EXPANDING UNIVERSE FROM THE LORENTZIAN MATRIX MODEL

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QUCS2012 @ NARA

BASED ON 1108. 1540 (PRL 108 011601,2012)
1110.4803 (PRD86 027901,2012)
1208.0711 (JHEP 1210 147,2012)

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Main spirit of lattice is

1st principle calculation with numerical method	→	Nonperturbative aspects of QCD

- It is now very well estabilshed thanks to the efforts of early pioneers like Wilson.

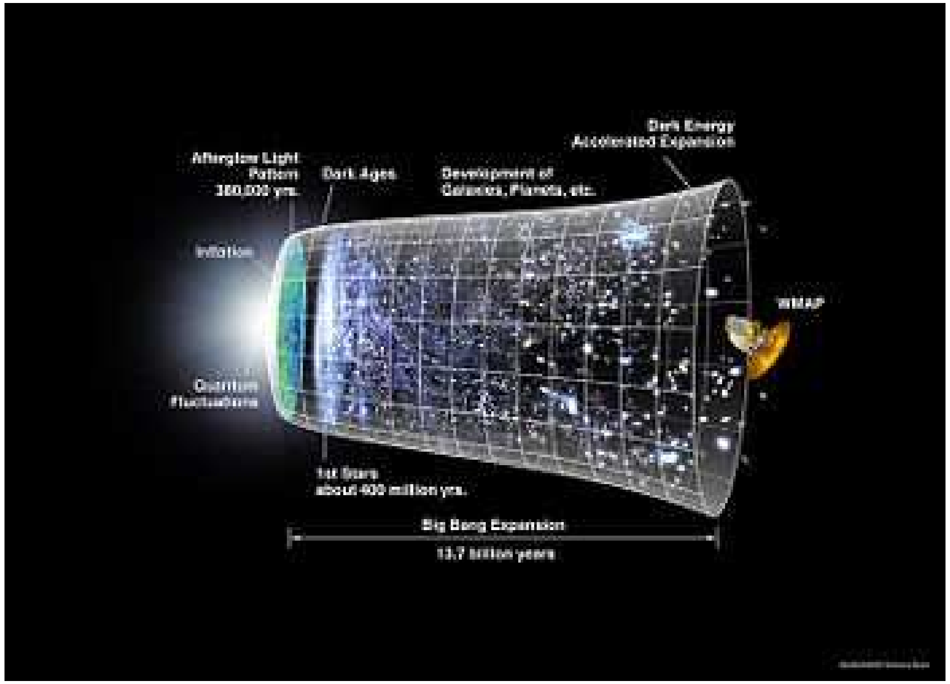
Topic of this talk aims

1st principle calculation with numerical method	→	Nonperturbative aspects, low energy implication of string theory

- Not established yet. Rely on matrix model conjectures.
- This topic is an exploration for future directions which is waiting for more courageous pioneers.

Our first target is cosmology,

where there are many interesting questions



- Initial singularity problem
- Spacetime dimensionality
- Inflaton, reheating, …
- CMB spectrum
- Dark energy/matter , etc

along with related experiments/observations.

. . .

Many analytic works/insights so far

- Quantum cosmology based on Wheeler-DeWitt eq,
  Vilenkin ('82), Hartle, Hawking ('83), …
- String gas cosmology,
  Brandenberger, Vafa ('89), …
- D-brane + Perturbative analysis, etc.
  Herdeiro, Hirano, Kallosh ('01), …

What we obtained in our study is

Monte Carlo study in Lorentzian signature.

Spontaneous breaking of rotational symmetry : SO(9)

Expanding 3d spaces emerge after SSB.

Expansion is consistent with exponential behavior.

Matrix Model (review)

Matrix Model

- Our starting point is type IIB matrix model in 0d.

$$Z = \int dA d\Psi e^{-S_b - S_f}$$

$$S_b = -\frac{1}{4g^2}\mathrm{tr}[A_\mu, A_\nu]^2$$

$$S_f = -\frac{1}{2g^2}\mathrm{tr}\left(\bar{\Psi}\Gamma^\mu[A_\mu, \Psi]\right)$$

Ishibashi, Kawai, Kitazawa, Tsuchiya ('96)

$N \times N$ hermitian matrices :
A_μ (Lorentz vector)
Ψ (10d Majorana-Weyl spinor)

A nonperturbative formulation conjectured for superstring theory.

- cf. 1d Matrix Quantum Mechanics, 2d Matrix String Theory

Banks, Fischler, Shenker, Susskind ('96) Dijkraaf, Verlinde, Verlinde ('97)

Symmetries

$\mathcal{N}=2$ SUSY $\left\{\begin{array}{l}\delta^{(1)}A_\mu = i\bar{\epsilon}\Gamma_\mu\Psi \\ \delta^{(1)}\Psi = \frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\epsilon\end{array}\right.$ $\left\{\begin{array}{l}\delta^{(2)}A_\mu = 0 \\ \delta^{(2)}\Psi = \xi\mathbf{1}\end{array}\right.$

Gauge symmetry

10d Lorentz symmetry

Bosonic shift symmetry $\left\{\begin{array}{l}\delta A_\mu = c_\mu\mathbf{1} \\ \delta\Psi = 0\end{array}\right.$

$$\begin{array}{l}\tilde{Q}^{(1)} = Q^{(1)} + Q^{(2)} \\ \tilde{Q}^{(2)} = i(Q^{(1)} - Q^{(2)})\end{array} \Rightarrow [\bar{\epsilon}\tilde{Q}^{(i)}, \bar{\xi}\tilde{Q}^{(j)}] = \delta^{(ij)}(\text{shift sym.})$$

If eigenvalues of bosonic matrix = spacetime coordinate,

$\mathcal{N}=2$ matrix SUSY Bosonic shift symmetry	$\Rightarrow$	10d $\mathcal{N}=2$ spacetime SUSY Translational symmetry

Relation to string theory

- IIB matrix model can be considered as a worldsheet regularization of Green-Schwarz action in Schild gauge.

$$S_{\text{string}} = \int d^2\sigma \sqrt{g} \left(\frac{1}{4}\{A_\mu, A_\nu\}^2 + \frac{1}{2}\bar{\Psi}\Gamma^\mu\{A_\mu, \Psi\} \right)$$

$$\{f, g\} \equiv \frac{\epsilon_{ab}}{\sqrt{g}}\partial_a f \partial_b g \quad \to \quad i[f, g]$$

$$\int d^2\sigma \sqrt{g} \quad \to \quad \mathrm{tr}$$

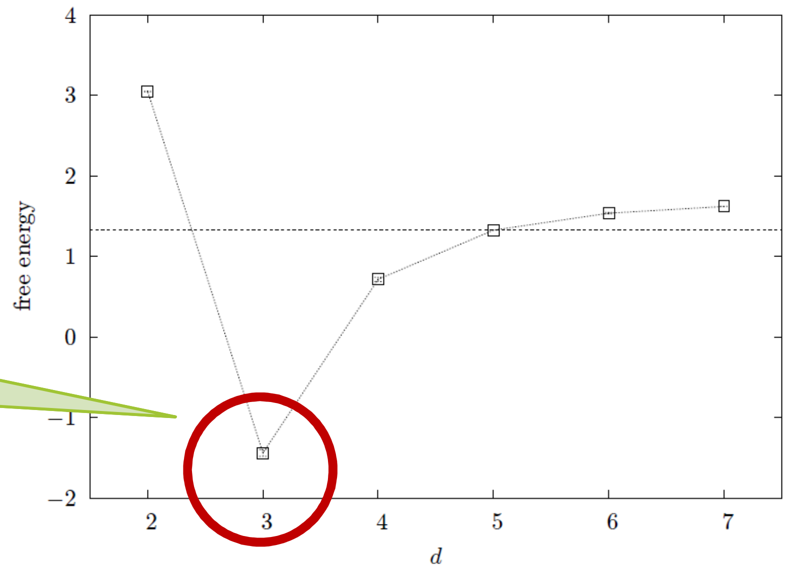
$$S_{\text{matrix model}} = -\frac{1}{4g^2}\mathrm{tr}[A_\mu, A_\nu]^2 - \frac{1}{2g^2}\mathrm{tr}\left(\bar{\Psi}\Gamma^\mu[A_\mu, \Psi]\right)$$

