$

consequence for ${}^{\forall}l$ integer

$$\langle\mathcal{O}(A)^l\rangle_m = m^{2(k+1)}\int\mathcal{D}A\,\hat{P}(m^2,A)\mathcal{O}(A)^l = O(m^{2(k+1)})$$

since $\mathcal{O}(A)$ and $\mathcal{O}(A)^l$ are both positive and share the same support.

Assumption 4 eigenvalues density can be expanded as

$$\rho^A(\lambda) \equiv \lim_{V\to\infty} \frac{1}{V} \sum_n \delta\left(\lambda - \sqrt{\bar{\lambda}_n^A \lambda_n^A}\right) = \sum_{n=0}^{\infty} \rho_n^A \frac{\lambda^n}{n!} \qquad \text{at } \lambda = 0 \ (\lambda < \epsilon\ )$$

More precisely, configurations which can not be expanded at the origin are "measure zero" in the configuration space.

4. Constraints on eigenvalue densities

$\boxed{\text{general N(odd)}}$

$\mathcal{O}_{1,0,0,N-1}$

$$\lim_{m\to 0}\lim_{V\to\infty}\left(-\langle\mathcal{O}_{0,0,0,N}\rangle_m + (N-1)\langle\mathcal{O}_{2,0,0,N-2}\rangle_m\right) = 0.$$

large volume

$m \to 0$

$$\Rightarrow \quad \frac{1}{V^N}\langle (S_0)^N\rangle_m = N_f^N \left\langle \left\{ \frac{N_{R+L}^A}{mV} + I_1 \right\}^N \right\rangle_m + O(V^{-1}) \to 0$$

$$I_1 = \frac{1}{Z_m}\int_0^{\Lambda_R} d\lambda\, \rho^A(\lambda) g_0(\lambda^2)\frac{2m_R}{\lambda^2+m_R^2} = \pi\rho_0^A + O(m)$$

$\Lambda_R = \frac{2}{Ra}$: cut-off

$$g_0(\lambda^2) = 1 - \frac{\lambda^2}{\Lambda_R^2},\quad m_R = m/Z_m$$

Both ρ_0^A and N_{R+L}^A are positive.

$$\Rightarrow \quad \langle \rho_0^A\rangle_m = O(m^2)$$

$\boxed{\text{1st constraint}}$

$$\lim_{V\to\infty}\left\langle \frac{(N_{R+L}^A)^N}{V^N}\right\rangle = O\left(m^{N+1}\right) \quad \underset{\forall N}{\Rightarrow} \quad \lim_{V\to\infty}\left\langle \frac{N_{R+L}}{V}\right\rangle_m = 0$$

for small but non-zero m

Example of calculations

$$S(x,y) = \sum_n \left[\frac{\phi_n(x)\phi_n^\dagger(y)}{f_m\lambda_n - m} + \frac{\gamma_5\phi_n(x)\phi_n^\dagger(y)\gamma_5}{f_m\bar{\lambda}_n - m} \right] - \sum_{k=1}^{N_{R+L}} \frac{1}{m}\phi_k(x)\phi_k^\dagger(y) + \sum_{K=1}^{N_D} \frac{Ra}{2}\phi_K(x)\phi_K^\dagger(y)$$

$$S_0 = -\int d^4x \,\mathrm{tr}\, F(D) S(x,x) = -\sum_n \left[\frac{F(\lambda_n)}{f_m\lambda_n - m} + \frac{F(\bar{\lambda}_n)}{f_m\bar{\lambda}_n - m} \right] + \frac{N^A_{R+L}}{m}$$

$$\frac{S_0}{V} = \underline{-\frac{1}{V}\sum_n \left[\frac{F(\lambda_n)}{f_m\lambda_n - m} + \frac{F(\bar{\lambda}_n)}{f_m\bar{\lambda}_n - m} \right]} + \frac{N^A_{R+L}}{Vm}$$

$$I_1 = \frac{1}{Z_m}\int_0^{\Lambda_R} d\lambda\, \rho^A(\lambda) g_0(\lambda^2) \frac{2m_R}{\lambda^2 + m_R^2} = \pi\rho_0^A + O(m)$$

$\boxed{\text{N=2}}$

$$\chi^{\sigma-\pi} = \frac{1}{V^2}\langle S_0^2 - P_a^2 \rangle_m, \qquad \chi^{\eta-\delta} = \frac{1}{V}\langle P_0^2 - S_a^2 \rangle_m$$

=0

$$\chi^{\eta-\delta} = N_f \left\langle \frac{1}{m^2 V}\{\underbrace{2N_{R+L}}_{=0} - N_f Q(A)^2\} + \underline{\frac{1}{Z_m}\left(\frac{I_1}{m_R} + I_2\right)} \right\rangle_m$$

topological charge

$$Q(A) = N_R^A - N_L^A$$

$$I_2 = \frac{2}{Z_m}\int_0^{\Lambda_R} d\lambda\, \rho^A(\lambda)\, \frac{m_R^2 - \lambda^2 g_0(\lambda^2) g_m}{(\lambda^2 + m_R^2)^2}, \quad g_m = \frac{1}{Z_m^2}\left(1 + \frac{m^2}{2\Lambda_R^2}\right)$$

$$\frac{I_1}{m_R} + I_2 = \rho_0^A \left(\frac{\pi_m}{m} + \frac{2}{\Lambda_R}\right) + 2\rho_1^A + O(m),$$

$$\uparrow$$

$$\langle \rho_0^A \rangle_m = O(m^2)$$

$$\lim_{m\to 0} \chi^{\eta-\delta} = 0 \quad \Longrightarrow \quad \lim_{m\to 0} \frac{N_f^2 \langle Q(A)^2 \rangle_m}{m^2 V} = 2 \lim_{m\to 0} \langle \rho_1^A \rangle_m$$

N=3

WT identities

$$\langle \mathcal{O}_{2001} \rangle_m \to 0, \quad \langle -\mathcal{O}_{0201} + 2\mathcal{O}_{1110} \rangle_m \to 0, \quad \langle \mathcal{O}_{0021} + 2\mathcal{O}_{1110} \rangle_m = 0$$

$$\langle -\mathcal{O}_{0003} + 2\mathcal{O}_{2001} \rangle_m \to 0, \quad \langle \mathcal{O}_{0021} - \mathcal{O}_{0201} + \mathcal{O}_{1110} \rangle_m \to 0,$$

$$\langle \rho_0^A \rangle_m = -\frac{m^2}{2} \langle \rho_2^A \rangle_m + O(m^4)$$

$$\Rightarrow \quad \lim_{V\to\infty} \frac{\langle Q(A)^2 \rho_0^A \rangle_m}{V} = O(m^4)$$

N=4

$$\langle \mathcal{O}_{4000} - \mathcal{O}_{0004} \rangle_m \to 0, \quad \langle \mathcal{O}_{4000} - 3\mathcal{O}_{2002} \rangle_m \to 0,$$
$$\langle \mathcal{O}_{0400} - \mathcal{O}_{0040} \rangle_m \to 0, \quad \langle \mathcal{O}_{0400} - 3\mathcal{O}_{0220} \rangle_m \to 0,$$
$$\langle \mathcal{O}_{2020} - \mathcal{O}_{0202} \rangle_m \to 0, \quad \langle \mathcal{O}_{2200} - \mathcal{O}_{0022} \rangle_m \to 0,$$
$$\langle 2\mathcal{O}_{1111} - \mathcal{O}_{0202} + \mathcal{O}_{0022} \rangle_m \to 0.$$

$$3N_f^2 \langle (I_2 + I_1/m)(I_1 - I_2/m) \rangle_m + \frac{6N_f^3}{m^3 V} \langle Q(A)^2 I_1 \rangle_m - \frac{N_f^4}{m^4 V^2} \langle Q(A)^4 \rangle_m \to 0.$$

$\sim \log m$ $\quad$ $\sim \log m$ $\quad$ $\sim \dfrac{1}{m^2}$

$$\lim_{V\to\infty} \frac{\langle Q(A)^2 \rangle_m}{V} = O(m^4)$$

$$\langle \rho_1^A \rangle_m = O(m^2)$$

$$\lim_{m\to 0} \frac{N_f^2 \langle Q(A)^2 \rangle_m}{m^2 V} = 2 \lim_{m\to 0} \langle \rho_1^A \rangle_m$$

2nd constraint

$$-3N_f^2 \frac{\pi^2}{m^2} \langle (\rho_0^A)^2 \rangle_m - \frac{N_f^4}{m^4 V^2} \langle Q(A)^4 \rangle_m \to 0.$$

negative semi-definite

$$\lim_{V\to\infty} \frac{\langle Q(A)^2 \rangle_m}{V} = O(m^6)$$

$$\langle \rho_0^A \rangle_m = O(m^4) \qquad \langle \rho_2^A \rangle_m = O(m^2)$$

$$\langle \rho_0^A \rangle_m = -\frac{m^2}{2} \langle \rho_2^A \rangle_m$$

3rd constraint

+ result from N=4k (general)

Final results

$$\lim_{m\to 0}\langle\rho^A(\lambda)\rangle_m = \lim_{m\to 0}\langle\rho_3^A\rangle_m \frac{|\lambda|^3}{3!} + O(\lambda^4)$$

No constraints to higher $\langle\rho_n^A\rangle_m$

$\langle\rho_3^A\rangle_m \neq 0$ even for "free" theory.

$$\langle\rho_0^A\rangle_m = 0$$

$$\lim_{V\to\infty}\frac{1}{V^k}\langle(N_{R+L}^A)^k\rangle_m = 0, \quad \lim_{V\to\infty}\frac{1}{V^k}\langle Q(A)^{2k}\rangle_m = 0$$

5. Discussion: Singlet susceptibility

Singlet susceptibility at high T

$$\lim_{m\to 0} \chi^{\pi-\eta} = \lim_{m\to 0} \lim_{V\to\infty} \frac{N_f^2}{m^2 V} \langle Q(A)^2 \rangle_m = 0$$

Both Cohen and Lee-Hatsuda are inaccurate.

