Tensor network approach for chiral symmetry restoration of 1-flavor Schwinger model at finite temperature

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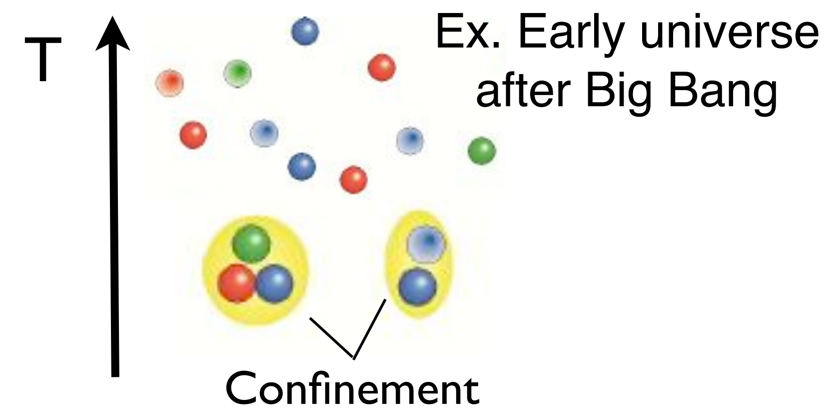
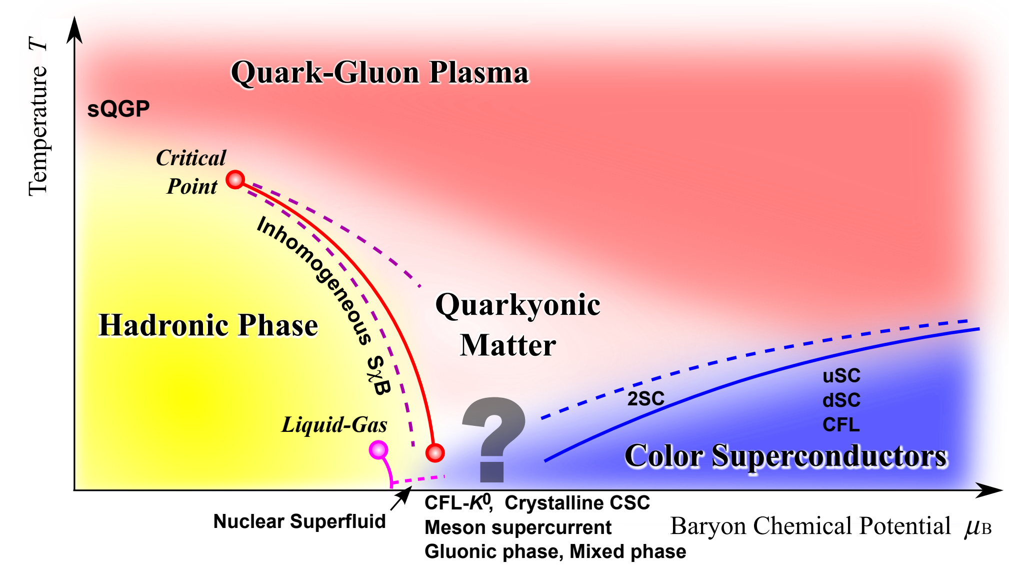
with M. C. Bańuls, K. Cichy, J. I. Cirac and K. Jansen

H. Saito et al. PoS LATTICE2014, 302, 2014, arXiv:1412.0596
M. C. Bańuls et al, arXiv:1505.00279

QCD Phase diagram

- Quark: one of elementary particles
- Confinement at low T (QCD Phase diagram)
  ⇒ non-perturbative aspect
- Lattice QCD simulation
  - ✴ At finite temperature and zero chemical potential : Successful!!
  - ✴ At finite chemical potential μ
    : failing
- But, a lot of interests in dense QCD
  - critical point at finite μ,
  - unknown phases at large μ





Fukushima, Hatsuda, Rept. Prog. Phys. 74, 014001 (2011), arXiv:1005.4814[hep-ph]

Sign problem

- Complex action at finite μ spoils Markov Chain Monte Carlo built on positive probability measure

  Complex quark determinant
- Techniques to avoid the problem in dense QCD on lattice
  - Reweighting: to change the distribution of Monte Carlo ensemble
    $\rightarrow$ overlap problem

    Z. Fodor *et al.* JHEP 0404(2004), 050, S. Ejiri *et al.* PRD82, 014508, K. Nagata *et al.* JHEP1204, 094(2012)
  - Taylor expansion of μ: coefficients $c_n(T)$ calculated with $\mu = 0$ ensemble
    $\rightarrow$ a problem of convergence

    C. R . Allton *et al.* PRD68, 014507
- For real time dynamics of high energy physics, alternative approaches required

  To establish a promising numerical tool
  $\Rightarrow$ our strategy: to employ Hamiltonian approach

Outline

- Motivation
- Schwinger model
  - Hamiltonian of Schwinger model
- Tensor Network
  - Matrix Product State (MPS)
  - Technical aspects of TN:
    Bond dimension, Variational search, Matrix Product Operator
- Chiral condensate of Schwinger model at zero T
- Thermal calculation in Schwinger model
- Summary

Schwinger model for $N_\mathrm{f} = 1$

J. Schwinger Phys.Rev. 128 (1962)

- 1+1 dimensional QED model

N. L. Pak and P. Senjanovic, Phys.Let.B71, 2 (1977), K. Johnson Phys.Let. 5, 4(1963)

  ✳ not QCD, but **similar to QCD** :

  confinement, chiral symmetry breaking (via anomaly for $N_\mathrm{f}=1$)

  ✳ exactly solvable in massless case

  ⇒ a good test case

- Hamiltonian of Schwinger model (staggered discrieization) in spin language

hopping term

mass term

$$H = x\sum_{n=0}^{N-2}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right] + \frac{\mu}{2}\sum_{n=0}^{N-1}\left[1+(-1)^n\sigma_n^z\right]$$

$$+\sum_{n=0}^{N-2}\left[l + \frac{1}{2}\sum_{k=0}^{n}\left((-1)^k + \sigma_k^z\right)\right]^2$$

T. Banks, L. Susskind and J. Kogut, PRD13, 4 (1973)

gauge part

$$= H_\mathrm{hop} + H_\mathrm{mass} + H_\mathrm{g}$$

Gauss law

where inverse coupling $x=1/a^2g^2$, dimensionless mass $\mu=2m/ag^2$ and $l = L(0)$

Tensor network (TN)

- Hamiltonian approach with exact diagonalization available for only small size:

  Ex. In 1D, with chain length $N \sim O(10)$, not enough to take thermodynamic limit

- Tensor Network: An efficient approximation of quantum many-body state from quantum information