- Wilson loop correlator is consistent with light-cone string field theory. Fukuma, Kawai, Kitazawa, Tsuchiya (98)

Difficulties and problems

- Interpretation of matrix : coordinate , momentum, or else ?
- The action look like around 10d flat spacetime.
  Can we describe curved geometry such as AdS ?
- Double scaling limit is not clear.
  When we take large-N limit, how should we take coupling constant as a function of N ?
- Emergence of 3d spacetime in Euclidean action.

d=3 has minimum free energy



Nishimura, Okubo, Sugino (11)

Lorentzian IIB Matrix Model

Lorentzian IIB Matrix Model

$$S = -\frac{1}{4g^2}\mathrm{tr}[A_\mu, A_\nu]^2 - \frac{1}{2g^2}\mathrm{tr}\left(\bar{\Psi}\Gamma^\mu[A_\mu, \Psi]\right)$$

A_μ , Ψ_α : $N \times N$ Hermitian matrices

- Let's avoid Wick rotation to study real time evolution.
- Pfaffian is real.
- Noncompact temporal direction requires an IR cutoff.

$$\frac{1}{N}\mathrm{tr}(A_0)^2 \leq \kappa \frac{1}{N}\mathrm{tr}(A_i)^2$$

- We want to study SSB of remaining SO(9) symmetry by Monte Carlo method.

Regularization of the model

- We can regularize oscillating phase by

a) introduce damping term, $\lim_{\epsilon\to 0}\int dA\,\mathrm{Pf}(\mathcal{M})\,e^{iS_b-\epsilon|S_b|}$

b) insert identity, $\int_0^\infty dr\,\delta\left(\frac{1}{N}\mathrm{tr}A_i^2 - r\right)$

c) and integrate out scale with $A_\mu \to \sqrt{r}A_\mu$

$$\int dA\,\delta(\ldots)\,\mathrm{Pf}(\mathcal{M}(A))\underbrace{\int dr\,r^{(D+d_F)(N^2-1)/2-1}\,e^{-r^2(\epsilon|S_b|-iS_b)}}_{\propto\,|S_b|^{-(D+d_F)(N^2-1)/4}}$$

d) Need to introduce L : $\int_0^{L^2} dr\,\delta(\ldots)$ $\boxed{\frac{1}{N}\mathrm{tr}(A_i)^2 \le L^2}$

Lorentzian Matrix Model

- We study following model by Monte Carlo method :

$$\boxed{Z = \int d\tilde{A}\, \mathrm{Pf}(\mathcal{M})\, \delta\left(\frac{1}{N}\mathrm{tr} F_{\mu\nu}^2\right)}$$

$$\begin{cases} d\tilde{A} = dA\ \delta\left(\frac{1}{N}\mathrm{tr}(A_i)^2 - 1\right)\ \theta\left(\kappa - \frac{1}{N}\mathrm{tr}(A_0)^2\right) \\ \int_0^{L^2} dr\, r^{(D+d_F)(N^2-1)/2-1}\, e^{-r^2(\epsilon|x|-ix)} \quad \propto \quad \delta(x) \quad \text{in the large } N,\ L. \end{cases}$$

- Note that

Lorentzian

$\mathrm{tr} F^2 = 0 \quad \rightarrow \quad 2\mathrm{tr} F_{0i}^2 = \mathrm{tr} F_{ij}^2$

noncommutative : Lie algebraic, ...

Euclidean

$\mathrm{tr} F^2 = 0 \quad \rightarrow \quad 2\mathrm{tr} F_{0i}^2 = \mathrm{tr} F_{ij}^2 = 0$

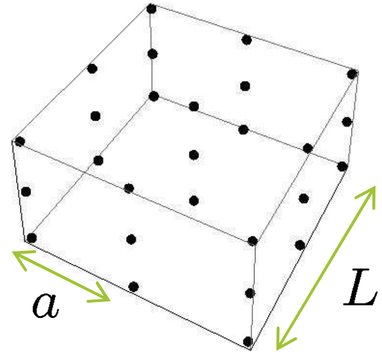
commutative

Comparison with lattice regularization

Lattice

lattice spacing $a \to 0$

volume of the box $L^d \to \infty$

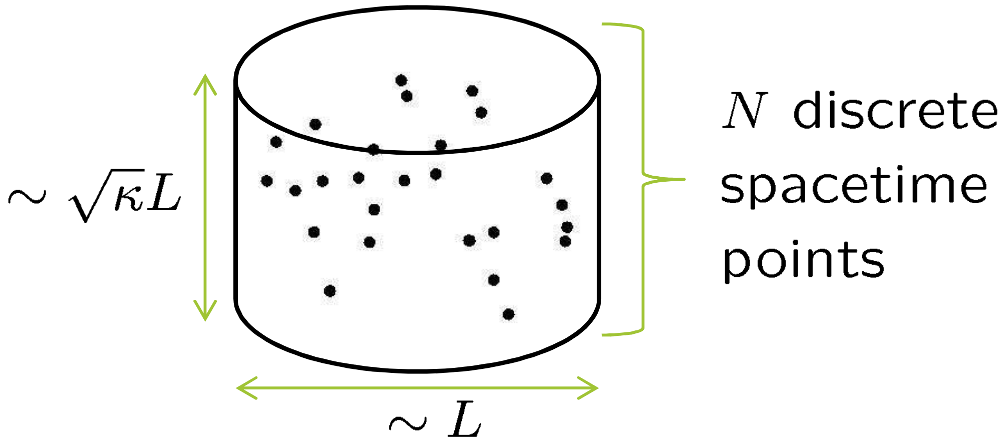


Lorentzian matrix model

matrix size $N \to \infty$

$\frac{1}{N}\mathrm{tr}A_i^2 \leq L^2 \to \infty$

$\frac{1}{N}\mathrm{tr}A_0^2 \leq \kappa L^2 \to \infty$



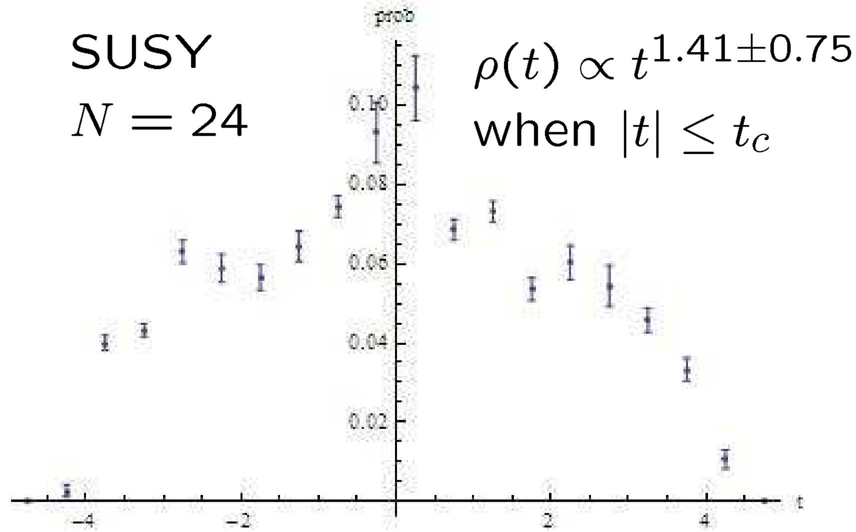
- After continuum limit (N) and infinite volume limit (L), only one parameter (kappa) remains.

