QCD phase transition at real chemical potential with canonical approach

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for

Zn Collaboration

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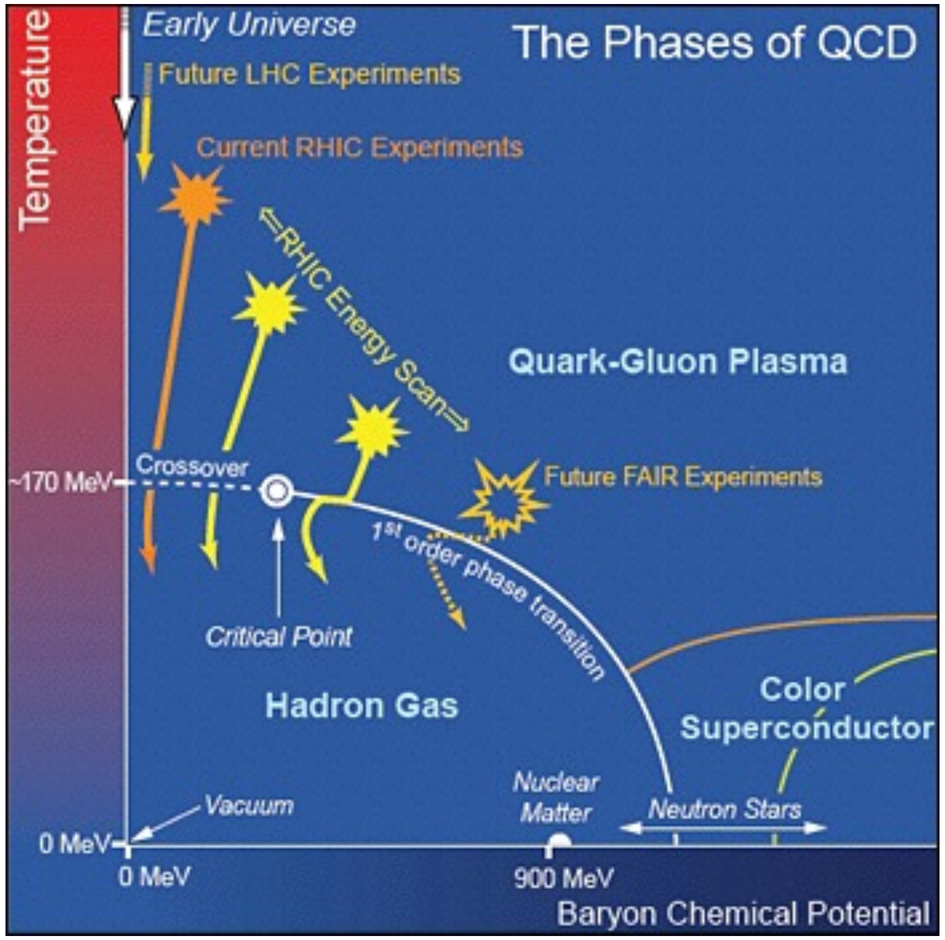
Finite density QCD