This, however, does not mean U(1)_A symmetry is recovered at high T.

$$\lim_{m\to 0} \chi^{\pi-\eta} = 0$$

is necessary but NOT "sufficient" for the recovery of U(1)_A .

More general Singlet WT identities

$$\langle \underline{J^0}\mathcal{O} + \underline{\delta^0}\mathcal{O}\rangle_m = O(m)$$

anomaly(measure) singlet rotation

We can show for $\mathcal{O} = \mathcal{O}_{n_1,n_2,n_3,n_4} = (P^a)^{n_1}(S^a)^{n_2}(P^0)^{n_3}(S^0)^{n_4}$

$$\lim_{V\to\infty}\frac{1}{V^k}\langle J^0\mathcal{O}\rangle_m = \lim_{V\to\infty}\left\langle \frac{Q(A)^2}{mV}\times O(V^0)\right\rangle_m = 0$$

where k is the smallest integer which makes the $V\to\infty$ limit finite.

$S^0 \sim O(V),\ P^a, S^a, P^0 \sim O(V^{1/2})$

$$\Rightarrow \quad \lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V^k}\langle\delta^0\mathcal{O}\rangle_m = 0$$

Breaking of U(1)_A symmetry is absent for these “bulk quantities”.

Important consequence

Effect of U(1)_A anomaly is invisible in scalar and pseudo-scalar sector.

Pisarski-Wilczek argument

Chiral phase transition in 2-flavor QCD is likely to be of first order !?
(See Taniguchi-san's talk in detail.)

Final Comments

1. Large volume limit is required for the correct result.

2. If the action breaks the chiral symmetry, the continuum limit is also required.

3. We only use a part of WT identities. Therefore, our constraints are necessary condition.

4. We can extend our analysis to the eigenvalue density with fractional power. The conclusion remains the same. (See the next page.)

Fractional power for the eigenvalue density

$$\rho^A(\lambda) \simeq c_A \lambda^\gamma, \ \gamma > 0$$

If non-singlet chiral symmetry is recovered at high T

$\Rightarrow$ $\gamma \le 2$ is excluded. $\Rightarrow$ $\gamma > 2$

consistent with the integer case (n > 2)

Part II

Massless up quark and Dashen phase in Chiral Perturbation Theory

with Mike Creutz @ BNL

1. Introduction

θ term in QCD $\quad i\theta\frac{1}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x) \equiv i\theta q(x)$ $\quad$ CP odd

$\Downarrow$

Neutron Electric Dipole Moment(NEDM)

Experimental bound

$$|\vec{d_n}| \le 6.3\times 10^{-26} e\cdot cm$$

Model estimate

$$|\vec{d_n}|/\theta \simeq 10^{-15} \sim 10^{-17} e\cdot cm$$

$\Rightarrow$ $\theta = \theta_{\rm QCD} + \theta_{\rm EW} \le O(10^{-8})$

Strong CP problem !

One possible "solution" $\quad m_u = 0 \quad$ massless up quark

(Lattice QCD already ruled out this ?)

chiral rotation

$$u \longrightarrow e^{i\alpha\gamma_5}u, \quad \bar u \longrightarrow \bar u e^{i\alpha\gamma_5},$$

$$m_u\,\bar u u \longrightarrow m_u\,\bar u e^{i2\alpha\gamma_5}u$$

$\theta \longrightarrow \theta' = \theta + 2\alpha N_f \quad$ chiral anomaly

if $m_u = 0$, we can make

$$\theta' = 0$$

by $\alpha = -\frac{\theta}{2N_f}$

Mike Creutz, "Quark masses, the Dashen phase, and gauge field topology" arXiv:1306.1245[hep-lat]

Mike's Oracles

$m_d > 0$ fixed, then

1. Nothing special happens at $m_u = 0$.

2. Massless neutral pion: $m_{\pi^0} = 0$ at $m_u =^{\exists} m_c < 0$.

3. Pion condensation (Dashen phase): $\langle \pi^0 \rangle \neq 0$ at $m_u < m_c < 0$.

4. $\chi = \infty$ at $m_u = m_c$.

$$\chi = \frac{1}{V}\langle Q^2 \rangle$$ topological susceptibility

5. $\chi = 0$ at $m_u = 0$.

In the part II, I show the above properties by ChPT including the anomaly effect.
In addition, we discuss an interesting prediction related to these in 2-flavor QCD.

ChPT with "anomaly"

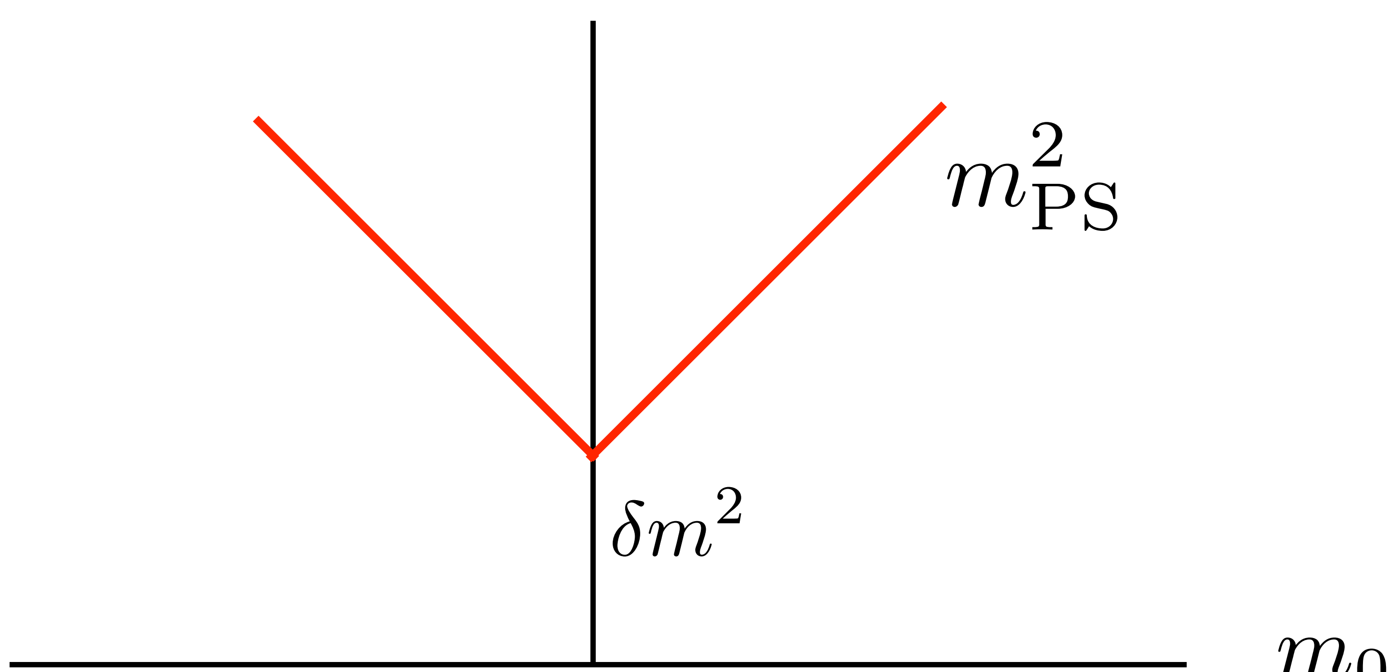
$$\mathcal{L} = \frac{f^2}{2}\mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) - \frac{1}{2}\mathrm{tr}\left(M^\dagger U + U^\dagger M\right) - \underbrace{\frac{\Delta}{2}\left(\det U + \det U^\dagger\right)}_{\text{effect of anomaly}}$$

Warm-up: $N_f = 1$ case

naive guess

$$m_{\rm PS}^2 = \frac{2B}{f^2}|m_0| + \delta m^2$$

No massless "pion"(eta)