Monte Carlo results

Time eigenvalues

- Let $t_1 \le t_2 \le \dots t_N$ be eigenvalues of A_0.

$$A_0 = \begin{pmatrix} \ddots & & 0 \\ & t & \\ 0 & & \ddots \end{pmatrix}$$

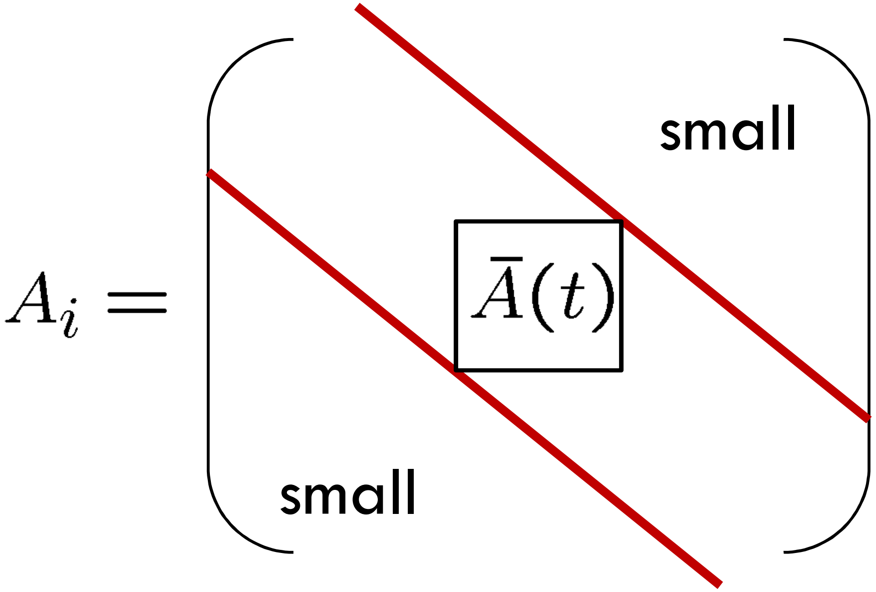


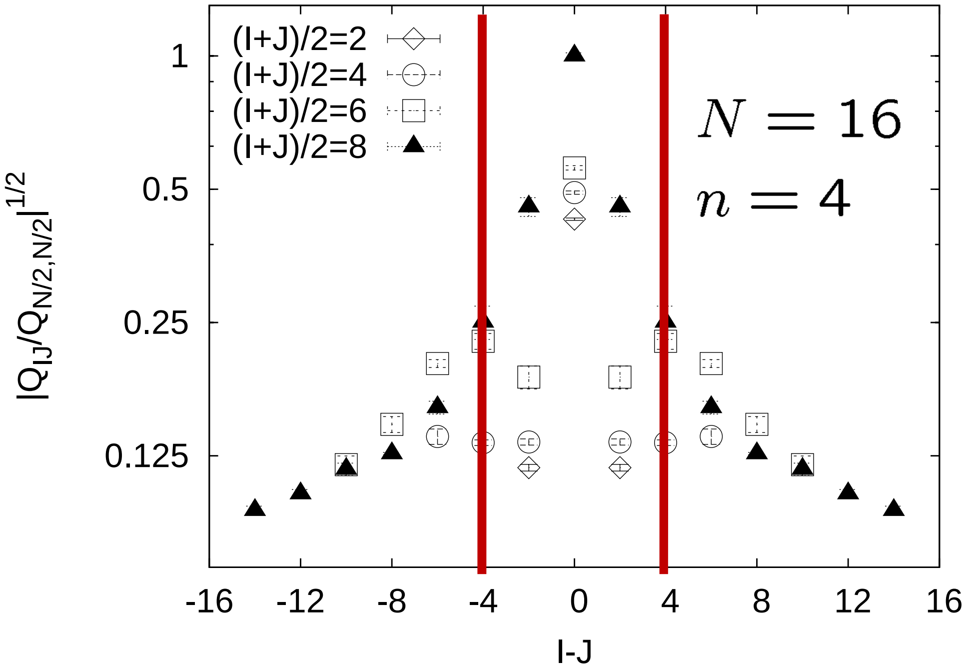
- Thanks to SUSY, their interaction cancels.

$$S_{eff}^{(1-\text{loop})} = (D - 2 - d_F) \sum_{m,n} \log(t_m - t_n)^2 = 0$$

Band diagonal structure

- Band diagonal structure appears dynamically for A_i in A_0's diagonal basis.



