二重指数関数型積分法の有効な使用法

1.収束の遅い解析解の計算
2.特殊函数の積分表示を使用した計算

$$I_n = \int_0^1\int_0^1 \cdots\cdots \int_0^1 \frac{1}{(x_1 x_2 \ldots x_n (x_1 + x_2 + \ldots + x_n))^{\alpha}} dx_n dx_{n-1} \ldots dx_1$$

$$\alpha = 0.2$$

の解析解の計算と　DEに関して

有限値を持つ事は,

$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$**の場合**

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$ (**等号は**$x_1 = x_2 = \ldots = x_n$)

の関係式を使用する。

$$n = 2 \qquad 2^{2\alpha} \times \frac{25}{14} \times (2^{1.4} - 2) \leq I_2 \leq \frac{2^{-\alpha}}{(1 - \frac{3}{2}\alpha)^2}$$

解析解 $= 1.66762994758580$

$$n = 3 \quad 3^{3\alpha} \times \frac{125}{66} \times (3^{2.2} - 3 \times 2^{2.2} + 3) \leq I_3 \leq \frac{3^{-\alpha}}{(1 - \frac{4}{3}\alpha)^3}$$

解析解 $= 1.88787303406103$

$$n = 4 \quad 4^{4\alpha} \times (\frac{32}{3}\log(4) - 18 \times \log(3) + 8 \times \log(2)) \leq I_4 \leq \frac{4^{-\alpha}}{(1 - \frac{5}{4}\alpha)^4}$$

解析解 $= 2.21178596178275$

$n=2$の場合を計算すると

$$\int_0^1\int_0^1 \frac{1}{(xy(x+y))^{\alpha}}dydx = \int_0^1\int_0^{1-x} \frac{1}{(xy(x+y))^{\alpha}}dydx$$

$$+\int_0^1\int_{1-x}^1 \frac{1}{(xy(x+y))^{\alpha}}dydx$$

$$=\int_0^1\int_0^y \frac{1}{(xy(x+y))^{\alpha}}dxdy+\int_0^1\int_0^x \frac{1}{(xy(x+y))^{\alpha}}dydx$$

$$=2\int_0^1\int_0^x \frac{1}{(xy(x+y))^{\alpha}}dydx$$

$$=2\int_0^1\int_0^1 \frac{x}{x^{3\alpha}t^{\alpha}(1+t)^{\alpha}}dtdx \quad (y=tx, dy=xdt)$$

$$=\frac{2}{2-3\alpha}\int_0^1 \frac{1}{t^{\alpha}(1+t)^{\alpha}}dt$$

$$=\frac{2}{2-3\alpha}\left(\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\alpha+n)}{n!\Gamma(\alpha)(n+1-\alpha)}\right)$$

解析解 $=1.66762994758580$

ただし、計算機で具体 的に計算できるか
という問題が発生する 。
$n!$の計算は指数部11ビット,15ビットで
それぞれ$n\approx 170,1700$でオーバーフローが発 生する。

このため $\int_0^1 \frac{1}{(t(1+t))^{\alpha}}dt$の展開式を

$$\frac{1}{1-\alpha}-\frac{\alpha}{2-\alpha}+\sum_{n=2}^{\infty}\frac{(-1)^n\alpha(\alpha+1)........(\alpha+n-1)}{n!(n+1-\alpha)}$$

$$=\frac{\alpha^2-2\alpha+2}{(1-\alpha)(2-\alpha)}+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)........(\alpha+2n-1)[\alpha^2-(2n+2)\alpha+2(2n+1)]}{(2n+1)!(2n+1-\alpha)(2n+2-\alpha)}$$

と変形し,

$$a_0=1$$

$$a_{i+1}=a_i\times\frac{(\alpha+2(i-1))(\alpha+2(i-1)+i)}{2i(2i+1)}$$

の漸化式を作成して

$$\int_0^1\frac{1}{(t(1+t))^{\alpha}}dt=\frac{\alpha^2-2\alpha+2}{(1-\alpha)(2-\alpha)}+\sum_{n=1}^{\infty}\frac{a_n[\alpha^2-(2n+2)\alpha+2(2n+1)]}{(2n+1-\alpha)(2n+2-\alpha)}$$

で計算する。この式の 収束は非常に遅く,

DE計算が非常に有効となる。

DEで$n=448, h=0.5^6$,倍精度演算と

展開式で倍精度演算,4倍精度演算で実行した 結果を

次ぎに示します。

展開式による計算

	計算結果	
N=2**J	倍精度	4倍精度
J	1.66762994758580	1.66762994758580
16	1.66762994749020	1.66762994749020
17	1.66762994755834	1.66762994755834
18	1.66762994757791	1.66762994757791
19	1.66762994758353	1.66762994758353
20	1.66762994758515	1.66762994758515
21	1.66762994758561	1.66762994758561
22	1.66762994758574	1.66762994758574
23	1.66762994758578	1.66762994758578
24	1.66762994758579	1.66762994758579
25	1.66762994758580	1.66762994758580

DE (倍精度演算) X5570　1CPU計算

result=　1.66762994758580
gosa=　0.000000000000000E+000
elapse=　4.999999946448952E-004　sec