- **Matrix product state (MPS)**: tensor network for 1d

$$|\psi\rangle \approx \sum_{i_1,\cdots,i_N} \mathrm{Tr}\left[M[1]^{i_1}\cdots M[N]^{i_N}\right]|i_1\cdots i_N\rangle$$

i_k: physical indices at site k,
m, n (=1,…, D) : indices from this approximation, ***D*** **: bond dimension**

site

physical index

$$M[k]^{i_k}_{mn}$$

virtual indices

An example of MPS

- 1/2-spin 2 particle system: $i_k = \uparrow$ or $\downarrow$ for $k = 1, 2$
- Supposing $D = 2$, e.g. tensors $M[k]^{i_k}_{mn}$

$$M[1]^{i_1=\uparrow} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix},\ M[1]^{i_1=\downarrow} = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix},\ M[2]^{i_2=\uparrow} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ M[2]^{i_2=\downarrow} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- By computing trace of products $\mathrm{Tr}\left[M[1]^{i_1} M[2]^{i_2}\right]$

$$\boxed{\mathrm{Tr}\left[M[1]^{i_1} M[2]^{i_2}\right] = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } (i_1, i_2) = (\uparrow, \downarrow), (\downarrow, \uparrow) \\ 0 & \text{for the others} \end{cases}}$$

$$\sum_{i_1, i_2} \underline{\mathrm{Tr}\left[M^{i_1} M^{i_2}\right]} \left|i_1 i_2\right\rangle = \frac{1}{\sqrt{2}} \left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right)$$

Bond dimension

- An efficient way to express sub-space of Hilbert space
- From quantum information aspect, much smaller value D than $d^{N/2}$ is enough to derive ground state F. Verstraete et al. PRL 93, 227204

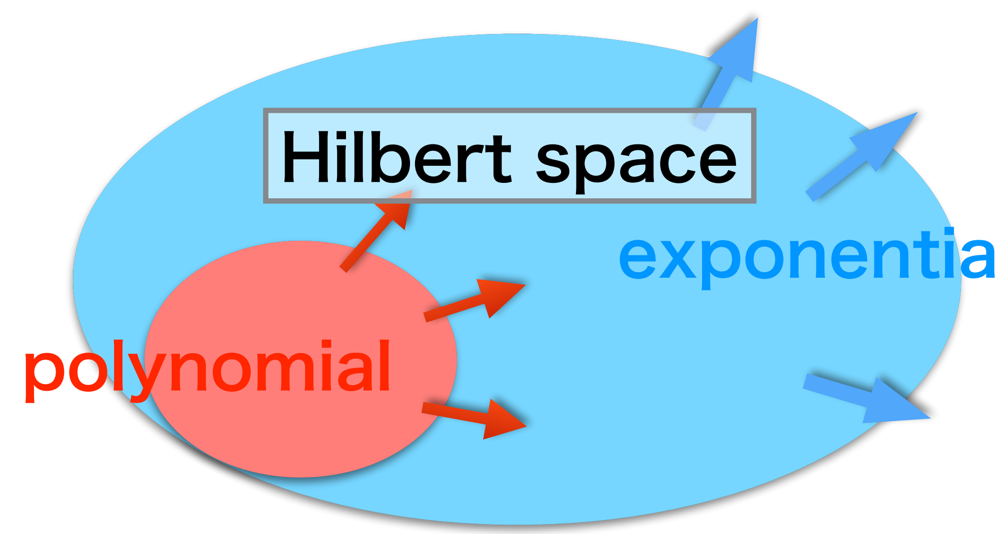
  Ex.) In our studies, $D \sim 100$ is enough ($\ll d^{N/2} \sim 10^{15}$)
- Hilbert space growing exponentially as increasing system size $\Leftrightarrow$ With TN, one can investigate sub-space growing polynomially
- As a systematic approximation

Ex.) 1d spin system

size of the whole Hilbert space		By using MPS with D,
d^{N}	$\Leftrightarrow$	NdD^2

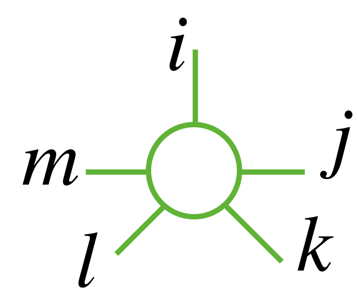
d : d.o.f of physical index at each site, N : chain length

$\Rightarrow$ If $D \sim d^{N/2}$, no advantage of TN

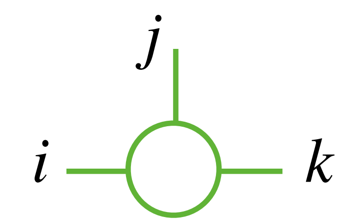


Graphical representation of TN

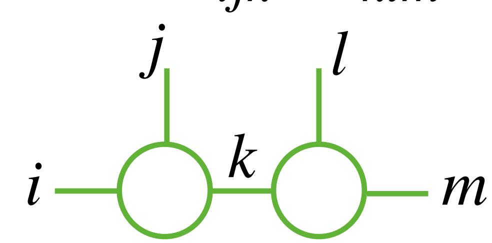
- Ex.1)
  - (i) A general tensor T_{ijklm}

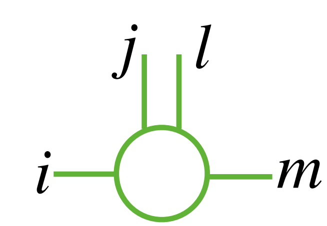


  - (ii) A tensor with three indices M_{ijk}



  - (iii) Product of two tensors $M_{ijk}\, A_{klm} \;=\; B_{ijlm}$



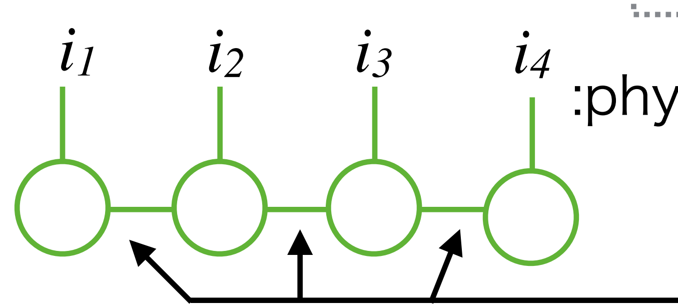


- Ex.2) MPS state:

$$\mathrm{Tr}\left[M[1]^{i_1}\cdots M[N]^{i_N}\right]$$

MPS state
$$|\psi\rangle \approx \sum_{i_1,\cdots,i_N} \mathrm{Tr}\left[M[1]^{i_1}\cdots M[N]^{i_N}\right]|i_1\cdots i_N\rangle$$