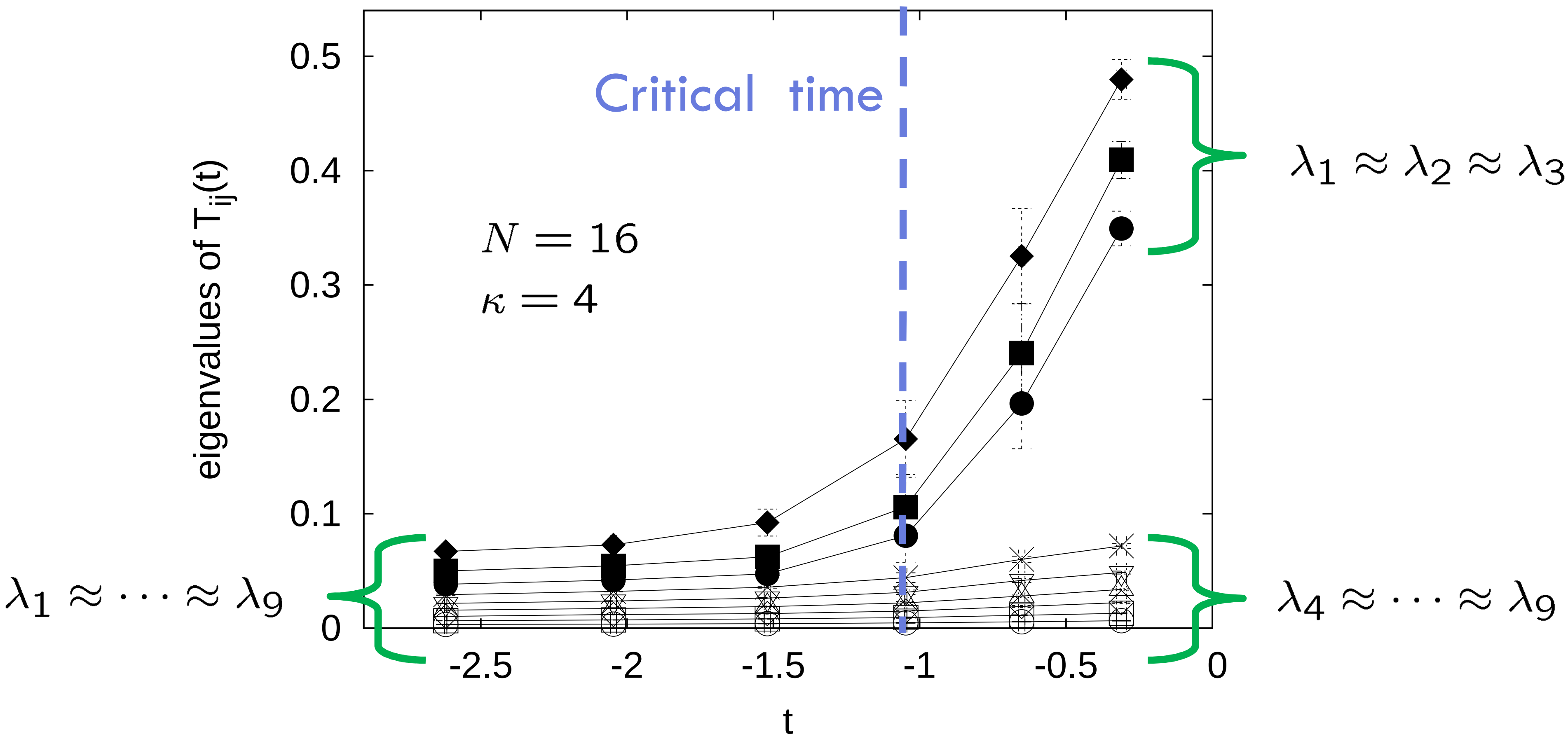


$n \times n$ subblock matrix represents space structure at given time.

$$[\bar{A}_i(t)]_{ab} = \langle t_{\nu+a}|A_i|t_{\nu+b}\rangle \quad , \quad t = \frac{1}{n}\sum_{k=1}^{n} t_{\nu+k}$$
$$(\nu = 1, ..., N-n \ , \quad a, b = 1, ..., n)$$

Results : SSB of SO(9) symmetry

order parameter : $T_{ij}(t) = \frac{1}{n}\mathrm{tr}(\bar{A}_i(t)\bar{A}_j(t))$ 9x9 real sym.



Mechanism of SSB

$$Z = \int d\tilde{A}\ Pf(\mathcal{M})\ f_{N,L}\left(\frac{1}{N}\mathrm{tr}F_{\mu\nu}^2\right)$$

$$\begin{cases} d\tilde{A} = dA\ \delta\left(\frac{1}{N}\mathrm{tr}(A_i)^2 - 1\right)\ \theta\left(\kappa - \frac{1}{N}\mathrm{tr}(A_0)^2\right) \\ \lim_{N,L\to\infty} f_{N,L}(x) = \delta(x) \end{cases}$$

$$\Rightarrow \quad \boxed{\frac{1}{N}\mathrm{tr}(A_0)^2 \leq \kappa\ , \quad \frac{1}{N}\mathrm{tr}(A_i)^2 = 1\ , \quad -2\mathrm{tr}F_{0i}^2 + \mathrm{tr}F_{ij}^2 = 0}$$

- Can we understand SSB in the large kappa ?

$\kappa \Uparrow \qquad \mathrm{tr}A_0^2 \Uparrow \qquad \mathrm{tr}F_{0i}^2 \Uparrow \qquad \mathrm{tr}F_{ij}^2 \Uparrow$

$\Rightarrow$ **Maximize** $\mathrm{tr}F_{ij}^2$ **with** $\frac{1}{N}\mathrm{tr}(A_i)^2 = 1$

Mechanism of SSB

- Large kappa for fixed N is described by

$$L = -\frac{1}{4N}\mathrm{tr}(F_{ij})^2 + \frac{\lambda}{2}\left(\frac{1}{N}\mathrm{tr}(A_i)^2 - 1\right)$$

- EOM is $[A_j,[A_j,A_i]] = \lambda A_i$

- With an Ansatz

$$A_i = \chi L_i \qquad \text{for } i = 1, ..., d$$

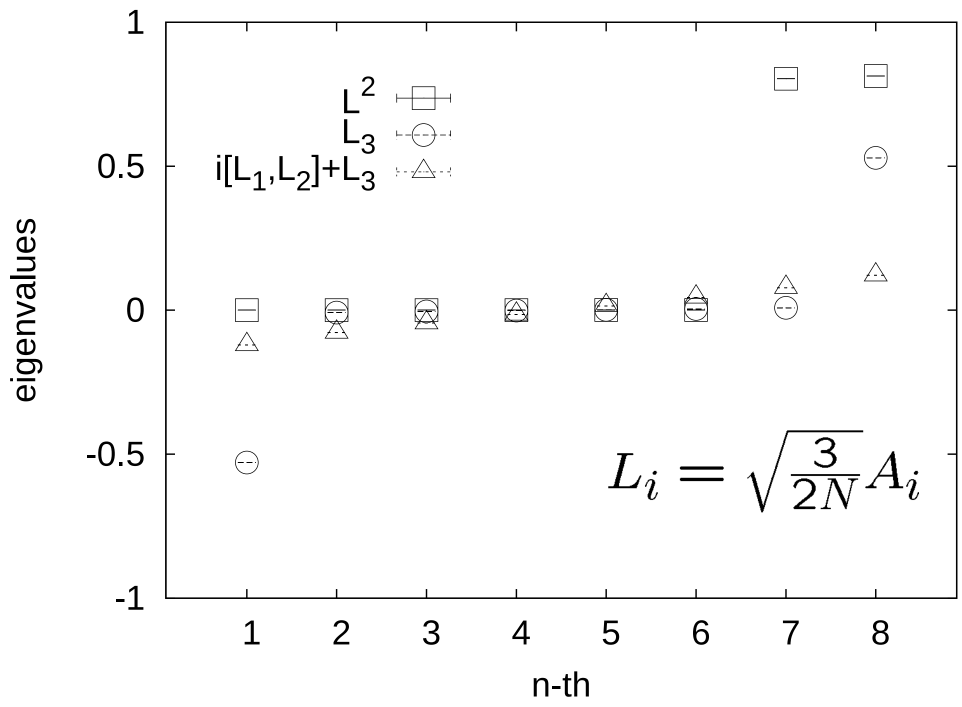
$$A_i = 0 \qquad \text{for } i = d+1, ...9$$

$$[L_i, L_j] = i f_{ijk} L_k \qquad \text{compact, semisimple}$$

$$\begin{cases} \dfrac{\chi^2}{N}\mathrm{tr}(L_i)^2 = 1 \quad \rightarrow \quad \chi = \sqrt{\dfrac{N}{\mathrm{tr}(L_i)^2}} \\ \mathrm{tr}F_{ij}^2 = \chi^4 \mathrm{tr}(f_{ijk}L_k)^2 \propto \dfrac{1}{\mathrm{tr}(L_i)^2} \leq \dfrac{2}{3} \end{cases} \quad \Rightarrow \quad su(2)$$

Mechanism of SSB

- We can solve this problem without any Ansatz by Monte Carlo method.

