2013/9/30 離散的手法による場と時空のダイナミクス at KEK

SUSY double-well matrix model as 2D IIA superstring and its dynamical SUSY breaking

Fumihiko Sugino

(Okayama Inst. Quantum Phys.)

Mainly based on

- T. Kuroki and F. S., Nucl. Phys. B **867** (2013) 448, arXiv 1208.3263
- T. Kuroki and F. S., arXiv 1306.3561
- M. G. Endres, T. Kuroki, F. S. and H. Suzuki, to appear in Nucl. Phys. B, arXiv 1308.3306

1 Introduction

◇ Solvable matrix models for 2D quantum gravity or noncritical string theories were vigorously investigated around 1990.

- as toy models for critical string theories, in particular focused on nonperturbative aspects.

- But, little has been known about (solvable) matrix models corresponding to noncritical superstrings with target-space SUSY.
  We would like to consider such matrix models.

- We hope our analysis helpful to analyze matrix models for critical superstrings.

◇ A simple SUSY matrix model we will discuss: [Kuroki-F.S. 2009]

$$S_{\rm MM} = N\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2-\mu^2) + \bar{\psi}(\phi\psi+\psi\phi)\right],$$

where

$$\left.\begin{array}{ll} B, \phi & : \text{Bosonic} \\ \psi, \bar{\psi} & : \text{Fermionic} \end{array}\right\} N \times N \text{ hermitian matrices.}$$

- SUSY:

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0,$$
$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0.$$

$\Rightarrow Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0$ (nilpotent)

- B, ψ, $\bar{\psi}$ integrated out

$$S_{\rm MM} \to N\mathrm{tr}\,\frac{1}{2}(\phi^2-\mu^2)^2 - \ln\det(\phi\otimes \mathbb{1}_N + \mathbb{1}_N \otimes \phi)$$

↑

Double-well scalar potential

◇ Large-N saddle point equation for $\rho(x) \equiv \frac{1}{N}\mathrm{tr}\,\delta(x-\phi)$:

$$\int dy\,\rho(y)\,\mathrm{P}\frac{1}{x-y} + \int dy\,\rho(y)\,\mathrm{P}\frac{1}{x+y} = x^3 - \mu^2 x$$

SUSY preserving large-N solution with filling fraction (ν_+, ν_-): → Fig. 1

$(\nu_+ + \nu_- = 1)$ [Kuroki-F.S. 2009]

$$\rho(x) = \begin{cases} \frac{\nu_+}{\pi}\,x\,\sqrt{(x^2-a^2)(b^2-x^2)} & (a < x < b) \\ \frac{\nu_-}{\pi}\,|x|\,\sqrt{(x^2-a^2)(b^2-x^2)} & (-b < x < -a) \end{cases}$$

with $a = \sqrt{\mu^2 - 2}$, $b = \sqrt{\mu^2 + 2}$.

- Exists for $\mu^2 > 2$.

(SUSY breaking one-cut solution for $\mu^2 < 2$. [Kuroki-F.S. 2010])

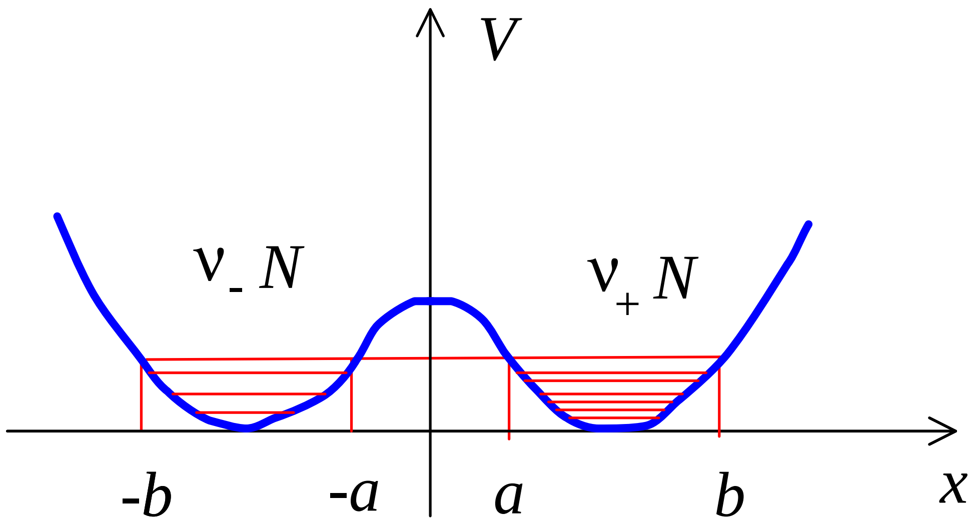

Figure 1: Double-well scalar potential.

- (large-N free energy) $= 0$, $\quad \langle \frac{1}{N} \mathrm{tr}\, B^n \rangle = 0 \ (n = 1, 2, \cdots)$
  strongly suggest that SUSY is preserved.

Note that $\mathrm{tr}\, B^n = Q\, \mathrm{tr}\, (i\bar{\psi} B^{n-1}) = \bar{Q}\, \mathrm{tr}(i\psi B^{n-1})$.

$\Rightarrow$ The SUSY minima are infinitely degenerate, parametrized by (ν_+, ν_-).

◇ In this talk,

- we compute correlation functions of this matrix model (in section 2).

  ($\longrightarrow$ Logarithmic critical behavior)

- We discuss correspondence between the matrix model and 2D type IIA superstring theory on a nontrivial RR background (in sections 3 & 4).

- Summary and Discussions so far (in section 5).

- We compute nonperturbative effects in the matrix model, and observe that the SUSY is spontaneously broken in the double scaling limit (in section 6).

$$\Downarrow$$

In the type IIA theory,

SUSY is dynamically broken by a nonperturbative effect.

To our knowledge, 1st explicit and analytic result for SUSY breaking by non-perturbative dynamics in superstring theory

◇ The logarithmic critical behavior is somewhat reminiscent of the $c = 1$ matrix model (matrix quantum mechanics) [Kazakov-Migdal 1988] or the Penner model (zero-dimensional matrix model). [Distler-Vafa 1991]

$$Z_{\mathrm{Penner}} = \int d^{N^2}M \, \exp[N t \, \mathrm{tr}\{M + \ln(1 - M)\}]$$

⇒ We expect
Our matrix model ∼ a SUSY version of the Penner model
∼ 2D superstring with target-space SUSY.

Note:

- This matrix model is equivalent to the $O(n=-2)$ model on a random surface [Kostov 1989]:

$$Z_{O(n)} = \int d^{N^2}\phi \, e^{-N\mathrm{tr}\, V(\phi)} \, \det(\phi \otimes \mathbb{1}_N + \mathbb{1}_N \otimes \phi)^{-n/2}$$

with $V(\phi) = \frac{1}{2}(\phi^2 - \mu^2)^2$.