reality

$$U = U_0 = e^{i\varphi_0}$$ vacuum ansatz $m = 2Bm_0$

$$V(\varphi_0) = -(m+\Delta)\cos\varphi_0$$

potential

$$\varphi_0 = \begin{cases} 0 & m+\Delta > 0 \\ \pi & m+\Delta < 0 \end{cases}$$

minimum

$$U(x) = U_0 e^{i\pi(x)/f}$$

PS meson field

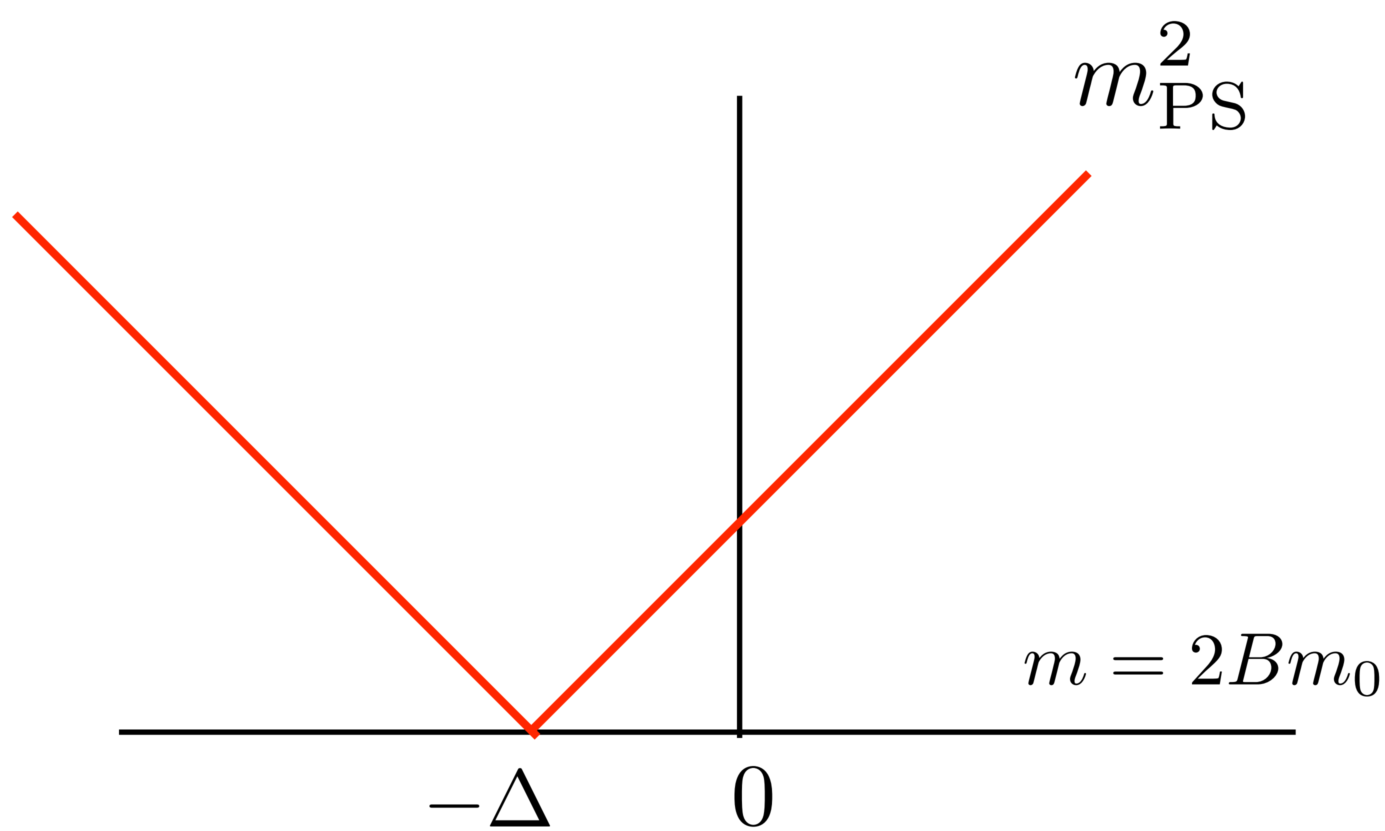
$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\pi(x)\partial^\mu\pi(x) - (m+\Delta)U_0\cos(\pi(x)/f) \\ &= \frac{1}{2}\left[(\partial_\mu\pi(x))^2 + \frac{|m+\Delta|}{f^2}\pi(x)^2\right] + O(\pi^4) \end{aligned}$$

$$m_{\rm PS}^2 = \frac{|m+\Delta|}{f^2}$$

$m=0$ is note special

non-symmetric under $m \to -m$

massless PS meson at $m = -\Delta$



2. Phase structure and pion masses at N_f=2

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \equiv 2B \begin{pmatrix} m_{0u} & 0 \\ 0 & m_{0d} \end{pmatrix}$$

mass term

$$U = U_0 = e^{i\varphi_0} \begin{pmatrix} e^{i\varphi_3} & 0 \\ 0 & e^{-i\varphi_3} \end{pmatrix}$$

vacuum

VEV

$$\langle\bar{\psi}\psi\rangle \equiv \frac{1}{2}\mathrm{tr}\,(U_0 + U_0^\dagger) = 2\cos(\varphi_0)\cos(\varphi_3),$$

$$\langle\bar{\psi}\tau^3\psi\rangle \equiv \frac{1}{2}\mathrm{tr}\,\tau^3(U_0 + U_0^\dagger) = -2\sin(\varphi_0)\sin(\varphi_3),$$

$$\langle\bar{\psi}i\gamma_5\psi\rangle \equiv \frac{1}{2i}\mathrm{tr}\,(U_0 - U_0^\dagger) = 2\sin(\varphi_0)\cos(\varphi_3),$$

$$\langle\bar{\psi}i\gamma_5\tau^3\psi\rangle \equiv \frac{1}{2i}\mathrm{tr}\,\tau^3(U_0 - U_0^\dagger) = 2\cos(\varphi_0)\sin(\varphi_3).$$

potential

$$V(\varphi_0,\varphi_3) = -m_u\cos(\varphi_0+\varphi_3) - m_d\cos(\varphi_0-\varphi_3) - \Delta\cos(2\varphi_0),$$

$$\Rightarrow \quad \frac{\partial V}{\partial\varphi_0} = m_u\sin(\varphi_0+\varphi_3) + m_d\sin(\varphi_0-\varphi_3) + 2\Delta\sin(2\varphi_0) = 0$$

$$\frac{\partial V}{\partial\varphi_3} = m_u\sin(\varphi_0+\varphi_3) - m_d\sin(\varphi_0-\varphi_3) = 0.$$

gap equations

Solutions

$0 < m_d < \Delta$

$$\sin^2(\varphi_3) = \frac{(m_d - m_u)^2\{(m_u + m_d)^2\Delta^2 - m_u^2 m_d^2\}}{4m_u^3 m_d^3}$$
$$\sin^2(\varphi_0) = \frac{(m_u + m_d)^2\Delta^2 - m_u^2 m_d^2}{4m_u m_d \Delta^2},$$

$$m_c^- < m_u < m_c^+$$

Dashen phase

$\Delta < m_d$

$$\sin^2(\varphi_3) = \frac{(m_d - m_u)^2\{(m_u + m_d)^2\Delta^2 - m_u^2 m_d^2\}}{4m_u^3 m_d^3}$$
$$\sin^2(\varphi_0) = \frac{(m_u + m_d)^2\Delta^2 - m_u^2 m_d^2}{4m_u m_d \Delta^2},$$

$$-m_c^- < m_u < m_c^+$$

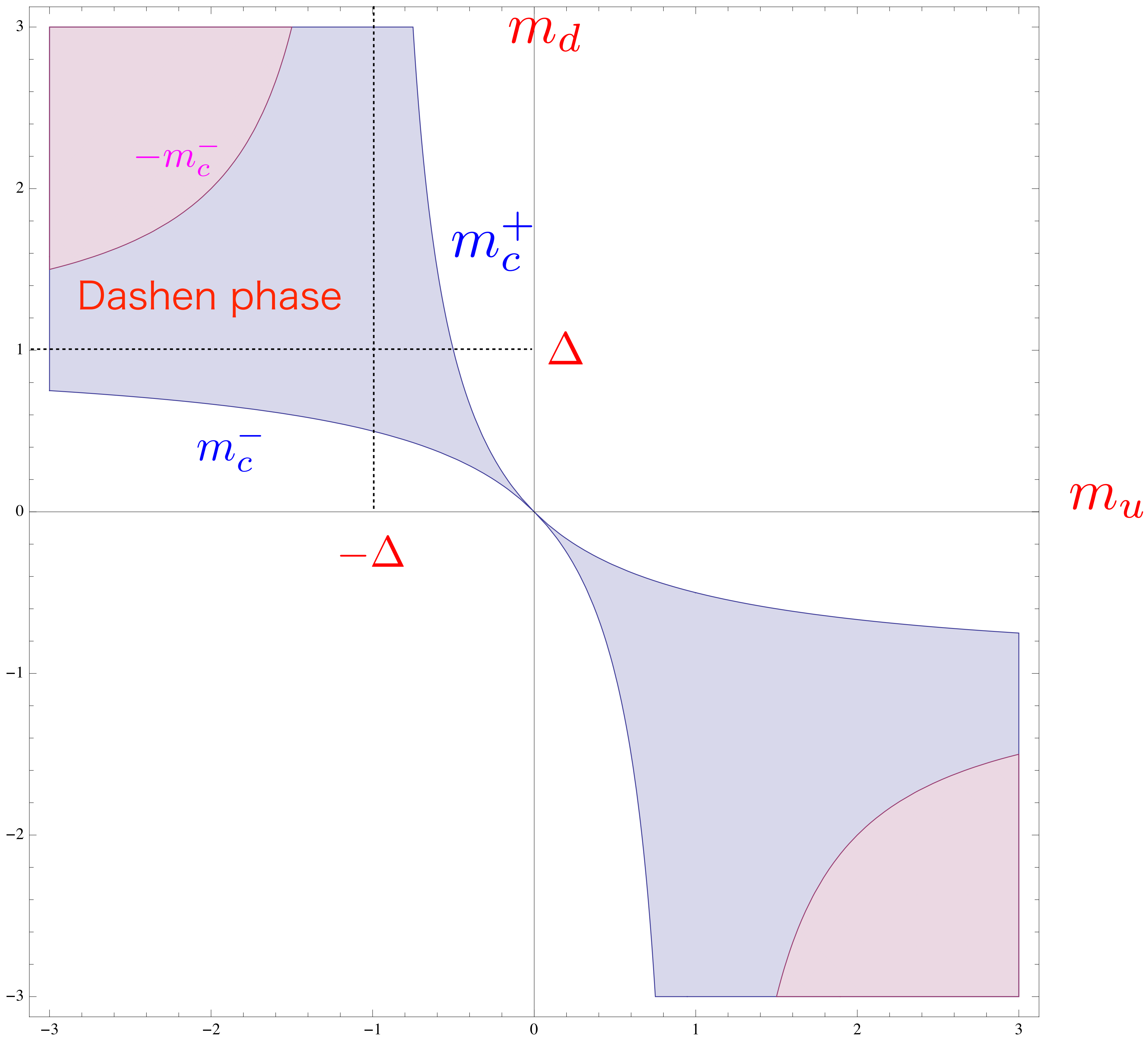
Dashen phase

$$U_0 = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\ \sin^2(\varphi_3) = \sin^2(\varphi_0) = 1 \ ).$$