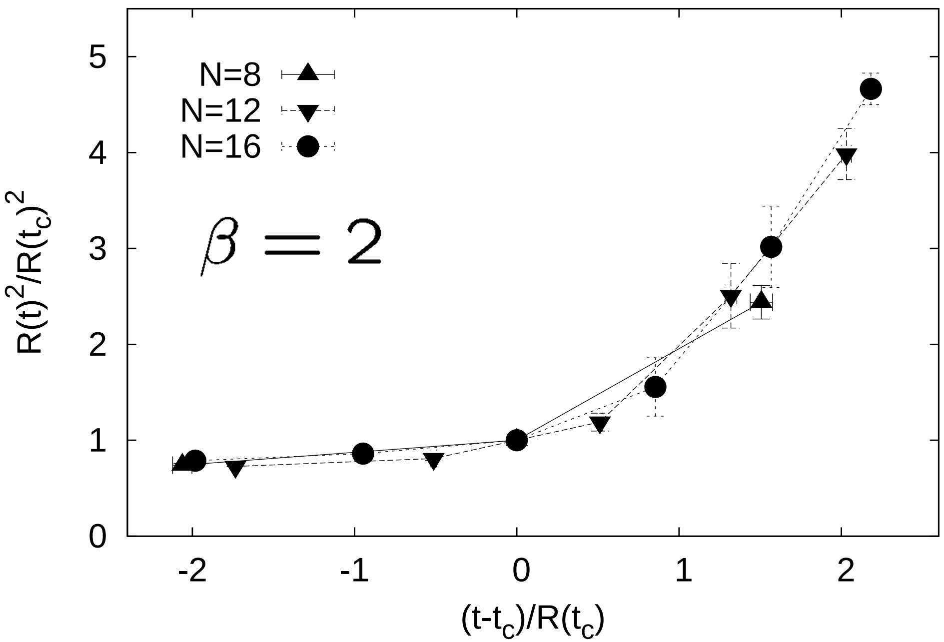


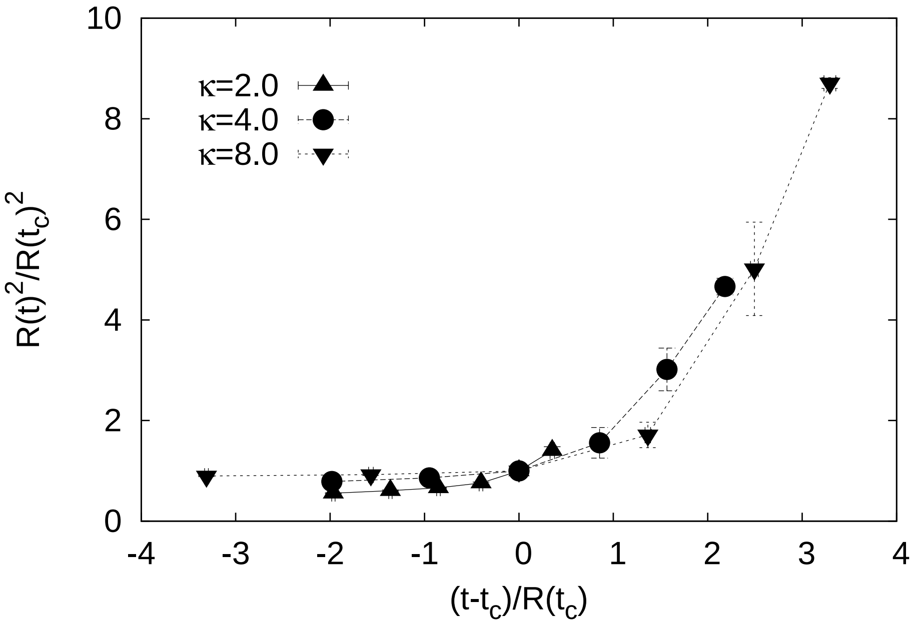
- 2x2 representation of SU(2) algebra gives the maximum, which explains 3d expanding spaces.

Continuum / Infinite volume limit

$N \to \infty$ with $\kappa = \beta N^p$, $p \sim 1/4$

$L, \ \beta \to \infty$





Different parameter values of N & kappa seem to converge to a single curve.

Summary and Discussion

Summary

0-d matrix model with SO(9,1)

$$S = -\frac{1}{4g^2}\mathrm{tr}[A_\mu, A_\nu]^2 + S_f$$

+

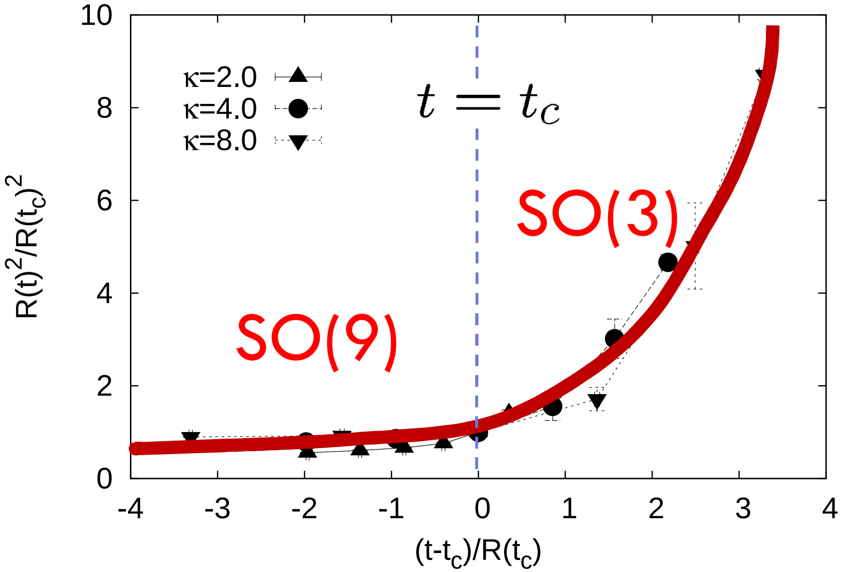
Two IR cutoffs

$$\frac{1}{N}\mathrm{tr}A_0^2 \leq \kappa L^2 \ , \qquad \frac{1}{N}\mathrm{tr}A_i^2 \leq L^2$$