ex. $N=4$
open boundary condition



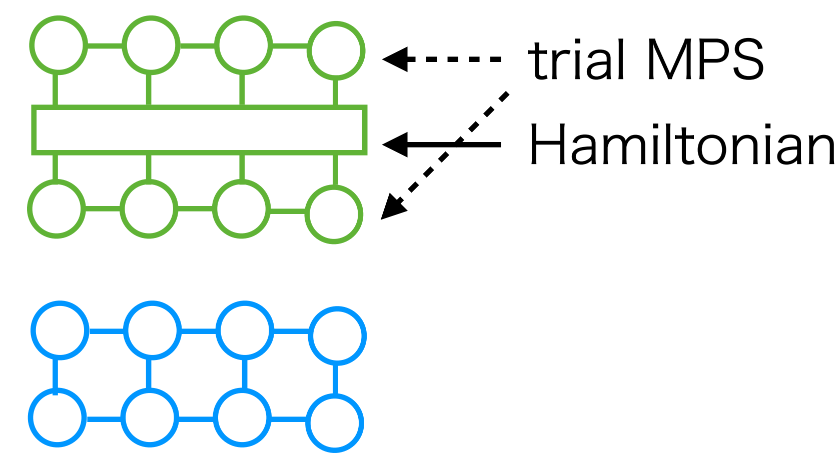
:physical indices

bond indices
from TN approximation

Variational search 1

- For ground (and some excited) state search
- Ground state derived by searching minimum of trial energy computed by trial MPS state:

$$E = \min E_{\text{trial}} = \min \frac{\langle\psi| H |\psi\rangle}{\langle\psi \mid \psi\rangle}$$



$|\psi\rangle$: a trial MPS state

- the minimum searched with variational approach

$$\frac{dE_{\text{trial}}}{dM[n]^{i_n}_{k_n k_{n+1}}} = 0$$

with fixing the other elements

Variational search 2

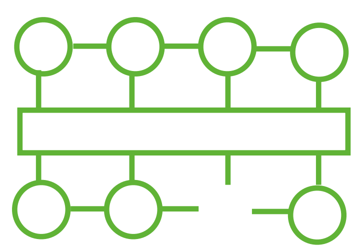
- More concretely,

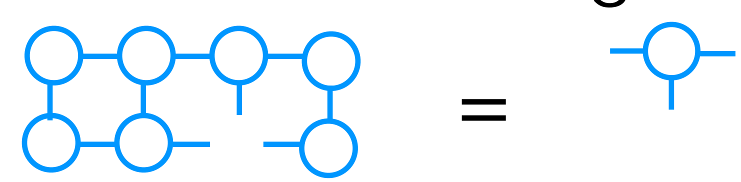
MPS state: $|\psi\rangle \approx \sum_{i_1,\cdots,i_N} \mathrm{Tr}\left[M[1]^{i_1}\cdots M[N]^{i_N}\right]|i_1\cdots i_N\rangle$

$\langle\psi|\,H\,|\psi\rangle$ $\langle\psi\,|\,\psi\rangle$: $M[n]^{i_n}_{k_n k_{n+1}}$ as a variable of trial energy, the others fixed

$$\frac{d\langle\psi|\,H\,|\psi\rangle}{dM[n]^{i_n}_{k_n k_{n+1}}} - E_{\text{trial}}\frac{d\langle\psi\,|\,\psi\rangle}{dM[n]^{i_n}_{k_n k_{n+1}}} = 0$$

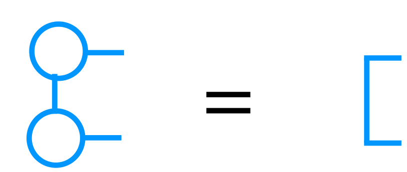
for given i_n, k_n, k_{n+1}

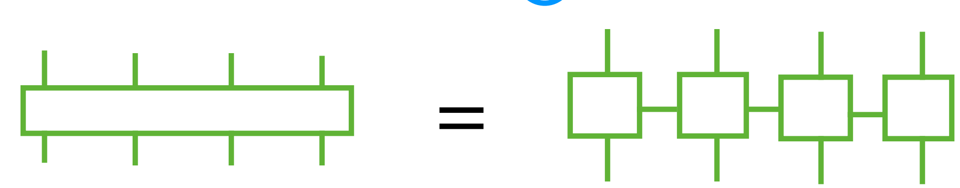




updating through the whole chain, until convergence

- Techniques to solve it efficiently,

i) canonical form derived by SVD: 

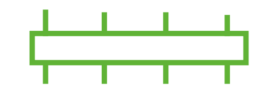
ii)MPO for Hamiltonian 

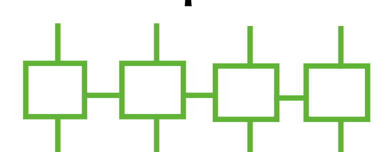
Matrix Product Operator (MPO)

- Hamiltonian of Schwinger model

hopping term, mass term, gauge part

$$H = x\sum_{n=0}^{N-2}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right] + \frac{\mu}{2}\sum_{n=0}^{N-1}\left[1+(-1)^n\sigma_n^z\right] + \sum_{n=0}^{N-2}\left[l + \frac{1}{2}\sum_{k=0}^{n}\left((-1)^k + \sigma_k^z\right)\right]^2$$

- Basis of Hamiltonian $d^N \Rightarrow$ Hamiltonian $d^N \times d^N$ matrix 
- MPO: a different way by mapping into operator space
- Ex.) MPO of one hopping term with $N = 4$, open boundary



$$H_{\text{hop,half}} = x\sum_{n=0}^{3}\sigma_n^+\sigma_{n+1}^- \quad \text{:hopping}$$

$$= \sum_{i_0,i_1,i_2,i_3} \mathrm{Tr}\left[A_0^{i_0}A_1^{i_1}A_2^{i_2}A_3^{i_3}\right]\left(\tilde{\sigma}_0^{i_0}\otimes\tilde{\sigma}_1^{i_1}\otimes\tilde{\sigma}_2^{i_2}\otimes\tilde{\sigma}_3^{i_3}\right)$$