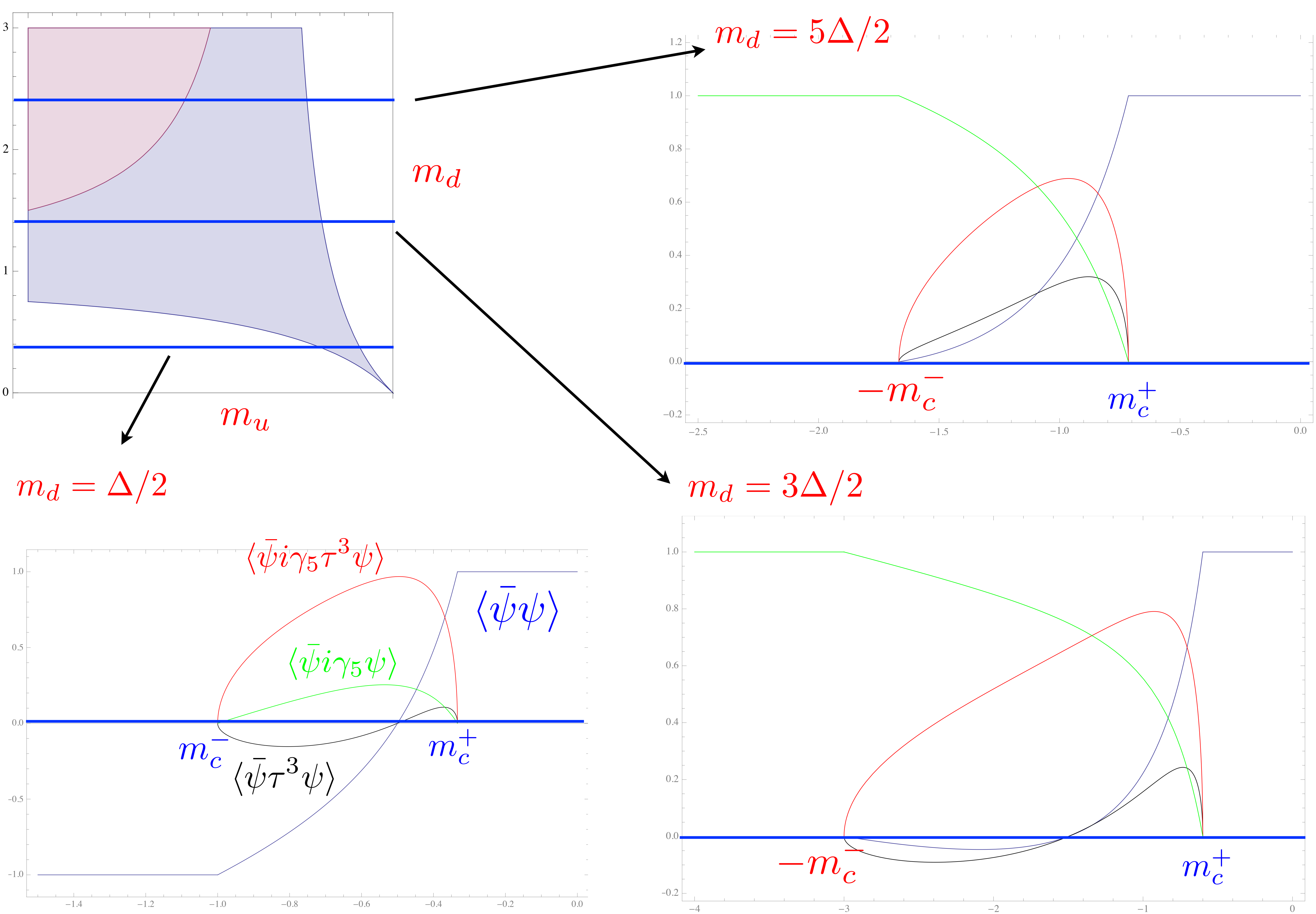
$$m_u < -m_c^-$$

$$m_c^{\pm} = -\frac{m_d\Delta}{\Delta \pm m_d} < 0,$$

Phase structure



VEV



m_d
m_u
$m_d = 5\Delta/2$
$-m_c^-$
m_c^+
$m_d = \Delta/2$
$\langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle$
$\langle \bar{\psi} \psi \rangle$
$\langle \bar{\psi} i \gamma_5 \psi \rangle$
m_c^-
m_c^+
$\langle \bar{\psi} \tau^3 \psi \rangle$
$m_d = 3\Delta/2$
$-m_c^-$
m_c^+

PS meson masses

$$U(x) = U_0 e^{i\Pi(x)/f}, \qquad \Pi(x) = \begin{pmatrix} \frac{\eta(x)+\pi_0(x)}{\sqrt{2}} & \pi_-(x) \\ \pi_+(x) & \frac{\eta(x)-\pi_0(x)}{\sqrt{2}} \end{pmatrix}$$

$$\mathcal{L}^{(2)} = \frac{1}{2}\{(\partial_\mu \pi_0(x))^2 + (\partial_\mu \eta(x))^2 + 2\partial_\mu \pi_+(x)\partial^\mu \pi_-(x)\} + \frac{\delta m}{2f^2}\eta^2(x)$$
$$+ \frac{m_+(\vec{\varphi})}{4f^2}\{\eta^2(x) + \pi_0^2(x) + 2\pi_+(x)\pi_-(x)\} - \frac{m_-(\vec{\varphi})}{2f^2}\underbrace{\eta(x)\pi_0(x)}_{\pi_0-\eta\ \text{mixing}}, \quad (26)$$

$$m_{\pi_\pm}^2 = \frac{m_+(\vec{\varphi})}{2f^2}$$ charged pion

$$m_{\tilde{\pi}_0}^2 = \frac{1}{2f^2}[m_+(\vec{\varphi}) + \delta m - X]$$ neutral pion

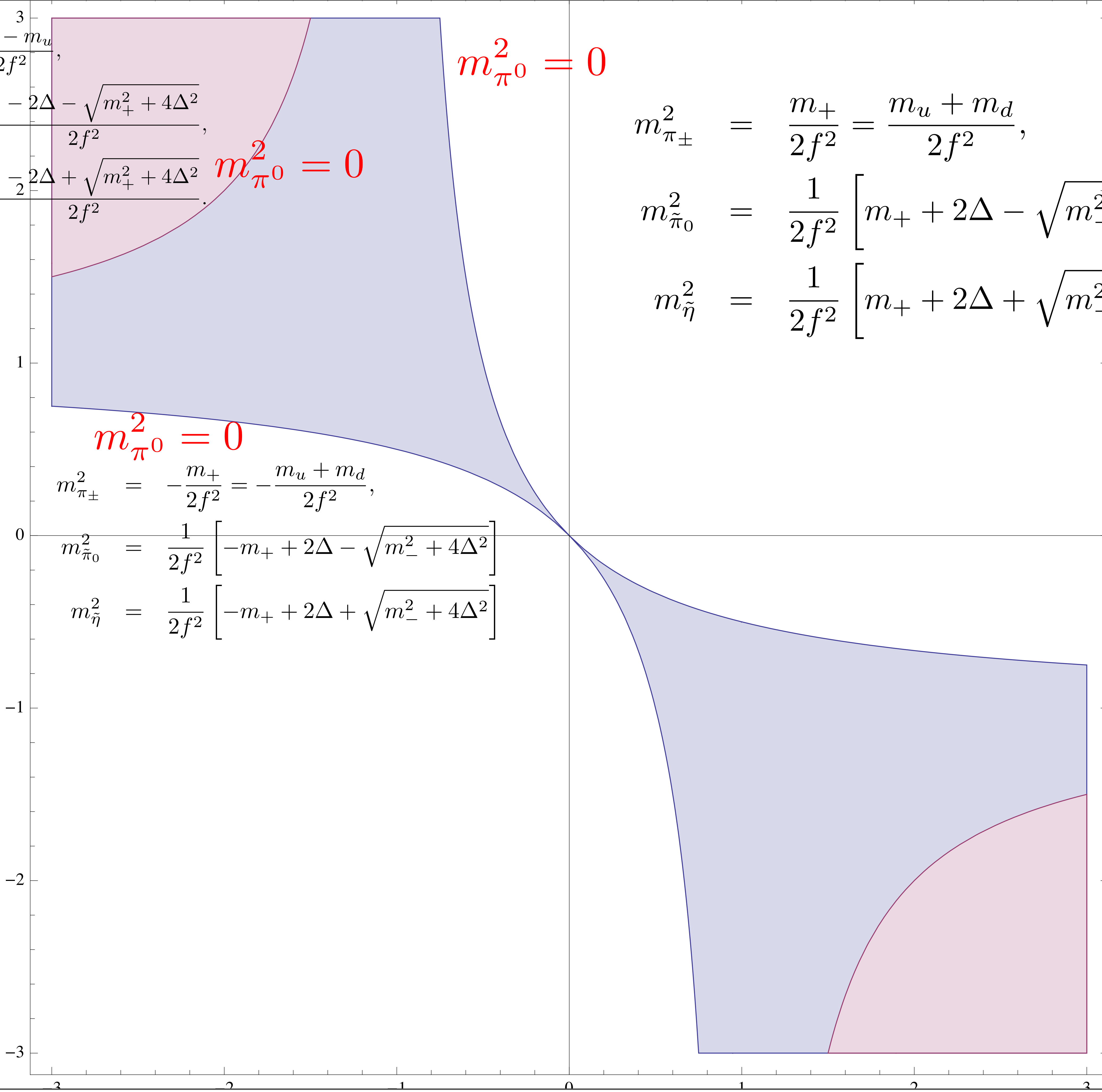
$$m_{\tilde{\eta}}^2 = \frac{1}{2f^2}[m_+(\vec{\varphi}) + \delta m + X]$$ eta meson