$$N \ , \ \kappa \ , \ L \quad \to \quad \infty$$

- Unique time history
- SSB from 9d to 3d spaces
- Noncommutative mechanism

Early Universe

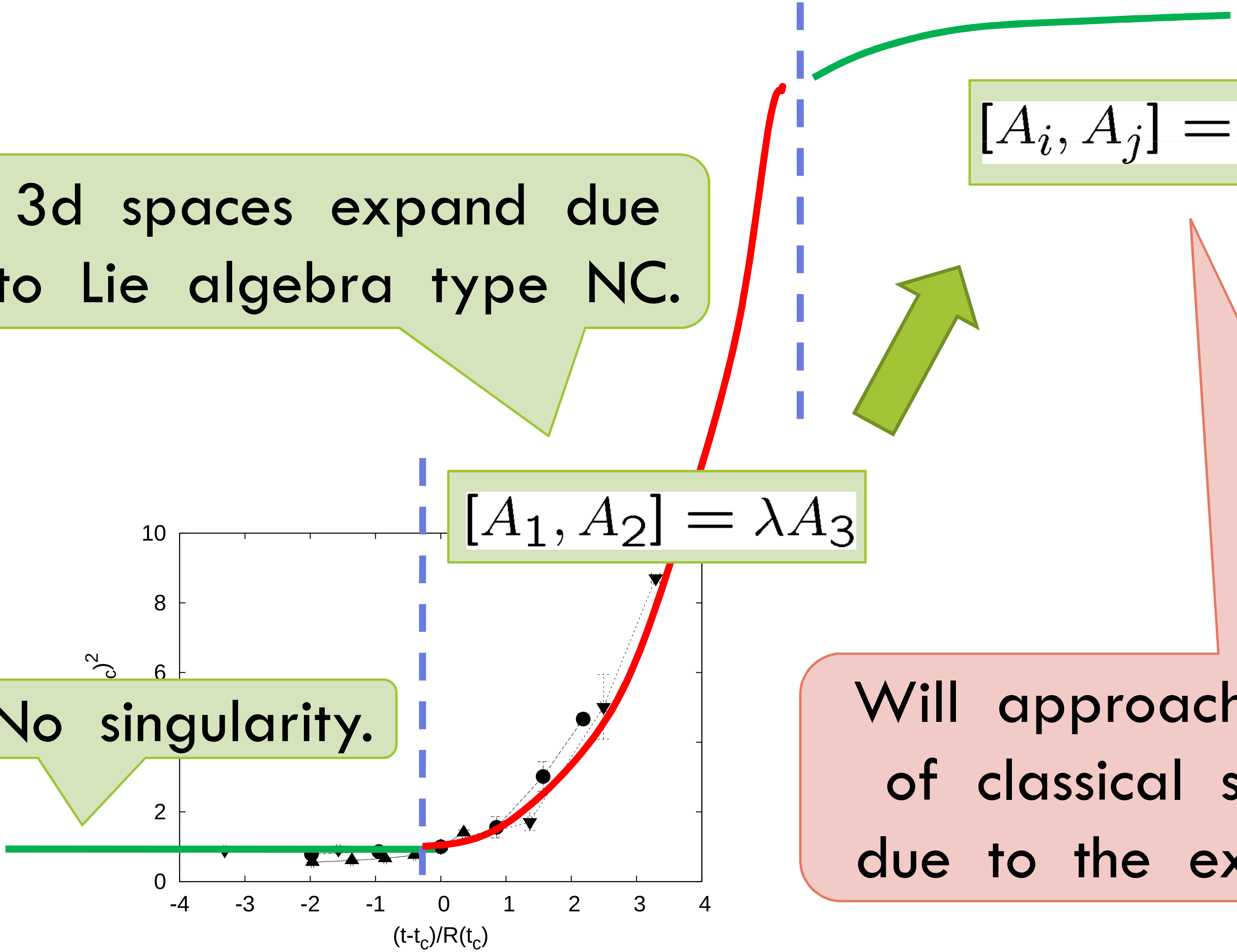


What happens in late time ?

3d spaces expand due to Lie algebra type NC.

$$[A_1, A_2] = \lambda A_3$$

No singularity.



$$[A_i, A_j] = 0 \ ?$$

Will approach to one of classical solutions due to the expansion.

Candidates for future

Classical EOM

$$-[A_0,[A_0,A_i]]+[A_j,[A_j,A_i]]=\lambda A_i$$
$$-[A_j,[A_j,A_0]]=\tilde{\lambda}A_0$$

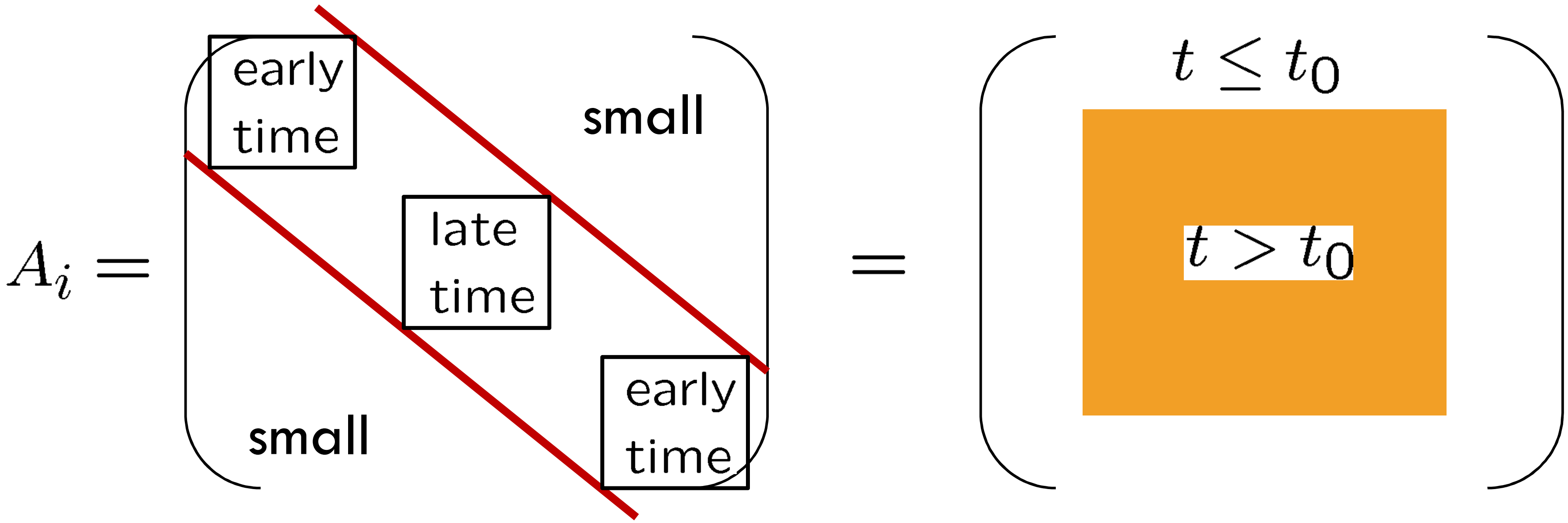
+

Ansatz based on Lie algebra

- We classified all possible solutions within the Ansatz. There are many expanding commutative classical solutions.
- Even with quite simple Ansatz, we can easily find classical solutions with power law or exponential behavior for expansion.
- With these candidates for future, we are now ready to proceed Monte Carlo study.

Effective model

- Effective model for $t > t_0$ will appear if we integrate over matrix elements corresponding to $t \leq t_0$.



- By changing t_0, we can obtain a RG flow to late time.