- Its critical behavior is described by $c = -2$ topological gravity (i.e. Gaussian one-matrix model). [Kostov-Staudacher 1992]

- It is easily seen by the Nicolai mapping $H = \phi^2$.
[Kostov 1990, Gaiotto-Rastelli-Takayanagi 2004]

Partition function in the (ν_+, ν_-) sector becomes

$$Z_{\mathrm{MM}}^{(\nu_+,\nu_-)} \Rightarrow (-1)^{\nu_- N} \int_{H_+} d^{N^2}H \, e^{N\mathrm{tr}\, \frac{1}{2}(H-\mu^2)^2}.$$

But, the H-integration is over positive definite hermitian matrices.

$\mathbf{tr}\,\phi^{2n}$ or $\mathbf{tr}\,B^{n}$ can be treated within the topological gravity (Gaussian one-matrix model) in $\frac{1}{N}$-expansion.

Boundary effect cannot be seen.

However, $\mathbf{tr}\,\phi^{2n+1}$, $\mathbf{tr}\,\psi^{2n+1}$, $\mathbf{tr}\,\bar{\psi}^{2n+1}$ etc are not observables in the topological gravity.

- $\mathbf{tr}\,\phi^{2n+1} = \mathbf{tr}\,H^{n+\frac{1}{2}}$ is singular at the origin.
- $(\mathbf{tr}\,\psi^{2n} = \mathbf{tr}\,\bar{\psi}^{2n} = 0.)$

Actually, we see nontrivial logarithmic critical behavior for these operators.

2 Planar correlation functions

$$\left\langle \frac{1}{N} \operatorname{tr} \phi^n \right\rangle_0 = \int dx\, x^n \rho(x)$$
$$= \left(\nu_+ + (-1)^n \nu_-\right) (2+\mu^2)^{n/2}\, F\left(-\frac{n}{2}, \frac{3}{2}, 3; \frac{4}{2+\mu^2}\right)$$

- reduces to a polynomial of μ^2 for n even:

$$\left\langle \frac{1}{N} \operatorname{tr} \phi^2 \right\rangle_0 = \mu^2, \qquad \left\langle \frac{1}{N} \operatorname{tr} \phi^4 \right\rangle_0 = 1 + \mu^4, \qquad \cdots.$$

($c = -2$ topological gravity)

- exhibits logarithmic singular behavior as $\mu^2 \to 2$ for n odd:

$$\omega \equiv \tfrac{1}{4}(\mu^2 - 2)$$

$$\left\langle \frac{1}{N} \operatorname{tr} \phi^{2k+1} \right\rangle_0 = (\nu_+ - \nu_-) \left[(\text{const.})\, \omega^{k+2} \ln \omega + (\text{less singluar}) \right].$$

- We also computed planar two- and three-point functions for

$$\Phi_{2k+1} \sim \frac{1}{N}\mathrm{tr}\,\phi^{2k+1}.$$

The results so far suggest

$$\langle \Phi_{2k_1+1} \cdots \Phi_{2k_n+1} \rangle_{C,0} = (\nu_+ - \nu_-)^n (\text{const.})\, \omega^{2-\gamma+\sum_{i=1}^{n}(k_i-1)} (\ln \omega)^n$$
$$+(\text{less singular})$$

with $\gamma = -1$.

↖ string susceptibility of $c = -2$ topological gravity

(nontrivial)

We will see that 2D superstring theory reproduces higher powers of $\ln \omega$ due to a RR-background.

- For fermions ($\Psi_{2k+1} \sim \frac{1}{N}\mathrm{tr}\,\psi^{2k+1}$, $\bar{\Psi}_{2k+1} \sim \frac{1}{N}\mathrm{tr}\,\bar{\psi}^{2k+1}$), ...

3 2D type IIA superstring [Kutasov-Seiberg 1990, Ita-Nieder-Oz 2005]

- (Target space) $= (x, \varphi)$,
  where $x \in S^1$ with self-dual radius ($R = 1$) and φ: Liouville.
  ($\nwarrow$ Same as the Penner model!)

- Holomorphic EM tensor (except ghost part) on string worldsheet:

$$T_m = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\psi_x \partial \psi_x - \frac{1}{2}(\partial \varphi)^2 + \frac{Q}{2}\partial^2 \varphi - \frac{1}{2}\psi_\ell \partial \psi_\ell$$

  with $Q = 2$.

- Target-space SUSY is nilpotent.

$$q_+(z) = e^{-\frac{1}{2}\phi - \frac{i}{2}H - ix}(z), \qquad Q_+ = \oint \frac{dz}{2\pi i}\, q_+(z),$$

$$\bar{q}_-(\bar{z}) = e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{H} + i\bar{x}}(\bar{z}), \qquad \bar{Q}_- = \oint \frac{d\bar{z}}{2\pi i}\, \bar{q}_-(\bar{z}),$$

  where $\psi_\ell \pm i\psi_x = \sqrt{2}e^{\mp iH}$.

  $\Rightarrow Q_+^2 = \bar{Q}_-^2 = 0$. ($\leftarrow$ Same as the matrix model!)

- Vertex operators (holomorphic sector):

$$\text{NS sector } (-1)\text{-picture}: \quad T_k(z) = e^{-\phi+ikx+p_\ell\varphi}(z)$$

$$\text{R sector } (-\tfrac{1}{2})\text{-picture}: \quad V_{k,\,\epsilon}(z) = e^{-\frac{1}{2}\phi+\frac{i}{2}\epsilon H+ikx+p_\ell\varphi}(z)$$

with $\epsilon = \pm 1$.

Locality with supercurrents, mutual locality, superconformal inv., level matching

$\Rightarrow$ physical vertex operators (on-shell particles)

$$p_\ell = 1 - |k|$$
$$k = \epsilon|k|$$

Winding background: [Ita-Nieder-Oz 2005]

(NS, NS) : $T_k(z)\,\bar{T}_{-k}(\bar{z})$ $(k \in \mathbf{Z} + \frac{1}{2})$ "tachyon" winding

(R+, R−) : $V_{k,+1}(z)\,\bar{V}_{-k,-1}(\bar{z})$ $(k = \frac{1}{2}, \frac{3}{2}, \cdots)$

(R−, R+) : $V_{-k,-1}(z)\,\bar{V}_{k,+1}(\bar{z})$ $(k = 0, 1, 2, \cdots)$

RR 2-form field strength winding

(NS, R−) : $T_{-k}(z)\,\bar{V}_{-k,-1}(\bar{z})$ $(k = \frac{1}{2}, \frac{3}{2}, \cdots)$ fermion$(-)$ momentum

(R+, NS) : $V_{k,+1}(z)\,\bar{T}_k(\bar{z})$ $(k = \frac{1}{2}, \frac{3}{2}, \cdots)$ fermion$(+)$ momentum

Note: We omit details of cocycle factors.