$$\Rightarrow \quad m_{\tilde{\pi}_0} < m_{\pi_\pm}$$

$$X = \sqrt{m_-(\vec{\varphi})^2 + \delta m^2},$$

$$m_\pm(\vec{\varphi}) = m_d \cos(\varphi_0 - \varphi_3) \pm m_u \cos(\varphi_0 + \varphi_3)$$
$$= m_\pm \cos(\varphi_0)\cos(\varphi_3) + m_\mp \sin(\varphi_0)\sin(\varphi_3),$$
$$\delta m = 2\Delta \cos(2\varphi_0).$$

$$m_\pm = m_d \pm m_u.$$



$$
\begin{aligned}
m_{\pi^\pm}^2 &= \frac{m_d - m_u}{2f^2}, \\
m_{\tilde{\pi}_0}^2 &= \frac{m_- - 2\Delta - \sqrt{m_+^2 + 4\Delta^2}}{2f^2}, \\
m_{\tilde{\eta}}^2 &= \frac{m_- - 2\Delta + \sqrt{m_+^2 + 4\Delta^2}}{2f^2}.
\end{aligned}
$$

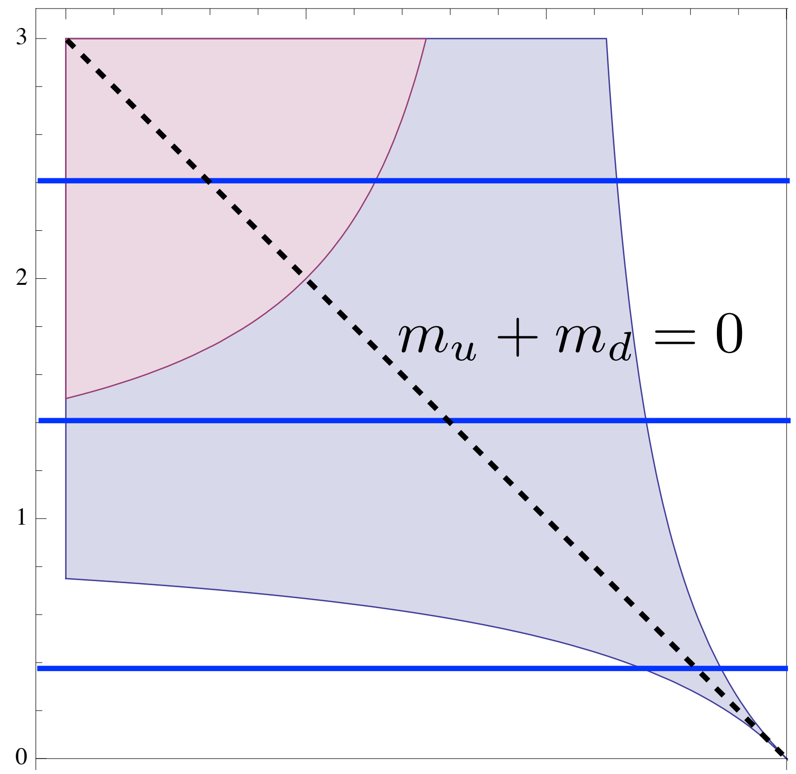
$m_{\pi^0}^2 = 0$

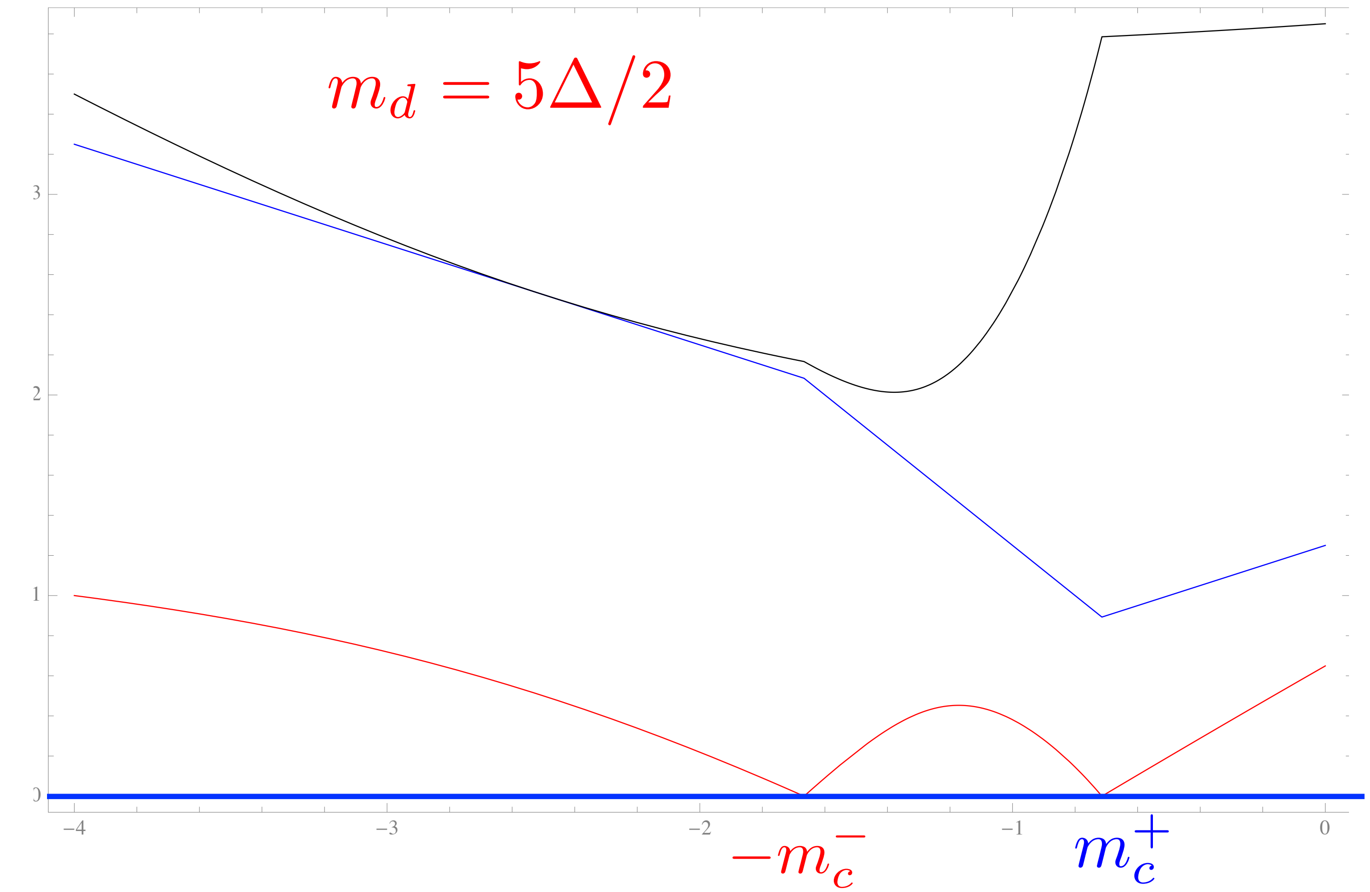
$$
\begin{aligned}
m_{\pi^\pm}^2 &= \frac{m_+}{2f^2} = \frac{m_u + m_d}{2f^2}, \\
m_{\tilde{\pi}_0}^2 &= \frac{1}{2f^2}\left[m_+ + 2\Delta - \sqrt{m_-^2 + 4\Delta^2}\right], \\
m_{\tilde{\eta}}^2 &= \frac{1}{2f^2}\left[m_+ + 2\Delta + \sqrt{m_-^2 + 4\Delta^2}\right],
\end{aligned}
$$