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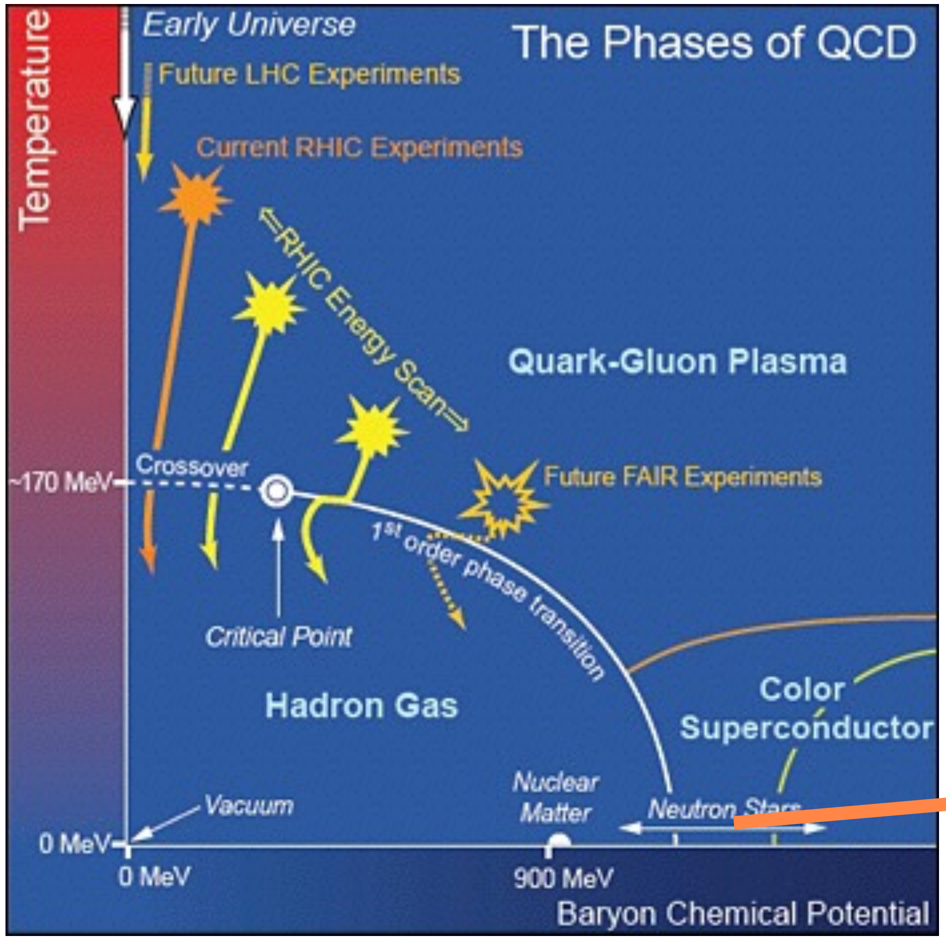
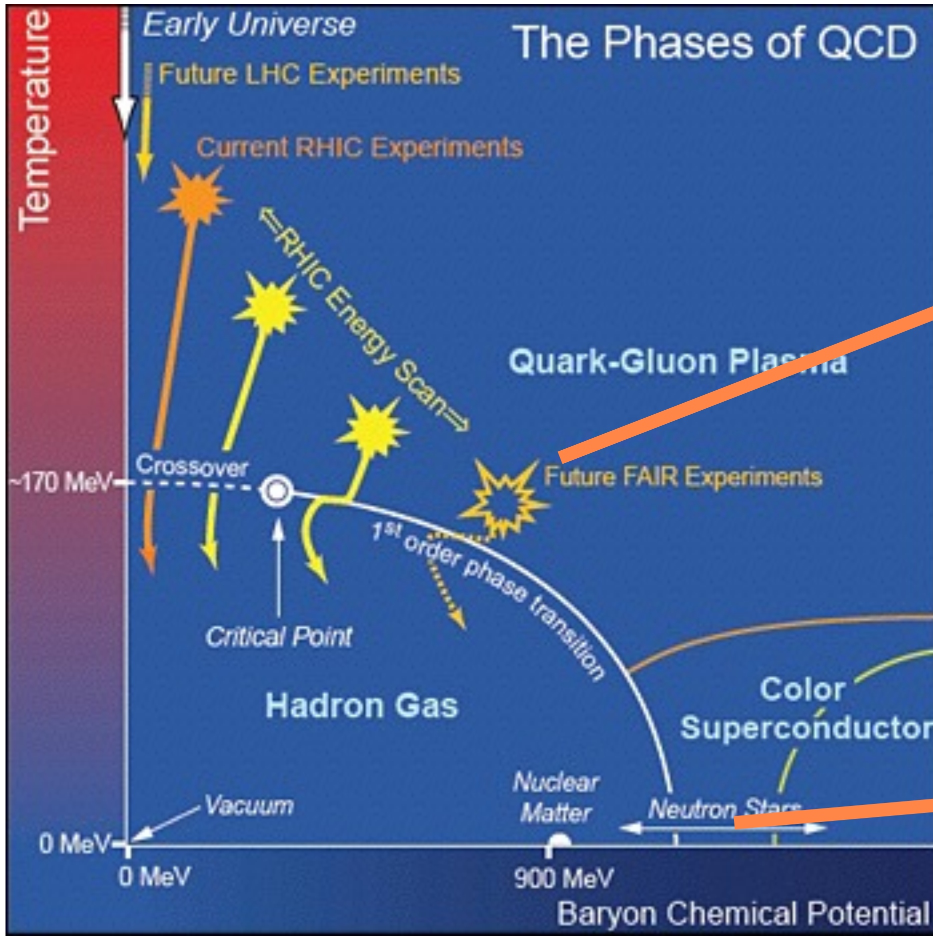