$\diamondsuit$ Let us assume the correspondence of supercharges between the matrix model and the type IIA theory:

$$(Q, \bar{Q}) \Leftrightarrow (Q_+, \bar{Q}_-).$$

$\Rightarrow$ SUSY transformation properties etc lead to

$$\Phi_1 = \frac{1}{N}\mathrm{tr}\,\phi \Leftrightarrow c_0\, g_s^2 \int d^2z\, V_{\frac{1}{2},+1}(z)\, \bar{V}_{-\frac{1}{2},-1}(\bar{z}) \qquad \text{(R+, R−)},$$

$$\Psi_1 = \frac{1}{N}\mathrm{tr}\,\psi \Leftrightarrow d_0\, g_s^2 \int d^2z\, T_{-\frac{1}{2}}(z)\, \bar{V}_{-\frac{1}{2},-1}(\bar{z}) \qquad \text{(NS, R−)},$$

$$\bar{\Psi}_1 = \frac{1}{N}\mathrm{tr}\,\bar{\psi} \Leftrightarrow \bar{d}_0\, g_s^2 \int d^2z\, V_{\frac{1}{2},+1}(z)\, \bar{T}_{\frac{1}{2}}(\bar{z}) \qquad \text{(R+, NS)},$$

$$\frac{1}{N}\mathrm{tr}(-iB) \Leftrightarrow g_s^2 \int d^2z\, T_{-\frac{1}{2}}(z)\, \bar{T}_{\frac{1}{2}}(\bar{z}) \qquad \text{(NS, NS)}.$$

Quartet w.r.t. $(Q, \bar{Q})$ $\Leftrightarrow$ Quartet w.r.t. $(Q_+, \bar{Q}_-)$

c_0, d_0, $\bar{d}_0$: numerical consts. , $\qquad \frac{1}{N} \Leftrightarrow g_s$

Furthermore, it is natural to extend it to higher $k(=1,2,\cdots)$ as

$$\Phi_{2k+1} = \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} + (\text{mixing}) \;\Leftrightarrow\; c_k\, g_s^2 \int d^2z\, V_{k+\frac{1}{2},\,+1}(z)\,\bar{V}_{-k-\frac{1}{2},\,-1}(\bar{z}),$$

$$\Psi_{2k+1} = \frac{1}{N}\mathrm{tr}\,\psi^{2k+1} + (\text{mixing}) \;\Leftrightarrow\; d_k\, g_s^2 \int d^2z\, T_{-k-\frac{1}{2}}(z)\,\bar{V}_{-k-\frac{1}{2},\,-1}(\bar{z}),$$

$$\bar{\Psi}_{2k+1} = \frac{1}{N}\mathrm{tr}\,\bar{\psi}^{2k+1} + (\text{mixing}) \;\Leftrightarrow\; \bar{d}_k\, g_s^2 \int d^2z\, V_{k+\frac{1}{2},\,+1}(z)\,\bar{T}_{k+\frac{1}{2}}(\bar{z}),$$

(Single trace operators in the matrix model) $\Leftrightarrow$ (Integrated vertex operators in IIA)

(Powers of matrices) $\Leftrightarrow$ (Windings or Momenta)

Note:

- RR 2-form field strength in (R$-$, R$+$) is a singlet under the target-space SUSYs $\boldsymbol{Q_+}$, $\boldsymbol{\bar{Q}_-}$, and appears to have no matrix-model counterpart.
- Expectation values of operators with nonzero Ramond charge (e.g. $\langle \boldsymbol{\Phi_{2k+1}} \rangle_{\boldsymbol{0}}$) are nonvanishing in the matrix model.

$\Rightarrow$ The matrix model is considered to correspond to IIA on a background of the RR 2-form.

Let us check the correspondence by computing amplitudes in IIA theory.

4 Correspondence between the matrix model and the IIA theory

◇ Correlation functions among integrated vertex operators in IIA on the trivial background:

$$\left\langle \prod_i \mathcal{V}_i \right\rangle = \frac{1}{\text{Vol.(CKV)}} \int \mathcal{D}(x, \varphi, H, \text{ghosts})\, e^{-S_{\rm CFT}} e^{-S_{\rm int}} \prod_i \mathcal{V}_i,$$

$$S_{\rm CFT} = \frac{1}{2\pi} \int d^2z \left[\partial x \bar{\partial} x + \partial\varphi\bar{\partial}\varphi + \frac{Q}{4}\sqrt{\hat{g}}\hat{R}\varphi + \partial H \bar{\partial} H + (\text{ghosts}) \right],$$

$$S_{\rm int} = \omega \int d^2z\, T^{(0)}_{-\frac{1}{2}}(z) \bar{T}^{(0)}_{\frac{1}{2}}(\bar{z}) \qquad (\leftarrow \text{0-picture (NS, NS) “tachyon”})$$

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$$S_{\text{CFT}} = \frac{1}{2\pi} \int d^2 z \left[\partial x \bar{\partial} x + \partial \varphi \bar{\partial} \varphi + \frac{Q}{4} \sqrt{\hat{g}} \hat{R} \varphi + \partial H \bar{\partial} H + (\text{ghosts}) \right],$$

$$S_{\text{int}} = \omega \int d^2 z\, T^{(0)}_{-\frac{1}{2}}(z) \bar{T}^{(0)}_{\frac{1}{2}}(\bar{z}) \qquad (\leftarrow \text{0-picture (NS, NS) “tachyon”})$$

◇ Correlation functions in IIA on (R−, R+) background:

$$\left\langle\!\left\langle \prod_i \mathcal{V}_i \right\rangle\!\right\rangle \equiv \left\langle \left(\prod_i \mathcal{V}_i \right) e^{W_{\text{RR}}} \right\rangle,$$

where $W_{\rm RR}$ is invariant under the target-space SUSYs:

$$W_{\rm RR} \;=\; (\nu_+ - \nu_-) \sum_{k\in \mathbf{Z}} a_k\, \omega^{k+1} \mathcal{V}_k^{\rm RR}, \qquad (a_k : \text{numerical consts.})$$

$$\mathcal{V}_k^{\rm RR} \equiv \begin{cases} \int d^2z\, V_{k,\,-1}(z) \bar{V}_{-k,\,+1}(\bar{z}) & (p_\ell = 1 - |k|,\; k = 0, -1, -2, \cdots) \\ \int d^2z\, V^{\rm (nonlocal)}_{-k,\,-1}(z) \bar{V}^{\rm (nonlocal)}_{k,\,+1}(\bar{z}) & (p_\ell = 1 + |k|,\; k = 1, 2, \cdots). \end{cases}$$

Note

- We treat the RR background for $(\nu_+ - \nu_-)$ small as exponentiated vertex operators: (Picture should be adjusted by hand.)

$$\left\langle\!\!\left\langle \prod_i \mathcal{V}_i \right\rangle\!\!\right\rangle \equiv \left\langle \left(\prod_i \mathcal{V}_i \right) e^{W_{\rm RR}} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(\prod_i \mathcal{V}_i \right) (W_{\rm RR})^n \right\rangle .$$