Quenched model

Ito, SWK, Koizuka, Nishimura, Tsuchiya
(a work in progress)

- Simple quenching is not enough : $S_{eff} = (D-2)\sum_{m,n}\log(t_m - t_n)^2$
- We include fermionic repulsion to temporal eigenvalues.

$$Z = \int dt dA_i\, \Delta^2\, \mathrm{Pf}(\mathcal{M})\, e^{iS_b}$$

$$\Delta^2 = \prod_{m,n}(t_m - t_n)^2$$

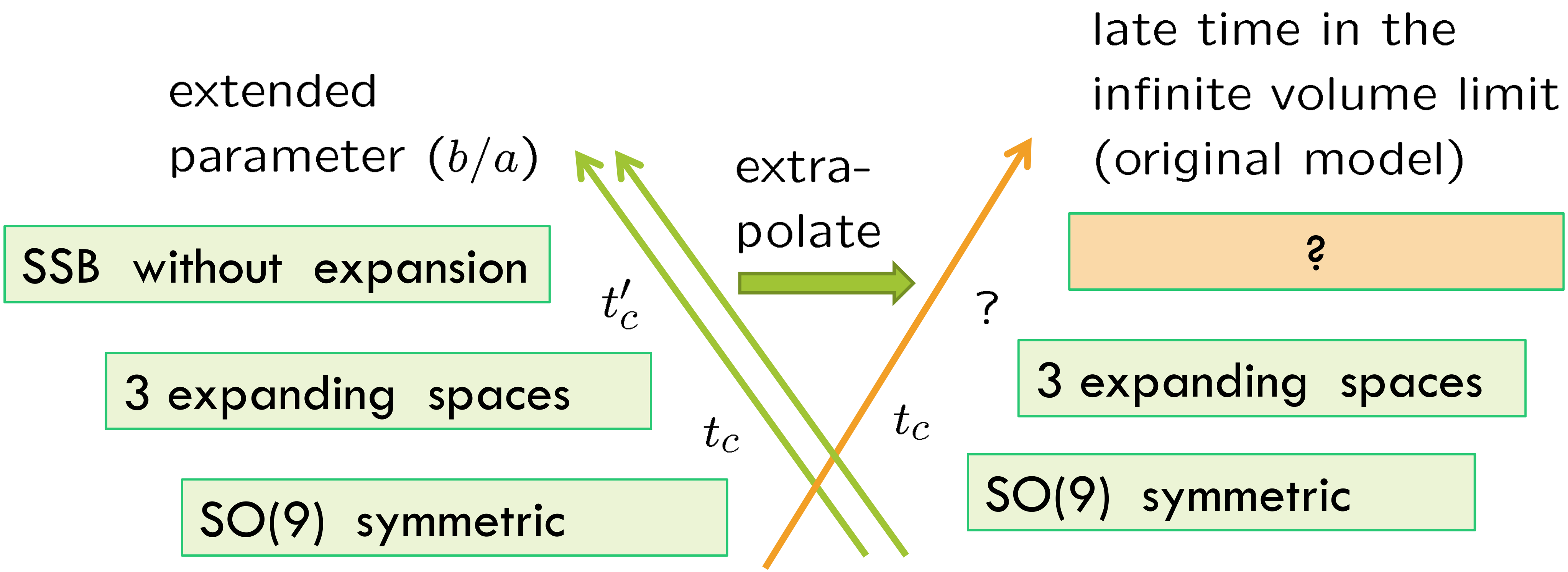
$$\boxed{Z_{VDM} = \int dt dA_i\, \Delta^D\, e^{iS_b}}$$

- Interesting properties such as SSB to 3d, expansion are kept.
- RG flow to late time is carefully studied in this model.
- But in late time, $\mathrm{tr}\left(\bar{\Psi}\Gamma^i[A_i,\Psi]\right)$ will become more important.
- However, we study this model first as a basic reference.

A preliminary idea

- We may consider a parameter extension of the model in a way to mimic the late time flow.

$$\mathrm{tr}F^2 \quad \rightarrow \quad a\,\mathrm{tr}F_{0i}^2 + b\,\mathrm{tr}F_{ij}^2$$



Backup (Classical solution)

General prescription

- variational function $(i = 1, \cdots, 9)$

$$\tilde{S} = \mathrm{tr}\left(-\frac{1}{4}[A_M, A_N][A^M, A^N] + \boxed{\frac{\tilde{\lambda}}{2}(A_0^2 - \kappa L^2)} - \boxed{\frac{\lambda}{2}(A_i^2 - L^2)} \right)$$

- classical equations of motion

$$-[A_0,[A_0,A_i]] + [A_j,[A_j,A_i]] - \lambda A_i = 0$$
$$[A_j,[A_j,A_0]] - \tilde{\lambda} A_0 = 0$$

- commutation relations

$$[A_i, A_j] = iC_{ij}$$
$$[A_i, C_{jk}] = iD_{ijk}$$
$$[A_0, A_i] = iE_i$$
$$[A_0, E_i] = iF_i$$
$$[A_i, E_j] = iG_{ij} \quad \cdots$$

Eq. of motion & Jacobi identity

⇓

Lie algebra

⇓

Unitary representation ⟺ classical solution

Ansatz

Ignore extra dimension $\Longrightarrow$ $A_i = 0 \quad \text{for} \quad i > d$

Assume commutative space $\Longrightarrow$ $[A_i, A_j] = 0$

$$[\boxed{A_i}, A_j] = i\cancel{C_{ij}}$$

$$[A_i, C_{jk}] = i\cancel{D_{ijk}}$$

$$[\boxed{A_0}, A_i] = i\boxed{E_i}$$

$$[A_0, E_i] = iF_i \Longleftarrow -[A_0, [A_0, A_i]] + [A_j, [A_j, A_i]] - \lambda A_i = 0$$

$$\boxed{F_i = \lambda A_i}$$

$$[A_i, E_j] = iG_{ij} \quad \cdots$$

$$G_{ij} = \boxed{M_{ij}} + \cancel{N_{ij}} + \frac{1}{d}\delta_{ij}H \Longleftarrow [A_j, [A_j, A_0]] - \tilde{\lambda} A_0 = 0$$

$$\boxed{H = \tilde{\lambda} A_0}$$

$$[A_0, [A_i, A_j]] + [A_i, [A_j, A_0]] + [A_j, [A_0, A_i]] = 0$$

Simplification

$M_{ij} = 0$ for $i \neq j$

$$M_i \equiv M_{ii} \qquad \sum_{i=1}^{d} M_i = 0$$

Lie algebra

$$[A_i, A_j] = 0 \ , \quad [A_0, A_i] = iE_i \ , \quad [A_0, E_i] = i\lambda A_i \ ,$$
$$[E_i, E_j] = 0 \ , \quad [A_i, E_j] = i\delta_{ij}\left(\frac{\tilde{\lambda}}{d}A_0 + M_i\right) \ , \quad [A_0, M_i] = 0 \ ,$$
$$[A_i, M_j] = i\frac{\tilde{\lambda}}{d}(1 - d\delta_{ij})E_i \ , \quad [E_i, M_j] = i\frac{\lambda\tilde{\lambda}}{d}(1 - d\delta_{ij})A_i \ , \quad [M_i, M_j] = 0$$

e.g.)

$d = 2, \quad \lambda > 0, \quad \tilde{\lambda} > 0$ $\Rightarrow$ $SO(2,2)$

d=1 case

$$[A_0, A_1] = iE \ , \quad [A_0, E] = i\lambda A_1 \ , \quad [A_1, E] = i\tilde{\lambda} A_0$$

Take a direct sum

$$A'_0 = A_0 \otimes 1\!\!1_K$$
$$A'_i = A_1 \otimes \mathrm{diag}(r_i^{(1)}, r_i^{(2)}, \cdots, r_i^{(K)})$$
$$\text{where} \quad r_i^{(m)2} = 1 \ \ (m = 1, \cdots, K)$$

$r^{(m)}$ can be distributed on a unit S^3

⇨ (3+1)D space-time $\sim$ $R \times S^3$

A complete classification of d=1 solutions has been done.
Below we only discuss a physically interesting solution.