$m_{\pi^0}^2 = 0$

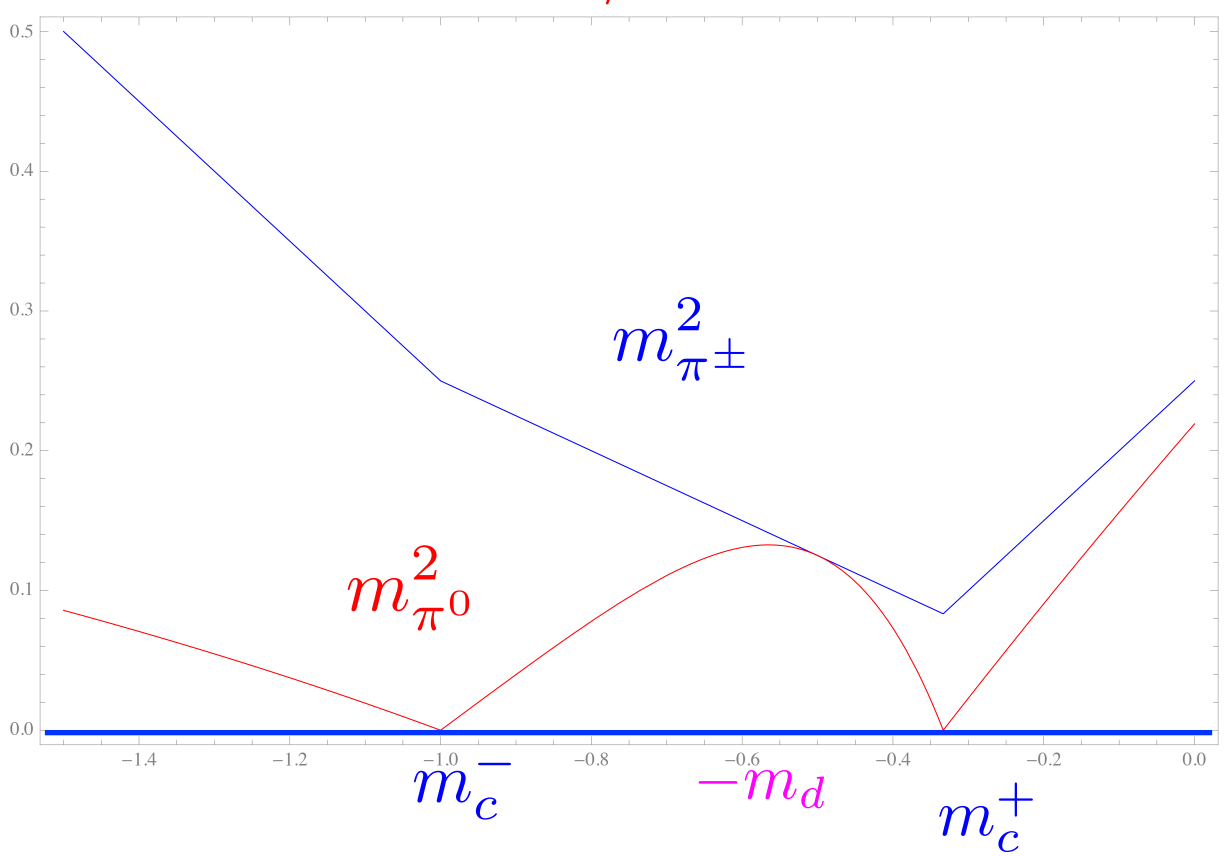
$m_{\pi^0}^2 = 0$

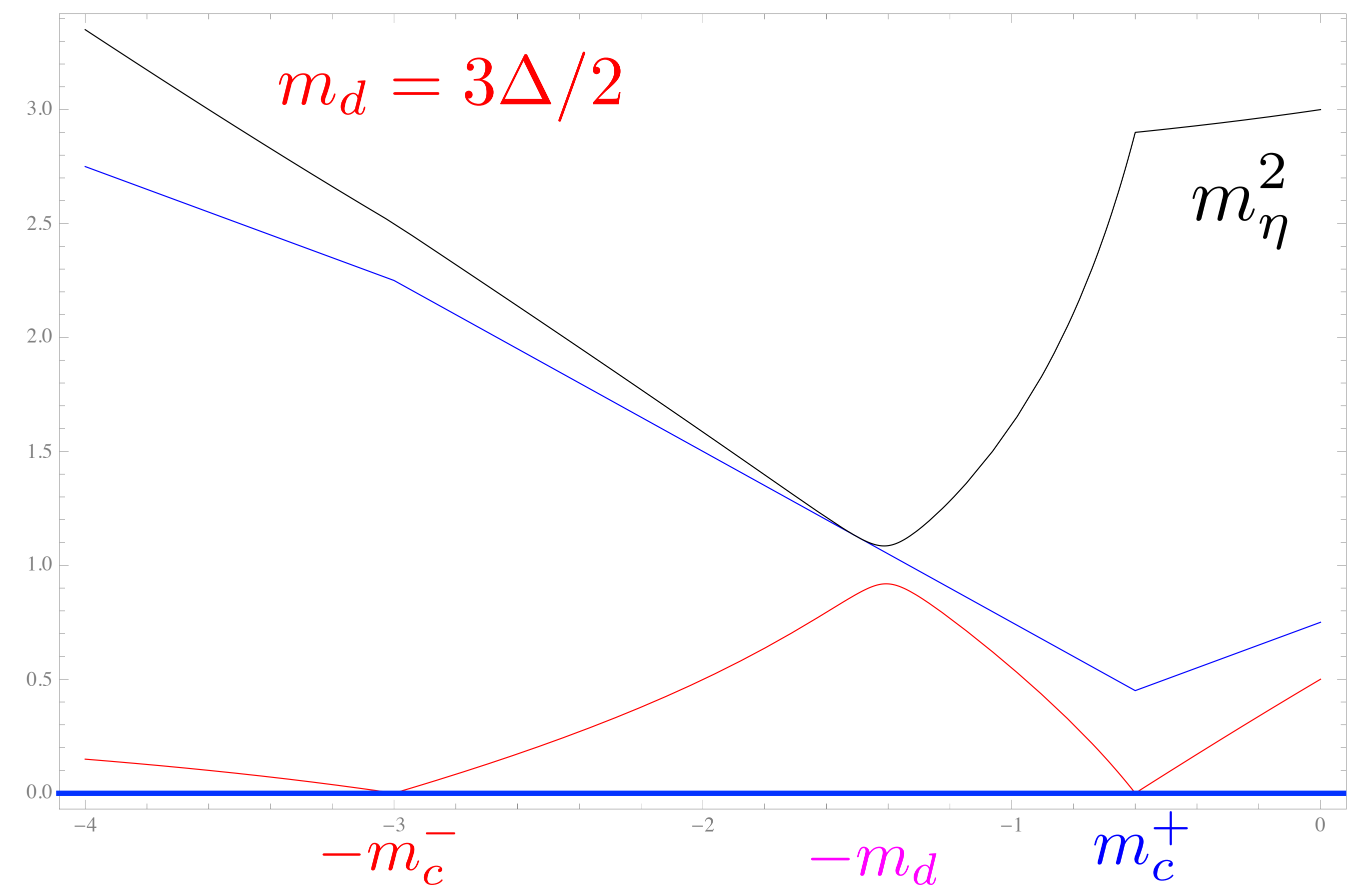
$$
\begin{aligned}
m_{\pi^\pm}^2 &= -\frac{m_+}{2f^2} = -\frac{m_u + m_d}{2f^2}, \\
m_{\tilde{\pi}_0}^2 &= \frac{1}{2f^2}\left[-m_+ + 2\Delta - \sqrt{m_-^2 + 4\Delta^2}\right] \\
m_{\tilde{\eta}}^2 &= \frac{1}{2f^2}\left[-m_+ + 2\Delta + \sqrt{m_-^2 + 4\Delta^2}\right]
\end{aligned}
$$


$m_u + m_d = 0$
3
2
1
0


$m_d = 5\Delta/2$
$-m_c^-$
m_c^+
−4
−3
−2
−1
0

$m_d = \Delta/2$


$m^2_{\pi^\pm}$
$m^2_{\pi^0}$
m_c^-
$-m_d$
m_c^+
0.5
0.4
0.3
0.2
0.1
0.0
−1.4
−1.2
−1.0
−0.8
−0.6
−0.4
−0.2
0.0


$m_d = 3\Delta/2$
m^2_η
$-m_c^-$
$-m_d$
m_c^+
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−4
−3
−2
−1
0

3. Topological susceptibility and massless up quark

(anomalous) WT identities

$$\langle \left[\partial^\mu A^a_\mu(x) + \bar\psi(x)\{M,T^a\}\gamma_5\psi(x) - 2N_f\delta^{a0}q(x)\right]\mathcal{O}(y)\rangle \quad = \quad \delta^{(4)}(x-y)\langle\delta^a\mathcal{O}(y)\rangle$$

$$q(x) \quad = \quad \frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}G_{\mu\nu}(x)G_{\alpha\beta}(x),$$ topological charge density

$\mathcal{O}(y) = q(y)$ with $a = 0,3$

Integrating over x,

$$\chi \quad \equiv \int d^4x\,\langle q(x)q(y)\rangle = \frac{2m_u}{N_f}\int d^4x \;\langle \bar u\gamma_5 u(x)q(y)\rangle\,.$$

?

$$\chi = \infty \text{ at } m_{\pi^0} = 0 \text{ and } m_u \neq 0$$

$$\chi = 0 \text{ at } m_{\pi^0} \neq 0 \text{ and } m_u = 0$$

Anomalous WT identities in N_f=2 ChPT

$$\langle \delta_x S \mathcal{O}(y)\rangle = \delta^{(4)}(x-y)\,\langle \delta\mathcal{O}(y)\rangle$$

WT identities

$$\delta U(x) = 2i\theta(x)U(x).$$

$$\delta_x S = i\theta(x)\left[\partial^\mu A_\mu(x) + \mathrm{tr}\left\{MU^\dagger(x) - M^\dagger U(x)\right\} - \underline{\Delta\{\det U(x) - \det U^\dagger(x)\}}\right],$$

$$A_\mu(x) = f^2 \mathrm{tr}\left\{U^\dagger(x)\partial_\mu U(x) - U\partial_\mu U^\dagger(x)\right\}$$

$2N_f q(x)$: topological charge density

$$2N_f\chi \equiv \int d^4x\,\langle \left[\partial^\mu A_\mu(x) + \mathrm{tr}\left\{MU^\dagger(x) - M^\dagger U(x)\right\}\right] q(y)\rangle$$

$$2N_f\chi = \underline{\frac{\Delta^2}{4}\int d^4x\,\langle \{\det U(x) - \det U^\dagger(x)\}\{\det U(y) - \det U^\dagger(y)\}\rangle}$$
$$+ \underline{\frac{\Delta}{2}\langle \det U(y) + \det U^\dagger(y)\rangle},$$

$$-\frac{2\Delta^2}{f^2}\int d^4x\,\langle \eta(x)\eta(y)\rangle$$

effect of contact term ? $= \Delta$

$$2N_f\chi = -\frac{4\Delta^2 m_+(\vec{\varphi})}{m_+^2(\vec{\varphi}) - m_-^2(\vec{\varphi}) + 2m_+(\vec{\varphi})\delta m} + \Delta.$$

$m_u = 0 \Rightarrow m_+(\vec{\varphi}) = m_-(\vec{\varphi}) = m_d$ and $\delta m = 2\Delta$

$$\Rightarrow \boxed{2N_f\chi = -\frac{4\Delta^2 m_d}{4m_d\Delta} + \Delta = 0,}$$

$$m_{\pi^0}^2 = 0 \Rightarrow \int d^4x\, \langle \eta(x)\eta(y)\rangle = \frac{1}{2X}\left(\frac{X_-}{m_{\tilde{\pi}_0}^2} + \frac{X_+}{m_{\tilde{\eta}}^2}\right) \rightarrow \infty$$

$$\Rightarrow \boxed{2N_f\chi \rightarrow -\infty, \quad m_{\tilde{\pi}_0} \rightarrow 0.}$$

4. Summary and Discussions

Using ChPT with anomaly effect, we show

1. $m_u = 0$ is nothing special if $m_d \neq 0$. (no symmetry)
2. At $m_u = m_c^{\pm}, -m_c^{-} \neq 0$, $m_{\pi^0} = 0$.
3. $\langle \pi^0 \rangle \neq 0$ at $m_c^{-}(-m_c^{-}) < m_u < m_c^{+}$. Dashen phase Staggered quark ?
4. $\chi = \infty$ at $m_u = m_c$.
5. $\chi = 0$ at $m_u = 0$.