◇ Standard Liouville theory computation for amplitudes leads to:

- $\langle N\mathrm{tr}(-iB)\,\Phi_{2k+1}\rangle_{\text{cylinder}} = -\frac{1}{4}\partial_\omega \langle \Phi_{2k+1}\rangle_{\text{disk}} \sim (\nu_+ - \nu_-)\,\omega^{k+1}\ln\omega$

$$\Updownarrow$$

$$\mathcal{N} g_s^{-2} \left\langle\!\!\left\langle \frac{1}{4}\left(\int T_{-\frac{1}{2}}\bar{T}_{\frac{1}{2}}\right)\left(c_k g_s^2 \int V_{k+\frac{1}{2},+1}\bar{V}_{-k-\frac{1}{2},-1}\right)\right\rangle\!\!\right\rangle$$

$$= \mathcal{N} c_k \frac{1}{4}(\nu_+ - \nu_-)\sum_{\ell\in\mathbf{Z}} a_\ell\,\omega^{\ell+1}\left\langle \left(\int T_{-\frac{1}{2}}\bar{T}_{\frac{1}{2}}\right)\left(\int V_{k+\frac{1}{2},+1}\bar{V}_{-k-\frac{1}{2},-1}\right)\mathcal{V}_\ell^{\mathrm{RR}}\right\rangle$$

$$= -\mathcal{N} c_k \frac{1}{2}(\nu_+ - \nu_-)\,a_k\,(\omega^{k+1}\ln\omega)\,e^{i2\pi\beta(-k^2-\frac{1}{2}k+\frac{1}{4})}$$

$$\uparrow$$

cocycle factor

$(\beta \in \mathbf{Z} + \frac{1}{2})$.

Similarly,

- $\langle \Phi_{2k_1+1}\Phi_{2k_2+1}\rangle_{\text{cylinder}} \sim \dfrac{1}{N^2}(\nu_+ - \nu_-)^2\, \omega^{k_1+k_2+1}(\ln\omega)^2$

$$\Updownarrow$$

$$\begin{aligned}
&\mathcal{N} g_s^{-2} \left\langle\!\!\left\langle \left(c_{k_1} g_s^2 \int V_{k_1+\frac{1}{2},+1}\bar{V}_{-k_1-\frac{1}{2},-1}\right)\left(c_{k_2} g_s^2 \int V_{k_2+\frac{1}{2},+1}\bar{V}_{-k_2-\frac{1}{2},-1}\right)\right\rangle\!\!\right\rangle \\
&= \mathcal{N} g_s^2 c_{k_1} c_{k_2}\,\frac{1}{2}(\nu_+ - \nu_-)^2 \sum_{\ell_1,\ell_2\in\mathrm{Z}} a_{\ell_1} a_{\ell_2}\,\omega^{\ell_1+\ell_2+2} \\
&\quad\times\left\langle \left(\int V_{k_1+\frac{1}{2},+1}\bar{V}_{-k_1-\frac{1}{2},-1}\right)\left(\int V_{k_2+\frac{1}{2},+1}\bar{V}_{-k_2-\frac{1}{2},-1}\right)\mathcal{V}^{\mathrm{RR}}_{\ell_1}\mathcal{V}^{\mathrm{RR}}_{\ell_2}\right\rangle \\
&= \mathcal{N} g_s^2 c_L\, c_{k_1} c_{k_2}\,(\nu_+ - \nu_-)^2\, 2\pi\, a_{k_1+k_2}\, a_{-1}\left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)^2 \omega^{k_1+k_2+1}(\ln\omega)^2 \\
&\quad\times e^{-i\pi\beta\{(k_1+\frac{1}{2})^2+(k_2+\frac{1}{2})^2+(k_1+k_2)^2+1\}}
\end{aligned}$$

with appropriate regularization of resonant singularities by the Liouville volume $c_L(-2\ln\omega)$.

- $\langle \Psi_{2k_1+1}\Psi_{2k_2+1}\rangle_{\text{cylinder}} \cdots$

Regularization:

For example, the formula for 4-pt. amplitude

$$\int d^2z\, z^{\alpha}\bar{z}^{\bar{\alpha}}(1-z)^{\beta}(1-\bar{z})^{\bar{\beta}} = \pi \frac{\Gamma(\bar{\alpha}+1)\Gamma(\bar{\beta}+1)}{\Gamma(\bar{\alpha}+\bar{\beta}+2)} \frac{\Gamma(-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)}$$

with

$$\alpha = \bar{\alpha} = k_3 k_4 - p_{\ell_3} p_{\ell_4} = k_1 + k_2,$$
$$\beta = \bar{\beta} = k_2 k_4 - p_{\ell_2} p_{\ell_4} - \frac{1}{2} = -k_1 - 1, \qquad (k_1, k_2 = 0, 1, 2, \cdots)$$

is indefinite.

We regularize it as

$$\alpha \to \alpha + \epsilon, \qquad \bar{\alpha} \to \bar{\alpha} + \epsilon, \qquad \beta \to \beta + \epsilon, \qquad \bar{\beta} \to \bar{\beta} + \epsilon$$

with $1/\epsilon = c_L(-2\ln\omega)$, and get the result $\frac{\pi}{2}\left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)^2 c_L(-2\ln\omega)$.

- This regularization preserves the mutual locality of vertex operators, i.e. does not change $\alpha - \bar{\alpha}$ and $\beta - \bar{\beta}$.

Remarks:

- Computation in the type IIA side reproduces the $(\nu_+ - \nu_-)$-dependence and the ω-dependence in the matrix model result.

- Moreover, relations among numerical coefficients seem consistent.
  In particular,

$$\hat{c}_k = \hat{c}_0\, e^{\gamma k}(2k+1)!, \qquad \hat{a}_k = \frac{\hat{a}_0\, e^{-\gamma k}}{k!(k+1)!} \qquad (k = 0, 1, 2, \cdots)$$

  with γ: const, $\hat{c}_k \equiv c_k\, e^{-i\pi\beta(k+\frac{1}{2})^2}$, $\hat{a}_k \equiv a_k\, e^{-i\pi\beta k^2}$,

$$d_0\bar{d}_0 = \frac{1}{4}c_0 \quad \to \text{consistent with SUSY.}$$

- Higher powers of $\ln \omega$ comes from resonances among external particles and the (R$-$,R$+$) background.

5 Summary and discussions so far

◇ We computed correlation functions in the double-well SUSY matrix model, and discussed its correspondence to 2D type IIA superstring theory on (R−,R+) background by computing amplitudes in both sides.

This is an interesting example of matrix models for superstrings with target-space SUSY, in which various amplitudes are explicitly calculable.