SL(2,R) solution

- SL(2,R) solution

$$[A_0, A_1] = iE \ , \quad [A_0, E] = i\lambda A_1 \ , \quad [A_1, E] = i\tilde{\lambda} A_0$$

$$A_0 = aT_2 \ , \quad A_1 = bT_0 \ , \quad E = cT_1$$

$$\lambda = a^2 \ , \quad \tilde{\lambda} = b^2 \ , \quad ab = c$$

$$\boxed{[T_0, T_1] = iT_2 \ , \quad [T_2, T_0] = iT_1 \ , \quad [T_1, T_2] = -iT_0}$$

- realization of the SL(2,R) algebra on $\{e^{in\theta}; n \in Z\}$

$$\mathcal{T}_0 = i\frac{d}{d\theta} + \epsilon$$

$$\mathcal{T}_1 = \frac{i}{2}\left[(\tau + \epsilon)e^{i\theta} + (\tau - \epsilon)e^{-i\theta} - 2\sin\theta\frac{d}{d\theta}\right]$$

$$\mathcal{T}_2 = \frac{1}{2}\left[-(\tau + \epsilon)e^{i\theta} + (\tau - \epsilon)e^{-i\theta} - 2i\cos\theta\frac{d}{d\theta}\right]$$

Space-time structure

- primary unitary series representation

$$(T_0)_{mn} = n\delta_{mn}$$
$$(T_1)_{mn} = -\frac{i}{2}(n - i\rho + \frac{1}{2})\delta_{m,n+1} + \frac{i}{2}(n + i\rho - \frac{1}{2})\delta_{m,n-1}$$
$$(T_2)_{mn} = -\frac{1}{2}(n - i\rho + \frac{1}{2})\delta_{m,n+1} - \frac{1}{2}(n + i\rho - \frac{1}{2})\delta_{m,n-1}$$

tri-diagonal

- 3K×3K diagonal block

Space-time noncommutativity disappears in the continuum limit.

$$\bar{A}_0(n) = a \begin{pmatrix} n-1 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n+1 \end{pmatrix} \otimes 1\!\!1_K$$

$$\bar{A}_1(n) = \frac{ib}{2} \begin{pmatrix} 0 & n + i\rho - \frac{1}{2} & 0 \\ -n + i\rho + \frac{1}{2} & 0 & n + i\rho + \frac{1}{2} \\ 0 & -n + i\rho - \frac{1}{2} & 0 \end{pmatrix} \otimes \mathrm{diag}(r_i^{(1)}, \cdots, r_i^{(K)})$$

Cosmological implications

- the extent of space

$$R(n) \equiv \sqrt{\frac{1}{3K}\mathrm{tr}(\bar{A}_1(n))^2} = \sqrt{\frac{b^2}{3}\left(n^2+\rho^2+\frac{1}{4}\right)}$$

$$\longrightarrow \quad \boxed{R(t) = \sqrt{\frac{\alpha^2}{3}(t^2+t_0^2)}}$$

$$t = na, \quad t_0 = \rho a$$

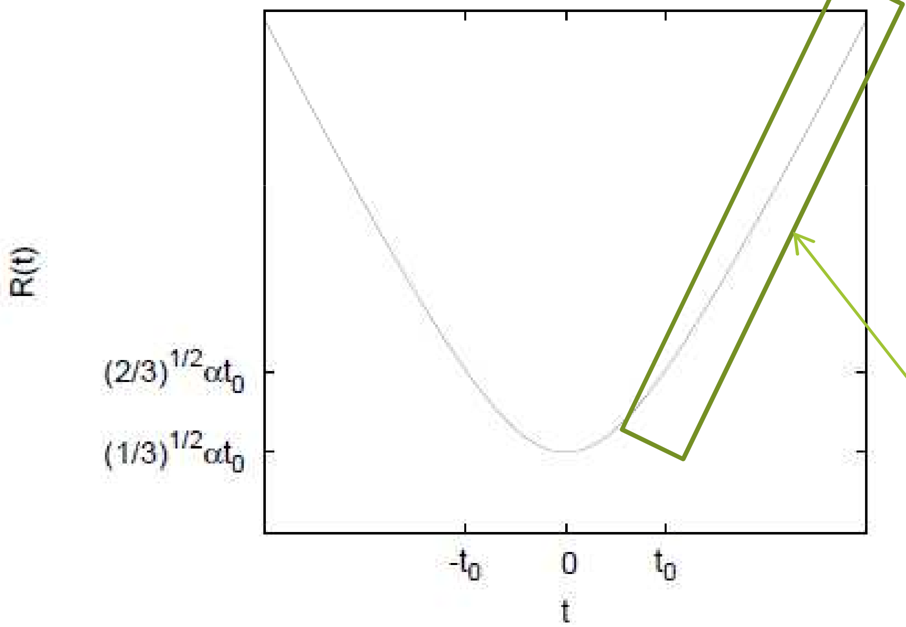
cont. lim. $\quad a \to 0, \quad n \to \infty, \quad \rho \to \infty$

- Hubble constant and the w parameter

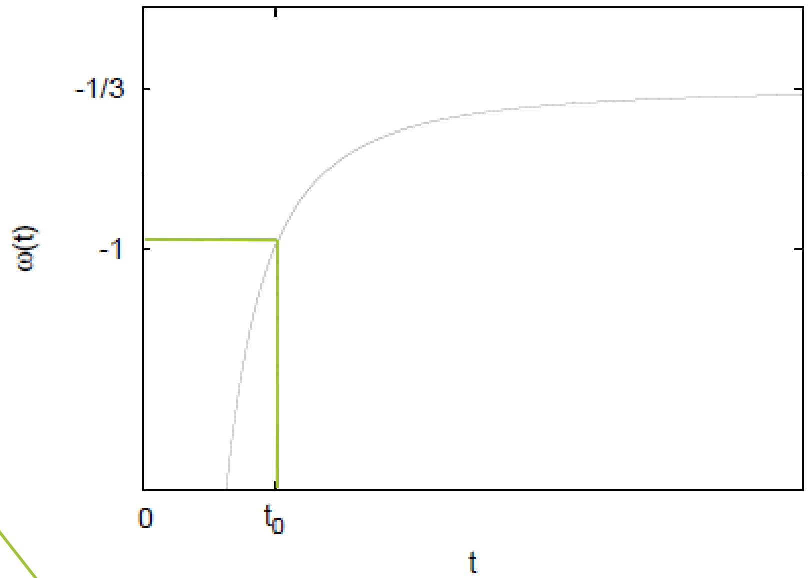
$$H(t) = \frac{\dot{R}(t)}{R(t)} = c\,R(t)^{-\frac{3}{2}(1+w)}$$

$$\begin{cases} w = \frac{1}{3} & \text{radiation dominant} \\ w = 0 & \text{matter dominant} \\ w = -1 & \text{cosmological constant} \end{cases}$$

$$R(t) = \sqrt{\frac{\alpha^2}{3}(t^2 + t_0^2)}$$



$$w = -\frac{2t_0^2}{3t^2} - \frac{1}{3}$$



This part is considered to give the late-time behavior of the matri x model

t_0 is identified with the present time.

$w = -1$ present accelerated ex pansion

cosmological const. $\sim (1/t_0)^4$ a solution to the cosmological constant problem

$w \to -\dfrac{1}{3}$ for $t \to \infty$ Cosmological constant disappears in the future.