If $m_d \neq 0$, massless up quark ($m_u = 0$) may not be universal (scheme-dependent). m_+ and m_- are renormalized differently.

Instead, massless up quark may be defined by $\chi = 0$.

On the lattice, we should first show that χ is universal.

We then check whether $\chi = 0$ or not at physical point.

Application

$$m_u = -m_d = -m$$
$$\theta = 0$$

chiral rotations

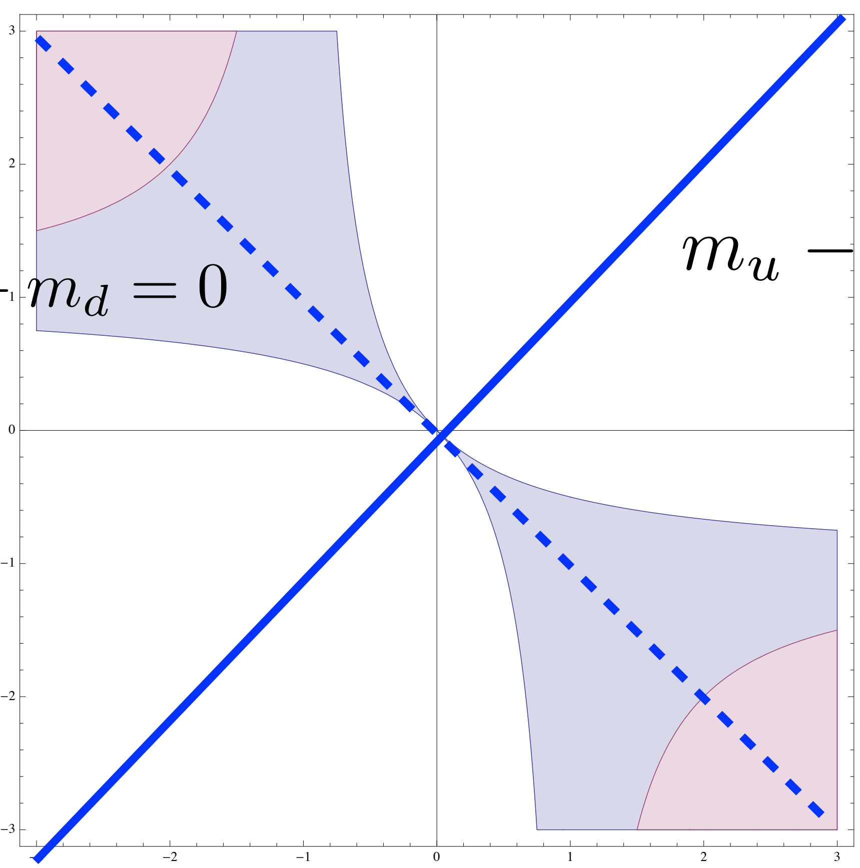
$$m_u = m_d = m$$
$$\theta = \pi$$

$$M = \begin{pmatrix} -m & 0 \\ 0 & m \end{pmatrix}$$

$$V_R = e^{i(\theta_0+\theta_3\tau^3)} = V_L^\dagger, \quad M' = V_L^\dagger M V_R = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$\theta_0 = \theta_3 = \pi/4,$$

$$\Delta \qquad\qquad -\Delta$$



ChPT analysis

VEV

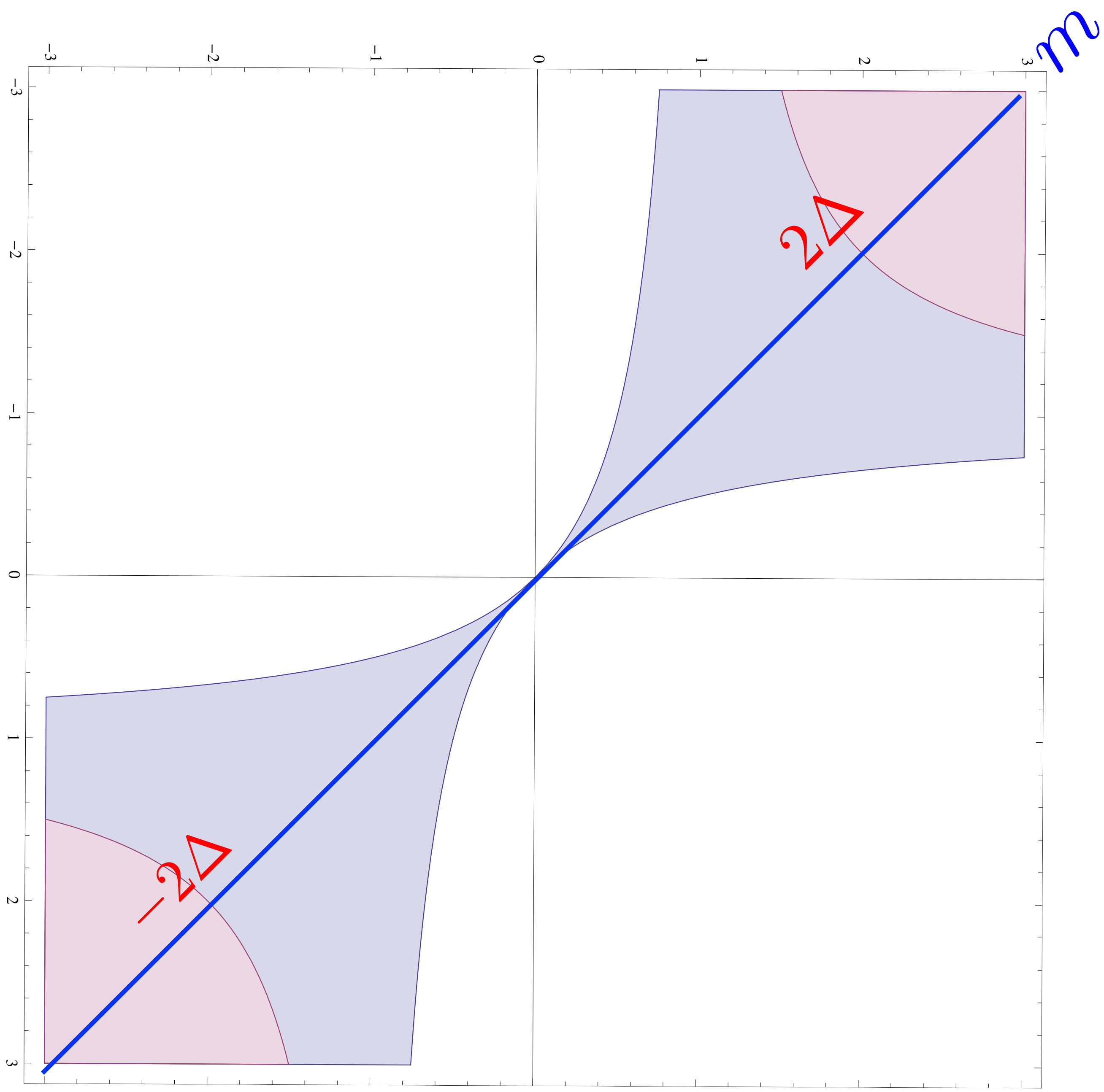
$$\langle\bar\psi\psi\rangle^2 + \langle\bar\psi i\gamma_5\psi\rangle^2 = 4$$