◇ Matrix-model counterpart of positive-winding "tachyons" $T_{k-\frac{1}{2}}\bar{T}_{-k+\frac{1}{2}}$ $(k = 1, 2, \cdots)$?

Similar to the Kontsevich-Penner model (introducing an external matrix source)?

[Imbimbo-Mukhi 1995]

◇ Other amplitudes (higher genus, higher point)?

◇ Case of $(\nu_+ - \nu_-)$ not small?

Related to black-hole (cigar) target space? cf. [Hori-Kapustin 2001]

$\diamondsuit$ Higher dimensional Cases $(\boldsymbol{D = 4, 6, 8, 10})$? (Liouville)$\times \boldsymbol{S^1} \times \mathbf{R}^{\boldsymbol{D-2}}$

[Kutasov-Seiberg 1990]

6 Nonperturbative SUSY breaking in the matrix model

◇ SUSY double-well matrix model

$$S_{\mathrm{MM}} = N\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2 - \mu^2) + \bar{\psi}(\phi\psi + \psi\phi)\right].$$

After integrating out matrices other than ϕ, the partition function is expressed in terms of eigenvalues λ_i $(i = 1, \cdots, N)$ as

$$Z = \tilde{C}_N \int \Big(\prod_{i=1}^{N} d\lambda_i\Big) \triangle(\lambda)^2 \prod_{i,j=1}^{N} (\lambda_i + \lambda_j)\, e^{-N\sum_{i=1}^{N}\frac{1}{2}(\lambda_i^2-\mu^2)^2}$$

$$= \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!}\, Z_{(\nu_+,\nu_-)},$$

where the partition function in the (ν_+, ν_-) sector is defined by the integration region

$$\int_0^{\infty} \prod_{i=1}^{\nu_+ N} d\lambda_i \qquad \int_{-\infty}^{0} \prod_{j=\nu_+ N+1}^{N} d\lambda_j.$$

By $\boldsymbol{\lambda}_j \to -\boldsymbol{\lambda}_j$ ($j = \nu_+ N + 1, \cdots, N$), it is easy to see

$$Z_{(\nu_+,\nu_-)} = (-1)^{\nu_- N} \, Z_{(1,0)}.$$

Thus, the total partition function vanishes:

$$Z = \sum_{\nu_- N = 0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!} \, Z_{(\nu_+,\nu_-)} = (1 + (-1))^N \, Z_{(1,0)} = 0.$$

$\Rightarrow$ Expectation values normalized by Z_{MM} become ill-defined.

Let us regularize it as

$$Z_\alpha \equiv \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!} e^{-i\alpha\nu_- N} Z_{(\nu_+,\nu_-)} = (1 - e^{-i\alpha})^N Z_{(1,0)}.$$

◇ Order parameter of spontaneous SUSY breaking:

$$\left\langle \frac{1}{N}\mathrm{tr}(iB) \right\rangle_\alpha = \left\langle \frac{1}{N}\mathrm{tr}\,(\phi^2 - \mu^2) \right\rangle_\alpha = \frac{1}{N^2}\frac{1}{Z_\alpha}\frac{\partial}{\partial(\mu^2)} Z_\alpha$$

$$= \left\langle \frac{1}{N}\mathrm{tr}\,(\phi^2 - \mu^2) \right\rangle^{(1,0)} \quad \leftarrow \text{VEV taken in the (1,0) sector}$$

is independent of α and well-defined in the limit $\alpha \to 0$.

6.1 Orthogonal Polynomials

Under the change of variables $x_i = \lambda_i^2 - \mu^2$, $Z_{(1,0)}$ reduces to a Gaussian matrix model

$$Z_{(1,0)} = \tilde{C}_N \int_{-\mu^2}^{\infty} \left(\prod_{i=1}^{N} dx_i \right) \triangle(x)^2 \, e^{-N \sum_{i=1}^{N} \frac{1}{2} x_i^2}.$$

Orthogonal polynomials

$$P_n(x) = x^n + \sum_{i=0}^{n-1} p_n^{(i)} x^i \qquad (n = 0, 1, 2, \cdots)$$

$$(P_n,\ P_m) \equiv \int_{-\mu^2}^{\infty} dx \, e^{-\frac{N}{2} x^2} \, P_n(x) \, P_m(x) = h_n \delta_{n,m}$$

satisfy the recursion relations

$$x P_n(x) = P_{n+1}(x) + S_n P_n(x) + R_n P_{n-1}(x), \qquad h_n = R_n h_{n-1}.$$

For example,

$$h_0 = \sqrt{\frac{2\pi}{N}}\left[1 - \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{N}{2}}\,\mu^2\right)\right], \quad S_0 = -p_1^{(0)} = \frac{1}{Nh_0}\,e^{-\frac{N}{2}\mu^4}, \cdots .$$

Note:

The coefficients S_n, R_n are expressed by the boundary value of the orthogonal polynomials:

$$\begin{aligned} S_n &= \frac{1}{N}\frac{1}{h_n}\,P_n(-\mu^2)^2\,e^{-\frac{N}{2}\mu^4}, \\ R_n &= \frac{n}{N} + \frac{1}{N}\frac{1}{h_{n-1}}\,P_n(-\mu^2)P_{n-1}(-\mu^2)\,e^{-\frac{N}{2}\mu^4}. \end{aligned}$$

What we want to compute is

$$\left\langle \frac{1}{N}\mathrm{tr}\,(\phi^2 - \mu^2)\right\rangle^{(1,0)} = \frac{1}{N}\sum_{n=0}^{N-1} S_n.$$

6.2 One-instanton contribution

Let us take into account the boundary effect iteratively.

Boundary $\Leftrightarrow$ Local maximum of the double-well potential $\frac{1}{2}(\lambda^2 - \mu^2)^2$

n eigenvalues on the local maximum $\Rightarrow$ n-instanton configuration

[Hanada et al 2004]

If we could ignore the boundary effect, the orthogonal polynomials would reduce to the Hermite polynomials

$$P_n^{(H)}(x) = \frac{1}{(2N)^{n/2}} H_n\left(\sqrt{\frac{N}{2}}\,x\right), \qquad H_n(x) \equiv (-1)^n\, e^{x^2} \frac{d^n}{dx^n}\, e^{-x^2},$$

$$h_n^{(H)} = \sqrt{2\pi}\frac{n!}{N^{n+\frac{1}{2}}}.$$

◇ The 1st order approximation ($\Leftrightarrow$ one-instanton effect):

$$S_n \Rightarrow S_n^{(H)} \equiv \frac{1}{N}\frac{1}{h_n^{(H)}} P_n^{(H)}(-\mu^2)^2\, e^{-\frac{N}{2}\mu^4}.$$

Then,

$$\left\langle \frac{1}{N}\mathrm{tr}\,(\phi^2 - \mu^2)\right\rangle^{(1,0)}\Bigg|_{1-\mathrm{inst.}} = \frac{1}{N}\sum_{n=0}^{N-1} S_n^{(H)}.$$