特殊函数の積分表示による計算

DE(倍精度計算)　分点数$N = 448$, 刻み幅$h = 0.5^6$

積分変数変換 区間$[10^{-30}, 1-2^{-53}]$

精度改善版

分点数$N = 550$, 刻み幅$h = 0.5^6$

積分変数変換 区間$[10^{-150}, 1-2^{-53}]$

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

$$= -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n+1}\right)$$

$$= -\gamma + \int_0^1 \frac{x - x^z}{(1-x)x}dx$$

$\psi(1) = -\gamma$

result = -0.577215664901533

gosa = 0.000000000000000E+000

$\psi(2) = 1-\gamma$

result = 0.422784335098467

gosa = 0.000000000000000E+000

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

$$\psi(3) = 0.922784335098467$$

result = 0.922784335098467

gosa = 4.44089209850062 6E-016

$$\psi\left(\frac{2}{3}\right) = -\gamma - \frac{3}{2}\log(3) + \frac{\sqrt{3}}{6}\pi$$

result = -1.31823441578659

gosa = 0.000000000000000E+000

$$\psi\left(\frac{1}{3}\right) = -\gamma - \frac{3}{2}\log(3) - \frac{\sqrt{3}}{6}\pi$$

result = -3.13203377989309

gosa = 1.27711174968681 0E-010

精度改善版

result = -3.13203378002081

gosa = 8.88178419700125 2E-016

$$\psi(\frac{1}{4}) = -\gamma - 3\log(2) - \frac{\pi}{2}$$

result = -4.22745346663955

gosa = 6.67367148210473 7E-008

精度改善版

result = -4.22745353337626

gosa = 8.88178419700125 2E-016

$$\psi(\frac{3}{4}) = -\gamma - 3\log(2) + \frac{\pi}{2}$$

result = -1.08586087978647

gosa = 2.22044604925031 3E-016

$$\psi'(z) = \frac{d}{dz}\psi(z) = \frac{d^2}{dz^2}\log\Gamma(z)$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z+n)^2} = \int_0^1 \frac{(-\log(t))t^{z-1}}{1-t}dt$$

$$\psi'(z+1) = \psi'(z) - \frac{1}{z^2}$$

$$\psi'(1) = \frac{\pi^2}{6}$$

result = 1.64493406684823

gosa = -2.22044604925031 3E-016

$\psi'(2) = 0.644934066848226$

result = 0.644934066848226

gosa = 4.44089209850063E-016

$\psi'(3) = 0.394934066848226$

result = 0.394934066848226

gosa = 3.88578058618805E-016

$$\psi'\left(\frac{1}{2}\right) = 3\zeta(2) = \frac{\pi^2}{2}$$

result = 4.93480220054464

gosa = 3.99680288865056E-014

精度改善版

result = 4.93480220054468

gosa = 8.88178419700125E-016

$$\gamma = \int_0^1 \left(\frac{1}{\log(t)} + \frac{1}{1-t}\right)dt$$

result = 0.577215664901533

gamma = 0.577215664901533

$$\gamma = -\int_0^1 \log|\log(t)|\,dt$$

result = 0.577215664901532

gamma = 0.577215664901533

$$\beta(z) = \psi(z) - \psi(\frac{z}{2}) - \log(2)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z+n} = \int_0^1 \frac{t^{z-1}}{1+t} dt$$

$$\beta(\frac{1}{2}) = \frac{\pi}{2}$$

result = 1.57079632679490

gosa = -4.44089209850063E-016

$$\beta(\frac{1}{3}) = \log(2) + \frac{\pi}{\sqrt{3}}$$

result = 2.50694654466645

gosa = 1.27711619057891E-010

精度改善版

result = 2.50694654479416

gosa = 0.00000000000000E+000

$$\beta(\frac{2}{3}) = -\log(2) + \frac{\pi}{\sqrt{3}}$$

result = 1.12065218367427

gosa = 2.22044604925031E-016

$$\int_0^1 \frac{1}{\log|\log(x)|}dx = -0.154479641320$$

$x = \frac{1}{e}$ で主値を取る。

result = - 0.154479641320290

eps = 4.698300000000000E - 010

gosa = 2.900180096077065E - 013

ε算法($\varepsilon_0 = 1.2^{-105}, \varepsilon_{i+1} = \varepsilon_i / 1.2, i = 0......., 14$)

eps = 0.37794771D - 09

result = - 0.15447963375535467 3D + 00

ex : 15 - 0.1544796400102996D + 00

0.285D - 09

ε算法$(\varepsilon_0 = 1.2^{-44}, \varepsilon_{i+1} = \varepsilon_i / 1.2, i = 0.......,14)$
倍精度演算精度改善

```
 eps= 0.25555697D-04
 result=  -0.15447964122042789 9D+00
ex -0.15447964132005601 0D+00   0.207D-09  -0.56011D-13
ex -0.15447964131985603 1D+00   0.453D-12   0.14397D-12
ex -0.15447964132071023 7D+00   0.177D-11  -0.71024D-12
ex -0.15447964132190081 2D+00   0.150D-11  -0.19008D-11
ex -0.15447964132036315 4D+00   0.303D-11  -0.36315D-12
ex -0.15447964132036443 0D+00   0.513D-13  -0.36443D-12
ex -0.15447964132036176 6D+00   0.745D-13  -0.36177D-12
ex:  15  -0.154479641320364 4D+00   0.619D-14
```

ε算法$(\varepsilon_0 = 1.2^{-105}, \varepsilon_{i+1} = \varepsilon_i / 1.2, i = 0.......,14)$

4倍精度演算

```
 eps= 0.37794771D-09
 result=  -0.154479641320933814D+00
ex -0.154479641319995629D+00   0.289D-11   0.43710D-14
ex -0.154479641320044100D+00   0.237D-12  -0.44100D-13
ex -0.154479641320043417D+00   0.186D-14  -0.43417D-13
ex -0.154479641320041434D+00   0.237D-14  -0.41434D-13
ex -0.154479641320042752D+00   0.292D-14  -0.42752D-13
ex -0.154479641320042742D+00   0.487D-16  -0.42742D-13
ex -0.154479641320042743D+00   0.514D-17  -0.42743D-13
ex:  15  -0.1544796413200427D+00   0.737D-18
```

永年0と考えられてきた（200年近く）積分値がここまで正しく計算されている。