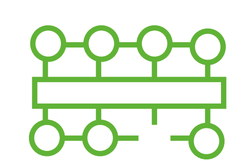
where $A_0^0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $A_0^1 = \begin{pmatrix} 0 & x & 0 \end{pmatrix}$, for left boundary

$$A_n^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_n^1 = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_n^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad A_3^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

for bulk $n = 2,3$ — for right boundary

$$\tilde{\sigma}^p = (1_{2\times 2}, \sigma^+, \sigma^-)$$

Useful to compute the trial energy

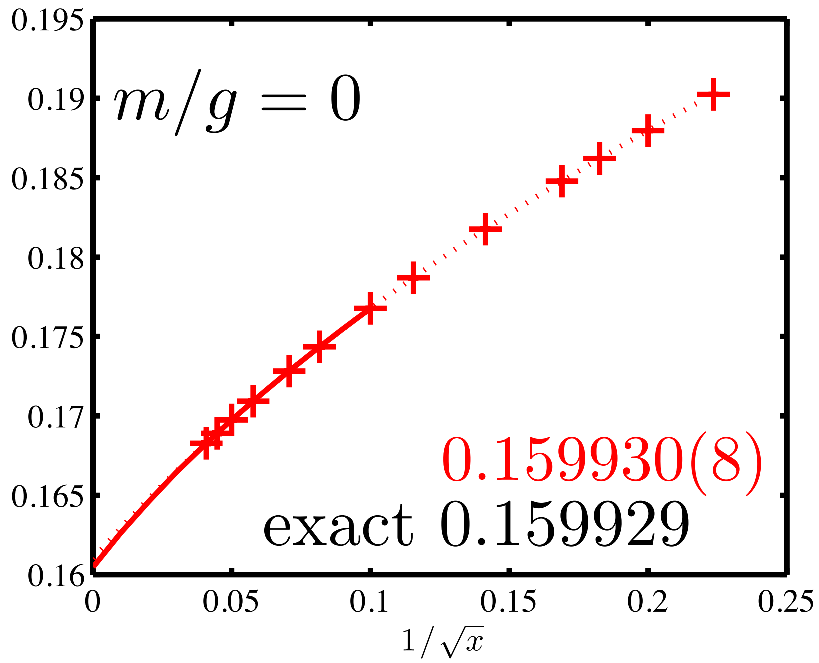


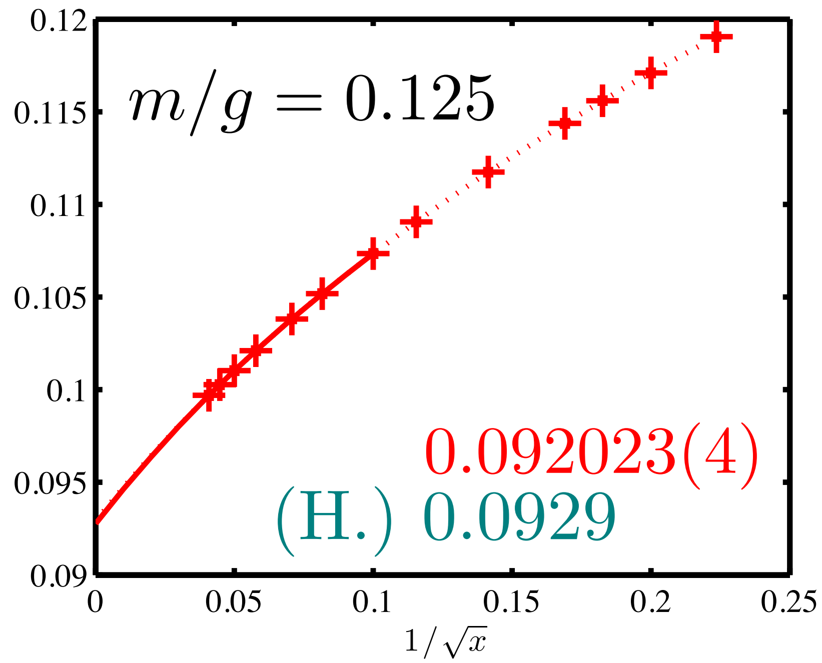
Our previous study

M. C. Bańuls et al JHEP 1311, 158, LAT2013, 332

- Schwinger model with MPS method
- With variational method, computing:
  - ✳ spectrum
  - ✳ (subtracted) chiral condensate: $\frac{\bar{\psi}\psi}{g} = \frac{\sqrt{x}}{N}\sum_n (-1)^n \left[\frac{1+\sigma_n^z}{2}\right]$ in spin language
- Continuum limit: $1/\sqrt{x} \to 0$

  with inverse coupling $x = 1/g^2a^2$





(H.) Y. Hosotani arXiv:9703153

Fit function:

$$f(x) = A + F\frac{\log(x)}{\sqrt{x}} + B\frac{1}{\sqrt{x}} + C\frac{1}{x}$$

Logarithmic correction from analytic form of free theory

Lattice gauge theory (LGT) with TN approach

- Earlier Study: critical behavior of Schwinger model with Density Matrix Renormalization Group

T. Byrnes, et al. PRD.66.013002 (2002)

- Nowadays: various branches

✳ Our previous studies

M. C. Bańuls et al JHEP 1311, 158, LAT2013, 332 (2013)

✳ Strong coupling exp.

K. Cichy, et al. Comput.Phys.Commun. 184 1666 (2013)

✳ LGT with TN on higher dimension

✳ Real time evolution

B. Buyens, et al. arXiv:1312.6654

✳ Quantum link model

P. Silvi, et al arXiv:1404.7439
E. Rico, et al. PRL112, 201601 (2014)

✳ Tensor Renormalization Group

Y. Shimizu, Y. Kuramashi arXiv:1403.0642 (With Lagrangian)

This study

Chiral symmetry restoration of Schwinger model for $N_f = 1$

- Chiral symmetry breaking at $T = 0$ (via anomaly)
  $\Leftrightarrow$ At high T, the symmetry restoration
- Order parameter : chiral condensate $\langle\bar{\psi}\psi\rangle$
- Analytic formula

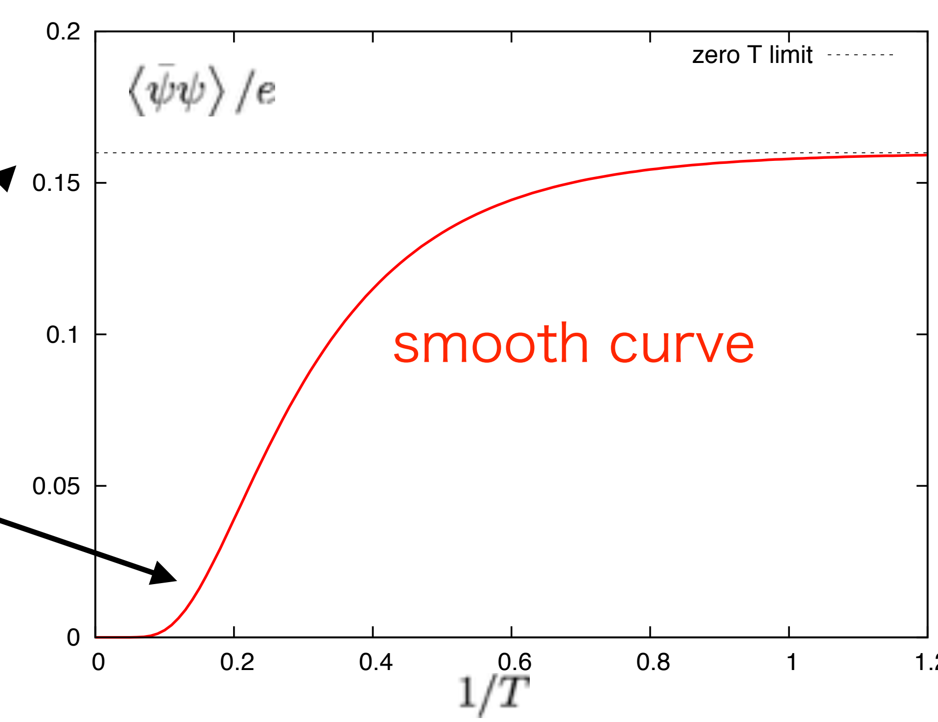
I. Sachs and A. Wipf, arXiv:1005.1822

$$\langle\bar{\psi}\psi\rangle = \frac{m_\gamma}{2\pi} e^\gamma e^{2I(\beta m_\gamma)}$$