Relevant formulas

$$\sum_{k=0}^{n-1}\frac{1}{2^k k!}\, H_k(x)^2 = \frac{1}{2^n\,(n-1)!}\left[H_n(x)^2 - H_{n-1}(x)\, H_{n+1}(x)\right]$$

and

$$e^{-x^2/2} H_n(x) = \pi^{\frac{1}{4}} 2^{\frac{n}{2}+\frac{1}{4}} n^{-\frac{1}{12}}\sqrt{n!}\left[\mathrm{Ai}(s) + \mathcal{O}(n^{-2/3})\right] \qquad (n \sim \infty)$$

for $x = \sqrt{2n+1} + \frac{s}{\sqrt{2}\, n^{1/6}}$

yield the result in the double scaling limit $(t \equiv N^{2/3}\omega)$:

$$\left\langle \frac{1}{N}\mathrm{tr}\,(\phi^2 - \mu^2) \right\rangle^{(1,0)}\Bigg|_{1-\mathrm{inst.}} = N^{-4/3}\left[\mathrm{Ai}'(4t)^2 - 4t\mathrm{Ai}(4t)^2 + \mathcal{O}(N^{-2/3})\right]$$
$$= N^{-4/3}\,\frac{1}{32\pi t}\,e^{-\frac{32}{3}t^{3/2}}\left[1 + \sum_{n=1}^{\infty} a_n^{(1)}\,t^{-\frac{3}{2}n}\right]$$

with $a_1^{(1)} = -\frac{17}{192}$, $a_2^{(1)} = \frac{1225}{73728}$, $a_3^{(1)} = -\frac{199115}{42467328}, \cdots$.

Notes:

- The double scaling limit is expected from the $c = -2$ topological gravity with the string susceptibility $\gamma = -1$.

$$\text{(Sphere free energy)} \sim N^2\omega^{2-\gamma} = t^{2-\gamma}$$

But, it is nontrivial that the nonperturbative contribution obeys this scaling.

- Instanton effects are suppressed in a simple large-N limit (ω fixed).
  But, this is not the case in the double scaling limit!

- The Airy function expression contains all perturbative contributions around the one-instanton configuration.

6.3 Two-instanton contribution

The 2nd order approximation ($\Leftrightarrow$ two-instanton effect):

$$P_n(x) = P_n^{(H)}(x) + \tilde{P}_n(x)$$

and linearize with respect to $\tilde{P}_n(x)$.

Leading order of two-instanton contribution is computed as

$$\left\langle \frac{1}{N}\mathrm{tr}\,(\phi^2 - \mu^2) \right\rangle^{(1,0)}\Bigg|_{2-\mathrm{inst.}} = N^{-4/3}\,\frac{1}{(64\pi)^2\,t^{5/2}}\,e^{-\frac{64}{3}\,t^{3/2}}\left[1 + \mathcal{O}(t^{-3/2})\right].$$

- The exponent $\frac{64}{3}\,t^{3/2}$ is consistent with the picture of two instantons.
  $\frac{1}{(64\pi)^2\,t^{5/2}}$: 1-loop fluctuation

- The one- and two-instanton effects are of the same order $\mathcal{O}(N^{-4/3})$.

- After wave function renormalization by the factor $N^{4/3}$, the SUSY breaking order parameter is nonzero due to instantons.

  $\Rightarrow$ SUSY is dynamically broken by the instanton effect.

Remarks:

- The free energy $F_{(1,0)}$ has no perturbative contribution, but we find

$$F_{(1,0)} = F_{(1,0)}\big|_{1-\text{inst.}} + F_{(1,0)}\big|_{2-\text{inst.}} + \cdots ,$$

where

$$\begin{aligned} F_{(1,0)}\big|_{1-\text{inst.}} &= \frac{1}{3}\left[32t^2\,\mathrm{Ai}(4t)^2 - \mathrm{Ai}(4t)\,\mathrm{Ai}'(4t) - 8t\,\mathrm{Ai}'(4t)^2\right] \\ &= \frac{1}{128\pi\, t^{3/2}}\, e^{-\frac{32}{3}t^{3/2}} \left[1 + \sum_{n=1}^{\infty} b_n^{(1)}\, t^{-\frac{3}{2}n}\right], \end{aligned}$$

with $b_1^{(1)} = -\frac{35}{192}$, $b_2^{(1)} = \frac{3745}{73728}$, $b_3^{(1)} = -\frac{805805}{42467328}, \cdots$.

$$\boldsymbol{F_{(1,0)}\Big|_{2-\mathrm{inst.}} = \frac{1}{2}\frac{1}{(128\pi)^2\,t^3}\,e^{-\frac{64}{3}\,t^{3/2}}\,\Big[1+\mathcal{O}(t^{-3/2})\Big]}$$

$\boldsymbol{F_{(1,0)}\Big|_{2-\mathrm{inst.}}}$ solely comes from interactions between instantons. Dilute gas approximation is not used.

- If matrix-model instantons correspond to D-brane like objects in the type IIA superstring, condensate of such D-branes seems to generate the nonperturbative vacuum.
  A linear combination of various condensates of the D-branes

$$F_{(1,0)} = F_{(1,0)}\big|_{1-\text{inst.}} + F_{(1,0)}\big|_{2-\text{inst.}} + \cdots ,$$

- Asymptotic expansion of $\mathrm{Ai}(s)$ for s large has a convergence radius zero, but it is Borel summable.
  $\Rightarrow$ Is $F_{(1,0)}\big|_{1-\text{inst.}}$ also?

  If so and higher instanton sectors has similar behavior, the nonperturbative vacuum would be relatively stable compared with usual string vacua (not Borel summable).

6.4 Numerical result for full nonperturbative effects

By using Mathematica, we can obtain $P_n(-\mu^2)$ up to a quite large n from the recursion relations.

$\Rightarrow$ One-point function $\left\langle \frac{1}{N} \mathrm{tr}\, (\phi^2 - \mu^2) \right\rangle^{(1,0)}$ is evaluated from $N = 1$ to $N = 1,000,000$.