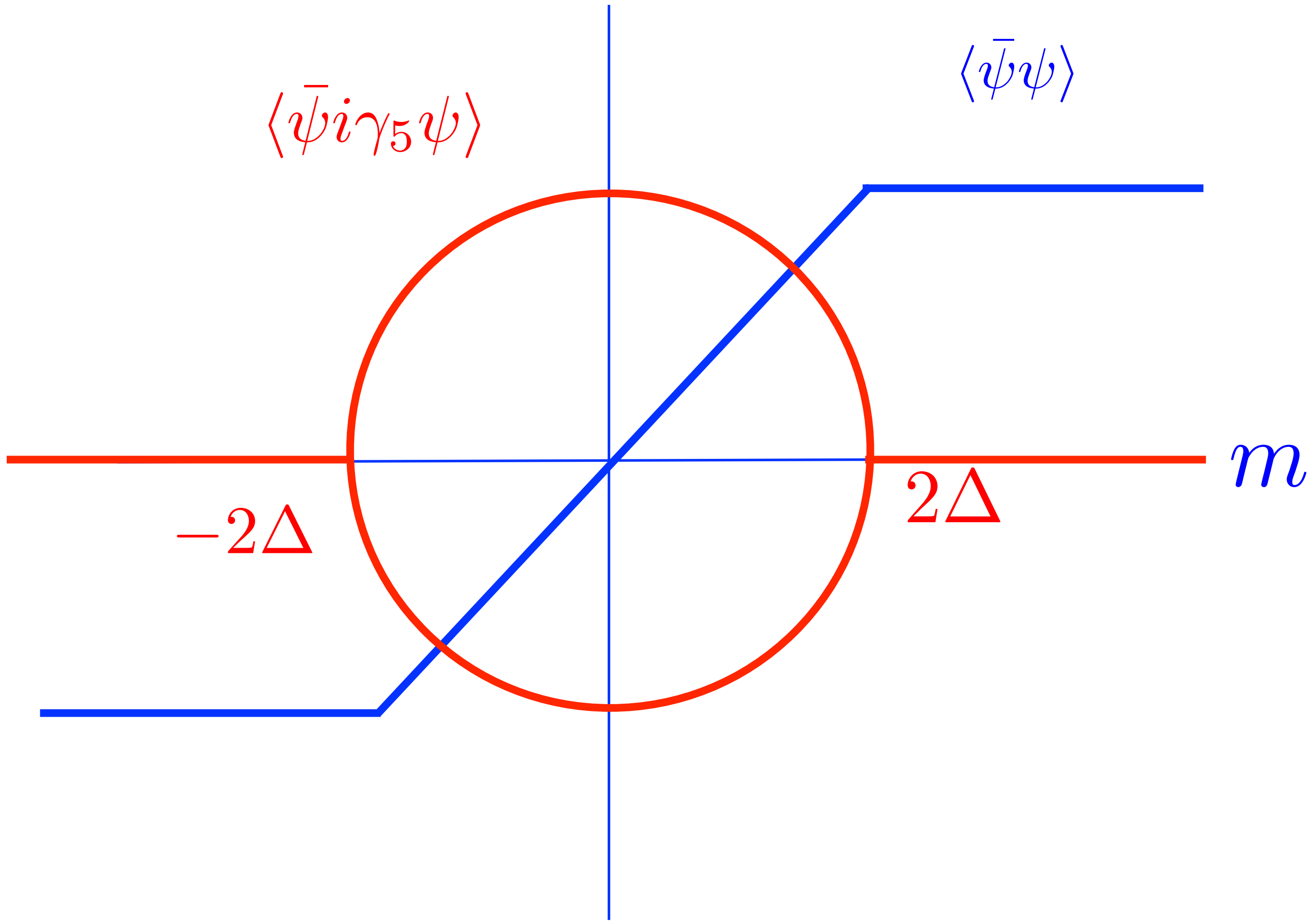
$$\cos\varphi_3 = 1$$

$$\cos\varphi_0 = \begin{cases} 1, & 2\Delta \le m \\ \dfrac{m}{2\Delta}, & -2\Delta < m < 2\Delta \\ -1, & m \le -2\Delta \end{cases}$$

$$\langle\bar\psi i\gamma_5\psi\rangle = 2\sin\varphi_0\cos\varphi_3 = \begin{cases} 0, & m^2 \ge 4\Delta^2 \\ \pm 2\sqrt{1-\dfrac{m^2}{4\Delta^2}}, & m^2 < 4\Delta^2 \end{cases}$$

$$\langle\bar\psi\psi\rangle = 2\cos\varphi_0\cos\varphi_3 = \begin{cases} 2, & 2\Delta \le m \\ \dfrac{m}{\Delta}, & -2\Delta < m < 2\Delta \\ -2, & m \le -2\Delta \end{cases},$$





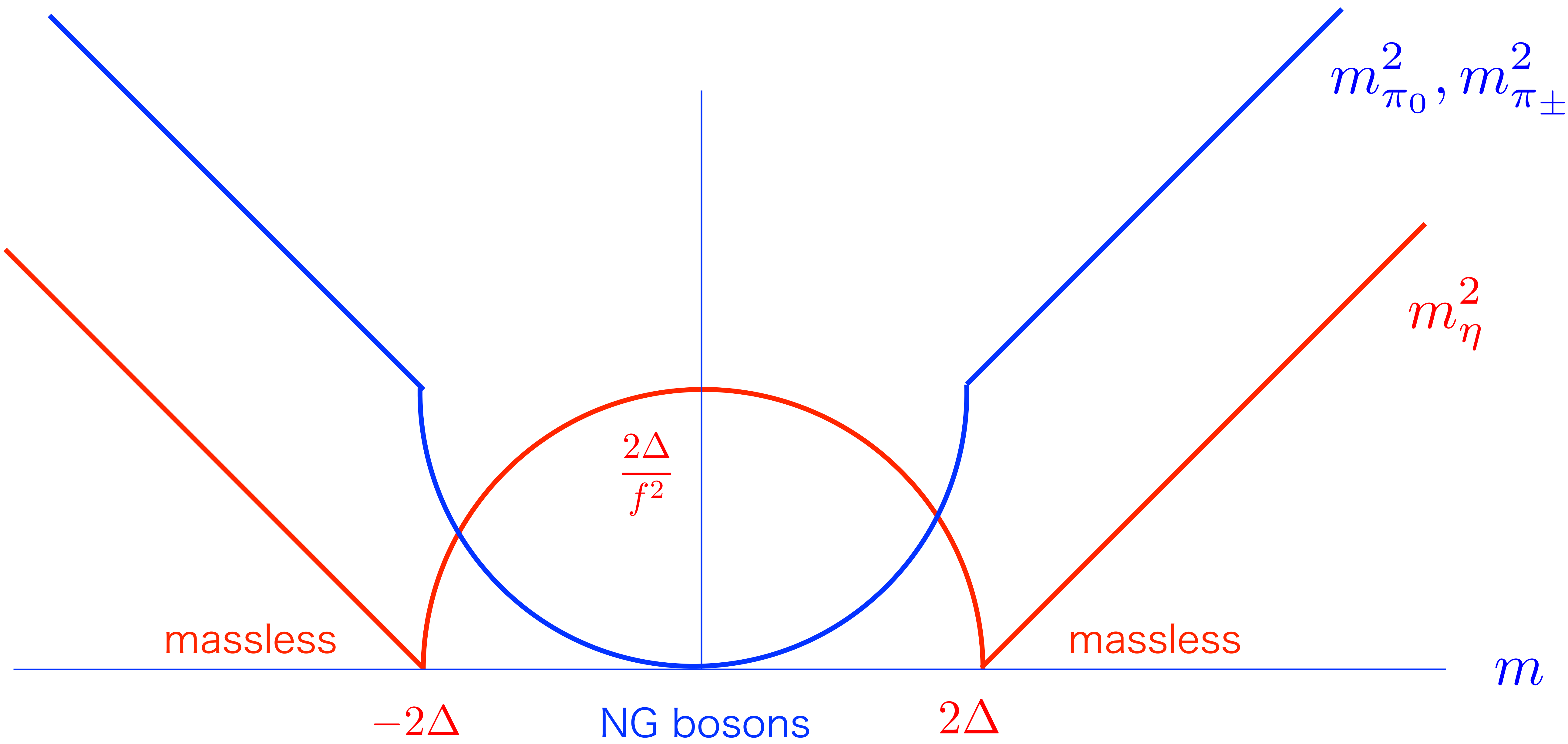
Spontaneous CP violation !

(eta condensation)

PS meson masses

$$m_{\pi_\pm}^2 = m_{\pi_0}^2 = \begin{cases} \dfrac{1}{2f^2} 2|m|, & m^2 \geq 4\Delta^2 \\ \underline{\dfrac{1}{2f^2}\dfrac{m^2}{\Delta}}, & m^2 < 4\Delta^2 \end{cases} \qquad m_\eta^2 = \begin{cases} \dfrac{1}{2f^2}\left[2|m| - 4\Delta\right], & m^2 \geq 4\Delta^2 \\ \dfrac{1}{2f^2}\dfrac{4\Delta^2 - m^2}{\Delta}, & m^2 < 4\Delta^2 \end{cases},$$

non-standard PCAC relation !



$$m_\pi^2 = \frac{1}{2f^2}\frac{m^2}{\Delta}$$

How can we get this from WT-identitities ?

$$\langle \{\partial^\mu A_\mu^3 + m\,\mathrm{tr}\tau^3(U^\dagger - U)\}(x)\mathcal{O}(y)\rangle = \langle \delta^x \mathcal{O}(y)\rangle$$

taking $\mathcal{O} = \mathrm{tr}\,\tau^3(U^\dagger - U)$ and integrating over x

$$\Rightarrow \quad m\int d^4x\, \langle \underbrace{\mathrm{tr}\,\tau^3(U^\dagger - U)(x)}_{= -i\frac{2\sqrt{2}}{f}\cos\varphi_0\,\pi_0(x)}\ \mathrm{tr}\,\tau^3(U^\dagger - U)(y)\rangle = -2\langle \underbrace{\mathrm{tr}(U + U^\dagger)(y)}_{= 4\cos\varphi_0}\rangle$$

$$\Rightarrow \quad m\frac{\cos^2\varphi_0}{f^2}\underbrace{\int d^4x\langle \pi_0(x)\pi_0(y)\rangle}_{= \frac{1}{m_{\pi_0}^2}} = \cos\varphi_0 \quad \Rightarrow \quad m_{\pi_0}^2 = \frac{m}{f^2}\underbrace{\cos\varphi_0}_{= \frac{m}{2\Delta}}$$

$$\Rightarrow \quad \boxed{m_{\pi_0}^2 = \frac{m}{f^2}\frac{m}{2\Delta}}$$

one m form WTI, the other m from VEV.