$$= \begin{cases} \dfrac{m_\gamma}{2\pi} e^\gamma & \text{for} \quad T \to 0 \\ 2T e^{-\pi T/m_\gamma} & \text{for} \quad T \to \infty \end{cases}$$

where $I(x) = \int_0^\infty \frac{1}{1 - e^{x \cosh(t)}} dt$ Euler constant $\gamma = 0.57721\ldots$

$m_\gamma = e/\sqrt{\pi}$



Thermal state calculation

F. Verstraete *et al* PRL 93, 20 (2004)

- Expectation value at finite T:

$$\langle \mathcal{O} \rangle_\beta = \frac{\mathrm{tr}\left[\mathcal{O}\, \rho(\beta)\right]}{\mathrm{tr}\left[\rho(\beta)\right]}$$

thermal density operator $\rho(\beta) \equiv e^{-\beta H}$
where $\beta = 1/T$

- How to calculate the $\rho(\beta)$
  - ✳ $\rho(\beta/2)$ to ensure positivity: $\rho(\beta) = \rho\left(\beta/2\right) \rho\left(\beta/2\right)^\dagger$
  - ✳ Evolution of temperature: $\rho(\beta/2) = \underbrace{e^{-\frac{\delta}{2}H} \cdots e^{-\frac{\delta}{2}H}}_{N = \beta/\delta}$

    $\boxed{\text{high } T \rightarrow \text{low } T}$

    Ex.) For fixed δ, larger N corresponds to lower T
  - ✳ Further details, a unit of our thermal density operator

$$e^{-\frac{\delta}{2}H} \approx e^{-\frac{\delta}{4}H_g} \underbrace{e^{-\frac{\delta}{2}(H_{\rm hop}+H_{\rm mass})}}_{\approx e^{-\frac{\delta}{4}H_e} e^{-\frac{\delta}{2}H_o} e^{-\frac{\delta}{4}H_e}} e^{-\frac{\delta}{4}H_g}$$

H_g : including long range int.

additional Trotter exp. for other terms into even site and odd site (2nd order Trotter exp.)

multiplication of five factors for each step of $\rho(\beta/2)$

Global optimization

- Global optimization: Updating each elements w/ fixing the others so that the distance $\epsilon = |\mathcal{O}_{\text{approx}} - \mathcal{O}|$ is minimum
- MPS approximation for $\rho(\beta)$

$$\rho(\beta) \approx \sum_{\substack{i_1,\cdots,i_N \\ j_1,\cdots,j_N}} \mathrm{Tr}\left[M[1]^{i_1 j_1} \cdots M[N]^{i_N j_N}\right] |i_1 \cdots i_N\rangle \langle j_1 \cdots j_N|$$

- How to obtain elements of tensors $M[1],\cdots, M[N]$:

  Ex. for $\rho(\beta) \to \rho'(\beta) \approx \rho(\beta) e^{-\frac{\delta}{2} H_g}$ so that the distance $\epsilon = \left|\rho'(\beta) - \rho(\beta) e^{-\frac{\delta}{2} H_g}\right|$ is minimum