$\Rightarrow$ Extrapolate the results to $N = \infty$. $\longrightarrow$ Figs. 2, 3

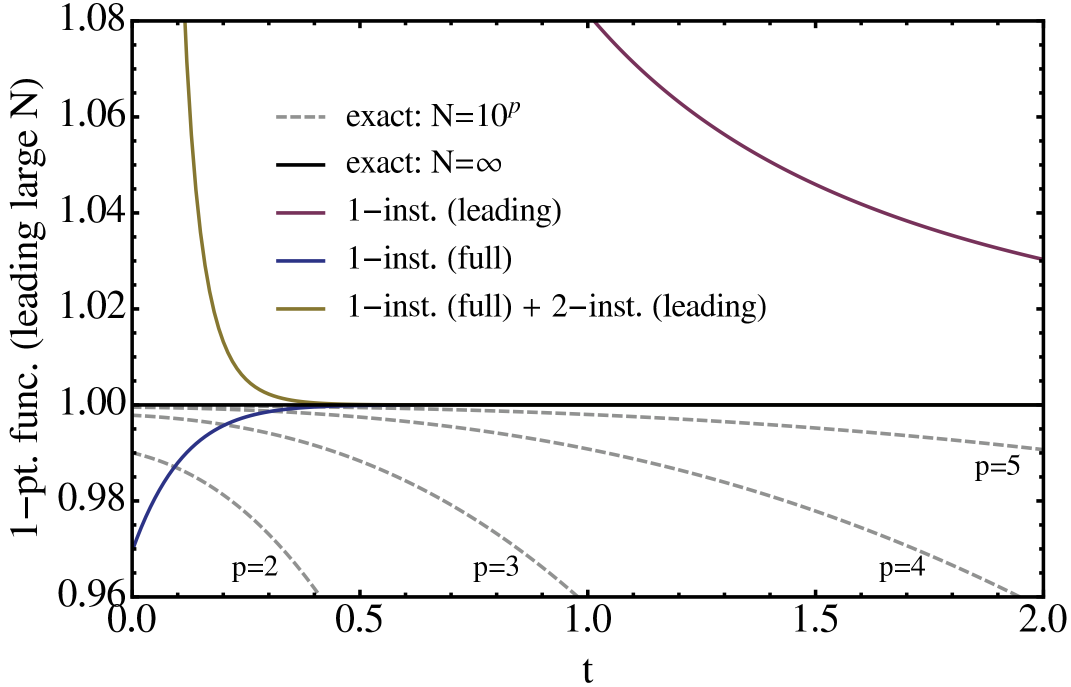

Figure 2: $\left\langle \frac{1}{N}\mathbf{tr}\,(\phi^2 - \mu^2)\right\rangle$ as a function of t. Everything is normalized by the $N = \infty$ result (Exact ($N = \infty$)), and thus the black solid line representing itself is flat. The gray dashed lines show the results for $N = 10^p$ ($p = 2, 3, 4, 5$). The red line (1-inst. (leading)) and the blue line (1-inst. (full)) show the behavior of the leading one-instanton contribution and the full one-instanton contribution, respectively. Finally, the yellow line (1-inst. (full) + 2-inst. (leading)) represents the sum of the full one-instanton result and the leading two-instanton result. The error associated to the extrapolation to $N = \infty$ is invisible.

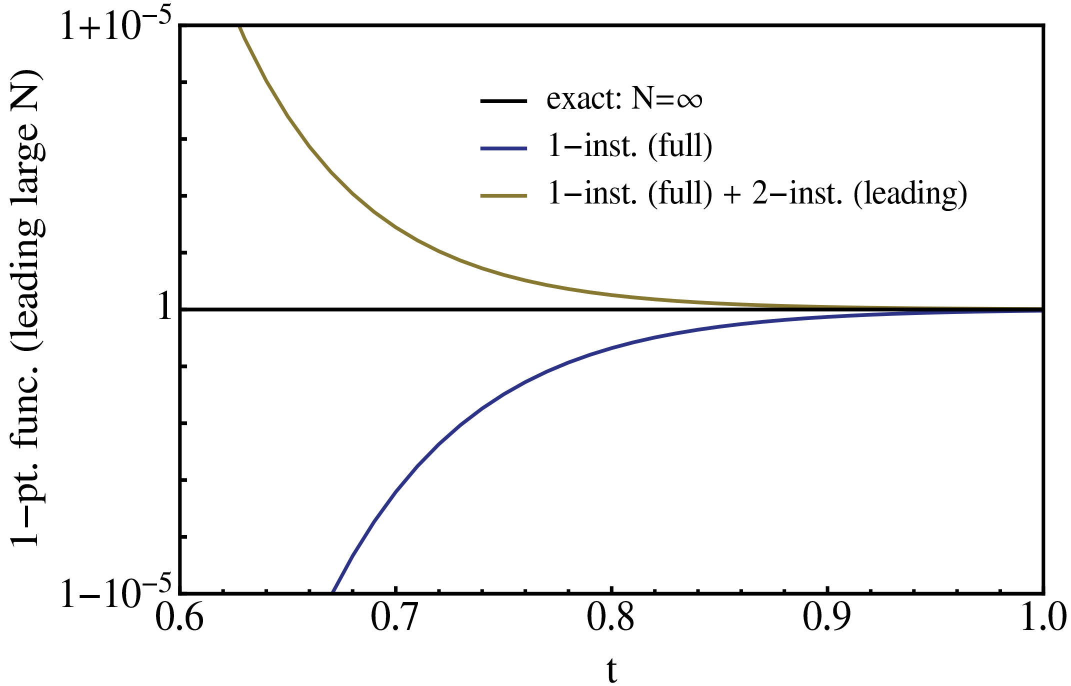

Figure 3: A magnified view of Fig. 2 around 1.00 in the vertical axis. Finite N results lie outside the plot range.

Finally we present full nonperturbative contribution to the free energy $F_{(1,0)} = -\ln Z_{(1,0)}$ by numerically integrating the $N = \infty$ result of the one-point function.