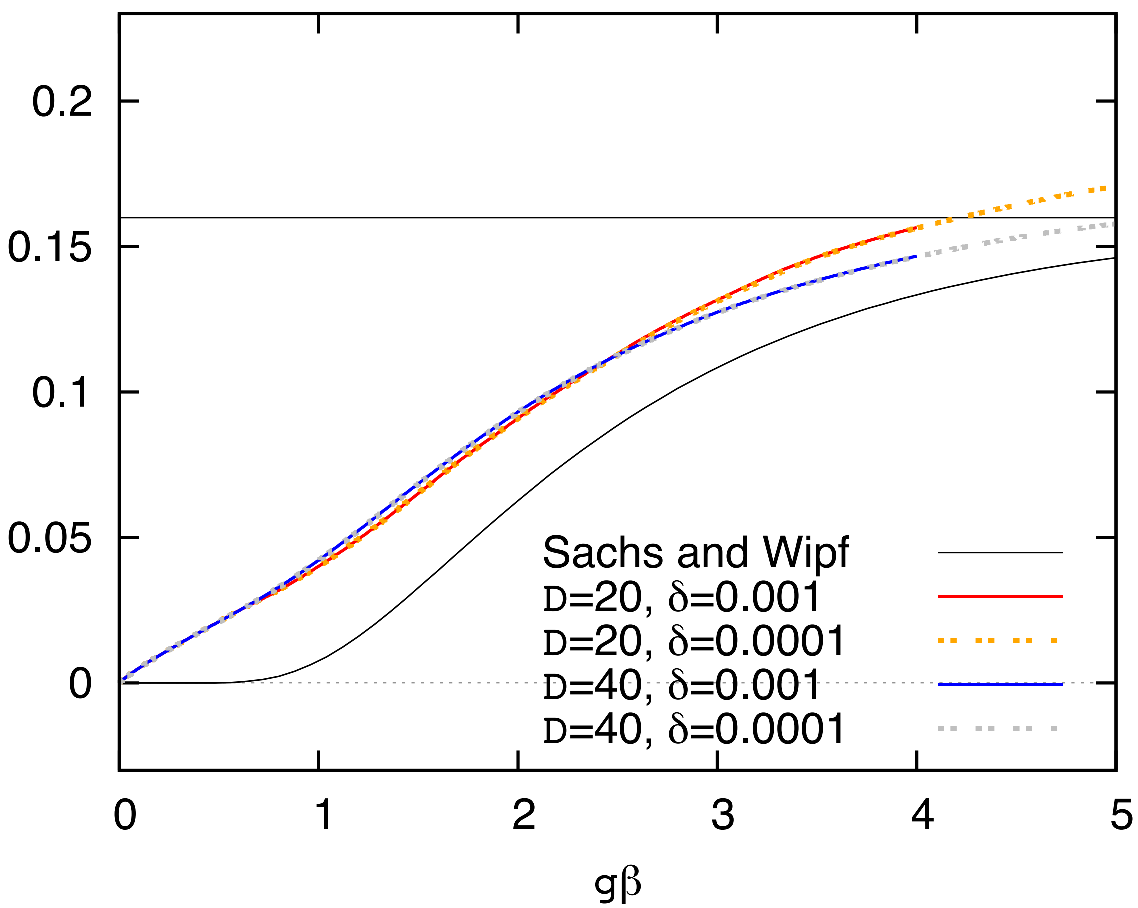
Simulation setup

- Open Boundary condition
- Four simulation parameters
  1. From MPS approx., bond dimension D $D \to \infty$
  2. From T evol., step size δ $\delta \to 0$
  3. chain length N $N \to \infty$
  4. inverse coupling x $1/\sqrt{x} \to 0$
- To avoid finite size effect: $N/\sqrt{x} > 15$
- Four extrapolation steps

Four systematic errors

<u>From bond dimension D,</u>

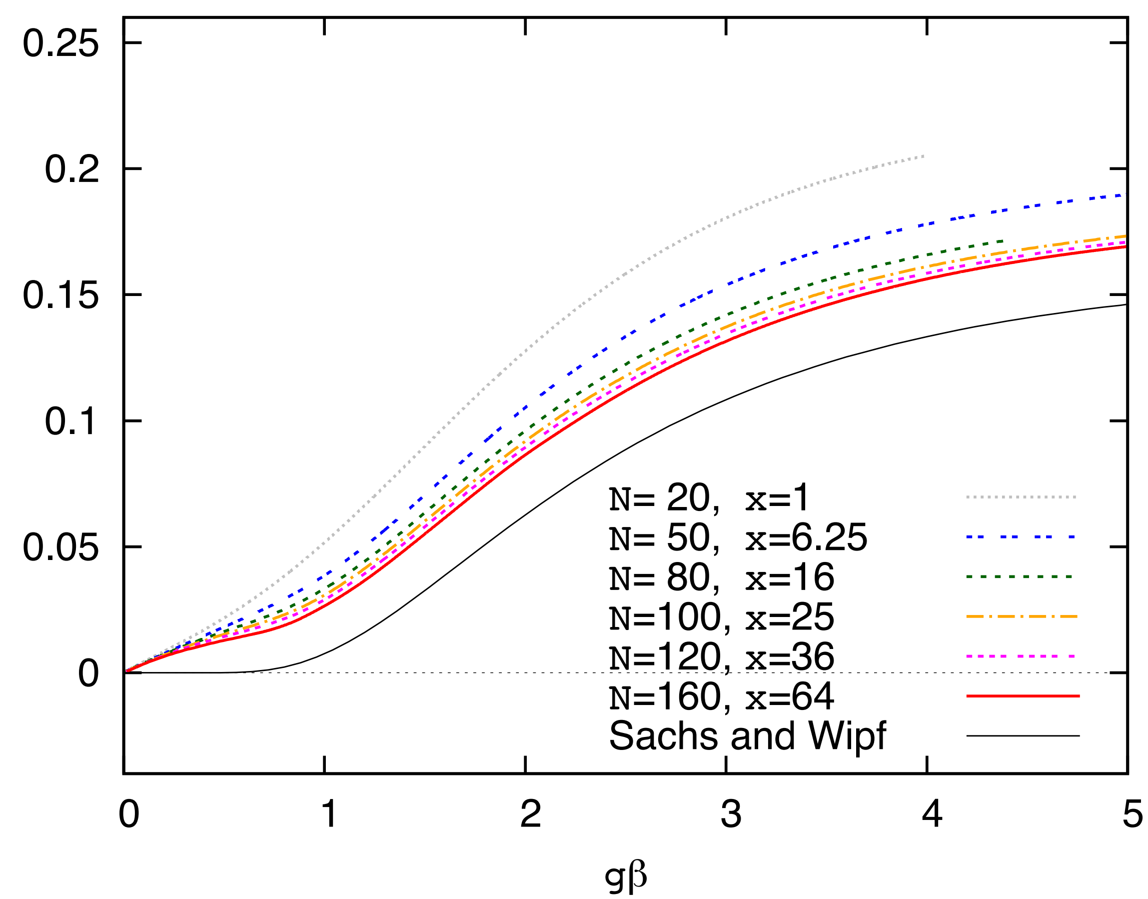
<u>step size δ</u>



<u>From chain length N,</u>

<u>inverse coupling x</u>

continuum limit with fixed physical length $N/\sqrt{x} = 20$



Extrapolations

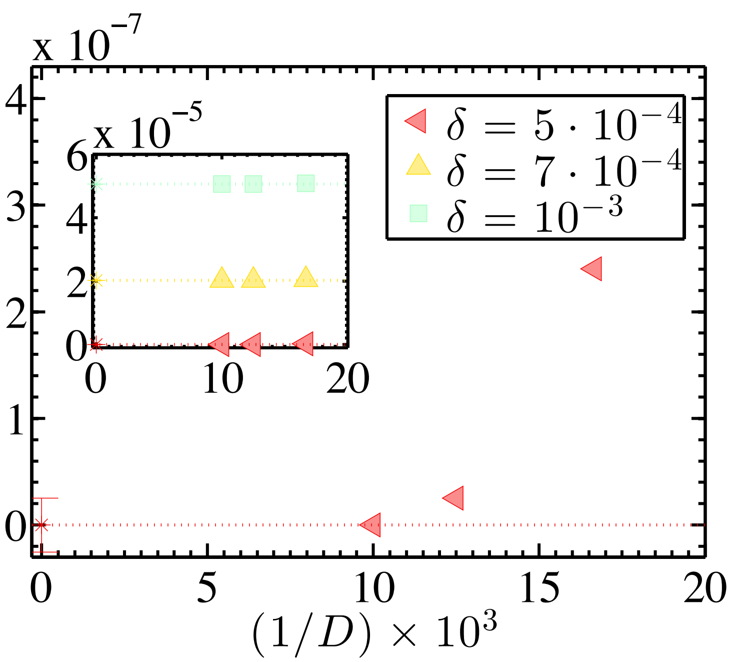
$D \rightarrow \infty$ with fixed δ, N, x

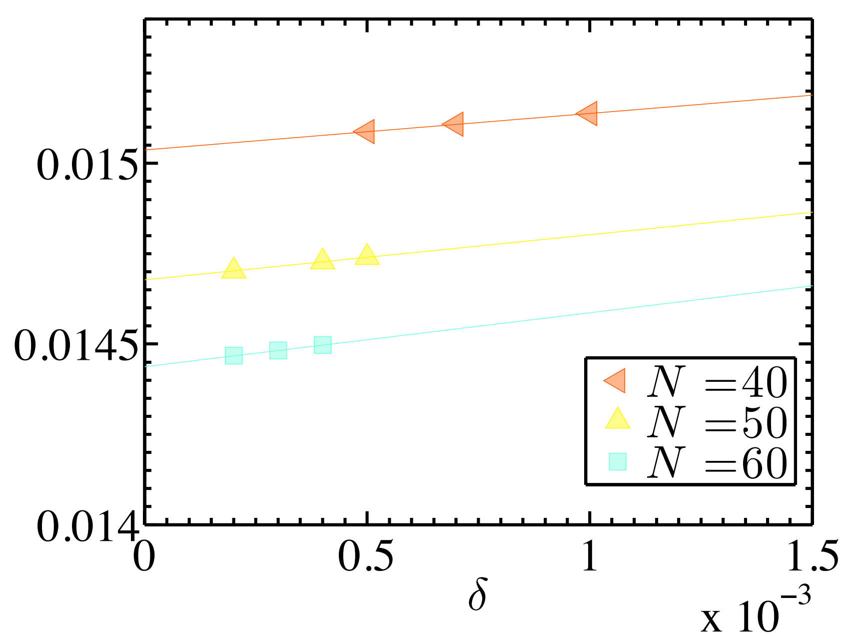
from a mathematical proof, convergence in D