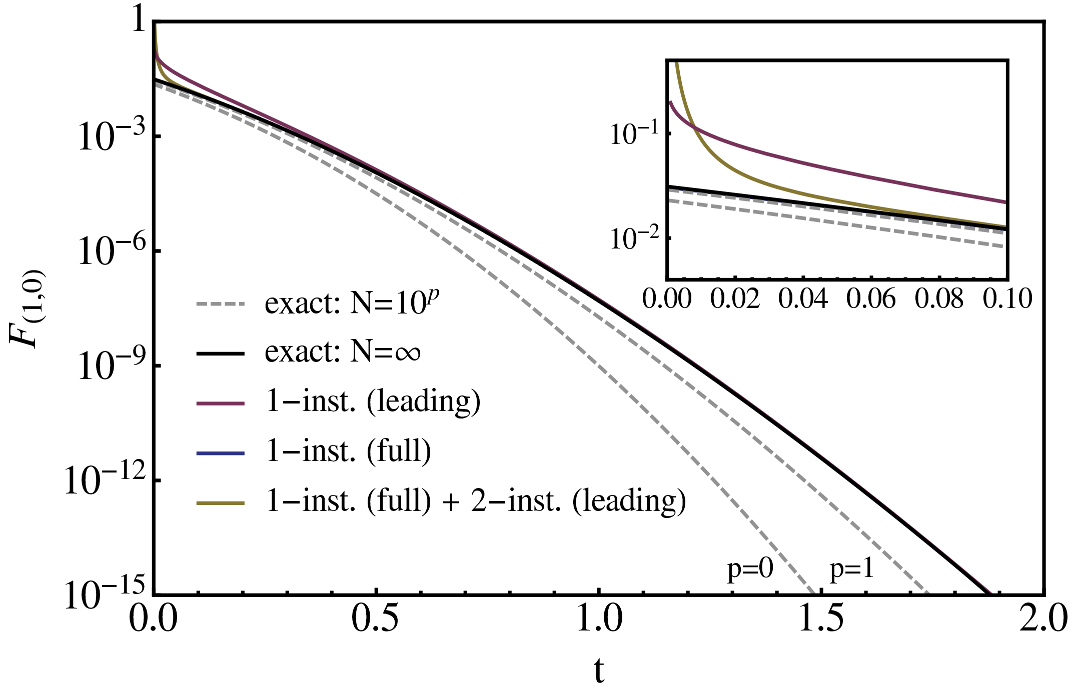

Figure 4: Full nonperturbative contribution to the free energy $F_{(1,0)}$ as a function of t. The black solid line (exact: $N = \infty$) represents the result. For comparison, finite N results are shown by the gray dashed lines (exact: $N = 10^p$). Also, the leading and full one-instanton contributions to $F_{(1,0)}$ are depicted by the red and blue lines, respectively. The yellow line represents the sum of the full one-instanton result and the leading two-instanton result.

- The free energy is a finite function of t even at the origin that corresponds to the strongly coupled limit of the type IIA superstring.
  $\Rightarrow$ S-dual theory (noncritical M-theory)?

- From the viewpoint of the perturbation theory, the free energy will be formally expressed as a double series with respect to $t^{-3/2}$ and $e^{-\frac{32}{3}t^{3/2}}$ (so-called trans-series [e.g. Schiappa, Mariño, Dunne, Ünsal,...]):

$$F_{(1,0)} = \sum_{k=1}^{\infty} e^{-\frac{32k}{3}t^{3/2}} \sum_{n=k}^{\infty} f_n^{(k)}\, t^{-\frac{3}{2}n}.$$

  In matrix models describing bosonic strings, it is extremely nontrivial to sum up the double series and obtain a well-defined result.

  However, in our matrix model for the IIA superstring, Fig. 4 indicates the well-defined result to be obtained after we manage the summation!

◇ It is interesting to obtain an analytic expression for the full nonperturbative contribution (by using trans-series and resurgent analysis).

◇ D-brane computation in the type IIA side.

Thank you very much for your attention!

A The Penner model [Distler-Vafa 1991]

- Partition function

$$Z = \mathcal{N}_P \int d^{N^2}M \exp[N t \operatorname{tr}\{M + \ln(1-M)\}]$$

$$= \mathcal{N}_P \int d^{N^2}M \exp\left[-N t \operatorname{tr} \sum_{k=2}^{\infty} \frac{1}{k} M^k\right],$$

where $\frac{1}{\mathcal{N}_P} = \int d^{N^2}M \exp\left[-N t \operatorname{tr} \frac{1}{2} M^2\right]$.

- Free energy

$$\ln Z = \sum_{g=0}^{\infty} N^{2-2g} \mathcal{F}_g,$$

$$\mathcal{F}_g = \frac{B_{2g}}{2g(2g-2)} t^{2-2g} \left((1+t)^{2-2g} - 1\right) \qquad \text{for} \qquad g \geq 2$$

$\Rightarrow$ Double scaling limit: $N \to \infty$, $t \to -1$ with $N(1+t) = -\nu$ fixed.

After putting $\nu = -i\mu$, the free energy of $c = 1, R = 1$ string is obtained.

$$\mathcal{F}_g = \frac{|B_{2g}|}{2g(2g-2)}\,\mu^{2-2g} \qquad (g \geq 2)$$

$$|B_{2g}| = (-1)^{g-1} B_{2g}$$

B The Kontsevich-Penner model (W_∞ matrix model)

Extension of the Penner model to include source terms for "tachyon" operators in 2D string (with $\nu \to -\nu$). [Imbimbo-Mukhi 1995]

- Partition function (solution of the W_∞ constraint):

$$Z(t,\bar{t}) = (\det A)^{\nu} \int d^{N^2}M \, \exp\left[\mathrm{tr}\left\{-\nu M A + (\nu - N)\ln M - \nu \sum_{k=1}^{\infty} \bar{t}_k M^k\right\}\right]$$

$$= \int d^{N^2}M \, \exp\left[\mathrm{tr}\left\{-\nu M + (\nu - N)\ln M - \nu \sum_{k=1}^{\infty} \bar{t}_k (M A^{-1})^k\right\}\right].$$

- $\bar{t}_k$ is a source for "tachyons" of negative momentum $-k$ ($\sim \mathrm{tr}\, M^k$).

- A: external $N \times N$ matrix

Source for positive-momentum "tachyons" t_k is given by the Kontsevich-Miwa transformation of A:

$$t_k = \frac{1}{\nu k} \mathrm{tr}\, A^{-k}.$$

$\Rightarrow$ Asymmetric treatment for positive/negative-momentum "tachyons"

- "Tachyon" amplitude

$$\langle \mathcal{T}_{k_1} \cdots \mathcal{T}_{k_n} \mathcal{T}_{-l_1} \cdots \mathcal{T}_{-l_m} \rangle = \frac{\partial}{\partial t_{k_1}} \cdots \frac{\partial}{\partial t_{k_n}} \frac{\partial}{\partial \bar{t}_{l_1}} \cdots \frac{\partial}{\partial \bar{t}_{l_m}} \ln Z(t, \bar{t}) \Big|_{t=\bar{t}=0}$$

C Observation for the correspondence between MM and 2D superstring

◇ Suppose that ψ and $\bar{\psi}$ correspond to target-space fermions in the corresponding superstring theory.

$$\psi \Leftrightarrow \text{(NS,R) sector}, \qquad \bar{\psi} \Leftrightarrow \text{(R,NS) sector}.$$

Then,

$$\begin{aligned} (-1)^{\mathrm{F}_L}: &\quad \psi \to \psi, \qquad \bar{\psi} \to -\bar{\psi}, \\ (-1)^{\mathrm{F}_R}: &\quad \psi \to -\psi, \qquad \bar{\psi} \to \bar{\psi}. \end{aligned}$$

In order for the matrix model action to be invariant under $(-1)^{\mathrm{F}_L}$ and $(-1)^{\mathrm{F}_R}$,

$$\begin{aligned} (-1)^{\mathrm{F}_L}: &\quad B \to B, \qquad \phi \to -\phi, \\ (-1)^{\mathrm{F}_R}: &\quad B \to B, \qquad \phi \to -\phi. \end{aligned}$$

Recall $S_{\mathrm{MM}} = N\,\mathrm{tr}\,\left[\frac{1}{2}B^2 + iB(\phi^2 - \mu^2) + \bar{\psi}\{\phi, \psi\}\right]$.

This means

$$B \Leftrightarrow \text{(NS,NS) sector}, \qquad \phi \Leftrightarrow \text{(R,R) sector}.$$