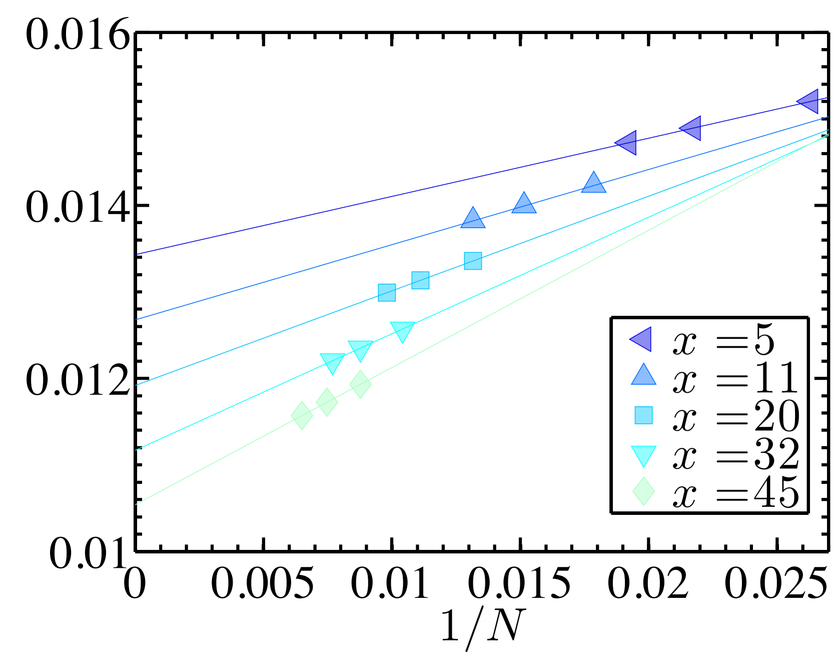
$\delta \rightarrow 0$ with fixed N, x

linear convergence in δ theoretically predicted

$N \rightarrow \infty$ with fixed x



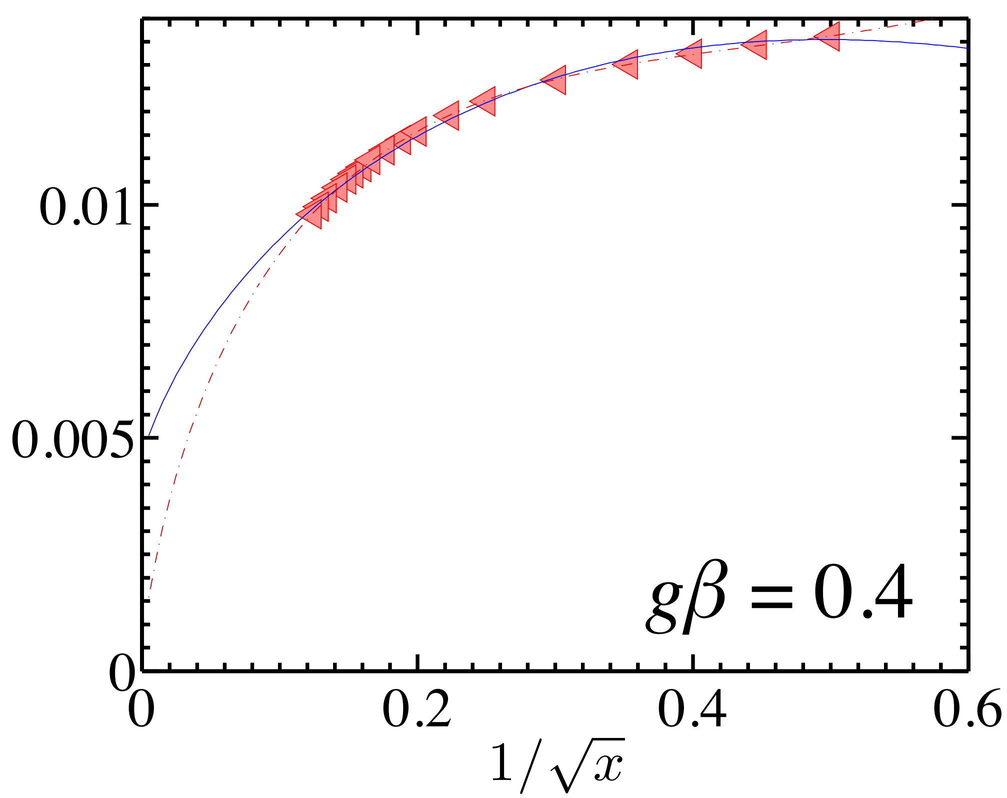


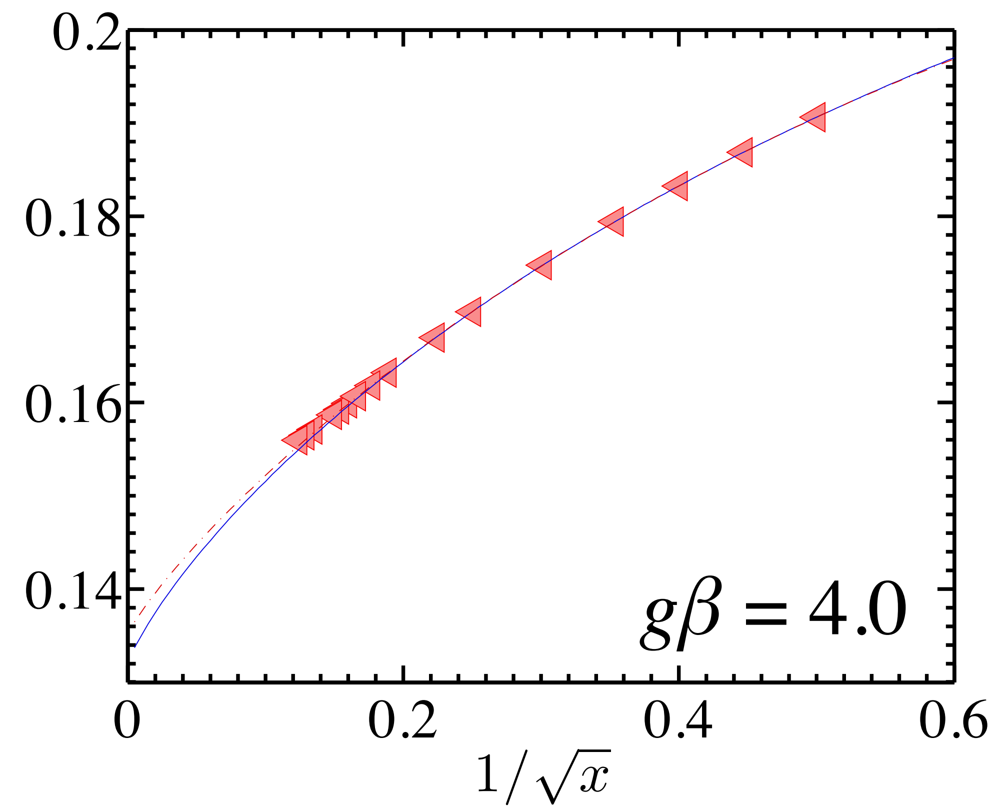


at $g\beta = 0.4$

Continuum extrapolation

- continuum limit extrapolation $1/\sqrt{x} \to 0$



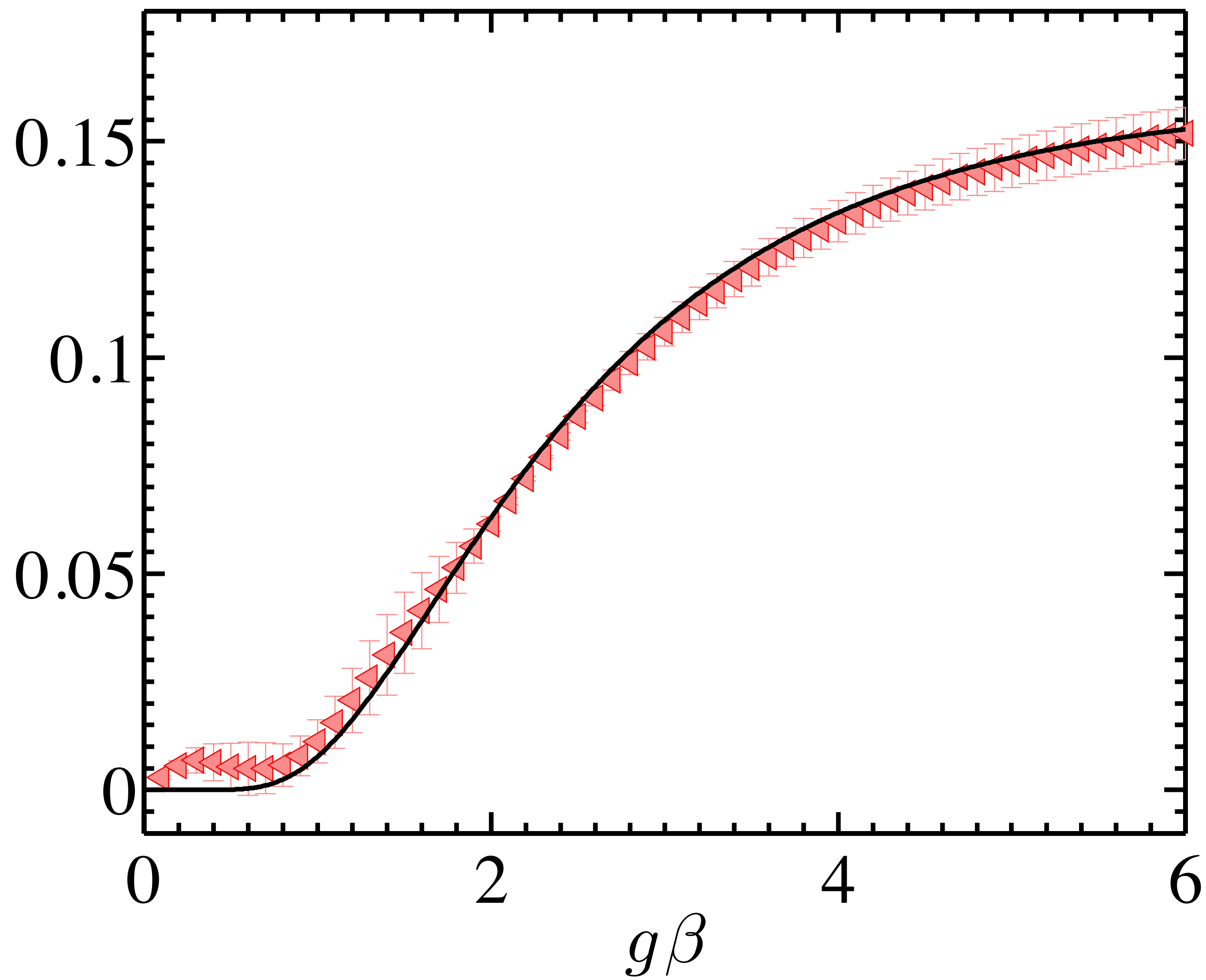


$$\Sigma = \Sigma_{\mathrm{cont}} + \frac{a_1}{\sqrt{x}}\log(x) + \frac{b_1}{\sqrt{x}}$$ (solid blue line)

$$\Sigma = \Sigma_{\mathrm{cont}} + \frac{a_2}{\sqrt{x}}\log(x) + \frac{b_2}{\sqrt{x}} + \frac{c_2}{x}$$ (bashed red line)

Chiral condensate at high T

After eliminating those systematic errors …



Summary

- Computing chiral condensate at finite T in Hamiltonian formalism with tensor network methods
- Evaluating dependence of parameters: bond dimension, step size, system size, inverse of coupling
- By taking continuum limit, we obtained results consistent with an analytic formula. I. Sachs and A. Wipf, arXiv:1005.1822
- Future plans

  i) Many flavor Schwinger model

  ii) Schwinger model at finite μ

  iii) Non-Abelian gauge theory

  iv) Real time evolution

  v) Higher dimension of TN