Neutron start with
2 x Solar mass

Finite density QCD



High density region
by experiments

J-PARC

RIKEN-RIBF

GSI-FAIR

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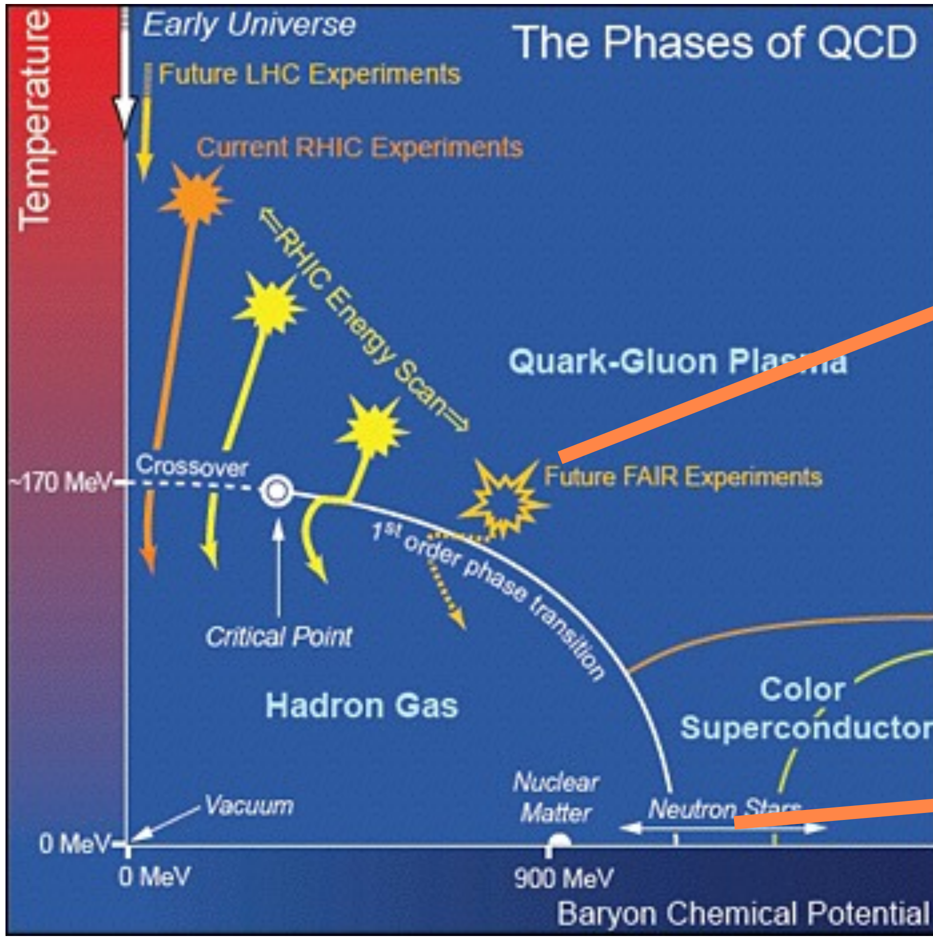

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Our tool: Lattice QCD

Complex action

Sign problem

Monte Carlo method

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Integrate out quark fields

$$Z = \int DU \int D\bar{\psi} D\psi e^{-\int \bar{\psi} D \psi} e^{-S_G} = \int DU \mathrm{Det} D(U) e^{-S_G(U)}$$

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Treat DetD as an observable = reweighting

Sign problem

re-weighting technique

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$$Z_G(\mu) = \int DU \frac{\mathrm{Det} D_W(\mu)}{\mathrm{Det} D_W(\mu_0)} \mathrm{Det} D_W(\mu_0) e^{-S_G}$$

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Utilize imaginary chemical potential

Solution = Canonical approach

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Grand canonical partition function

$$Z_G(T, \mu, V) = \mathrm{Tr}\left[\exp\left(-\frac{1}{T}\left(\hat{H} - \mu\hat{N}\right)\right)\right]$$

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for every energy and number of particles

For QCD $\left[\hat{H},\hat{N}\right] = 0$

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$$= \sum_n \sum_E \left\langle E, n \left| \exp\left(-\frac{\hat{H}}{T} + \frac{\mu}{T} n\right) \right| E, n \right\rangle$$

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How to extract $Z_C(n)$ from $Z_G(\mu)$?

$$Z_G(T,\mu,V) = \sum_{n=-\infty}^{\infty} Z_C(T,n,V)\xi^n$$

Solution = Canonical approach

How to extract Zc(n) from ZG(μ)?

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$$Z_C\left(T,n,V\right) = \oint \frac{d\xi}{2\pi i}\xi^{-n-1}Z_G(T,\xi,V)$$

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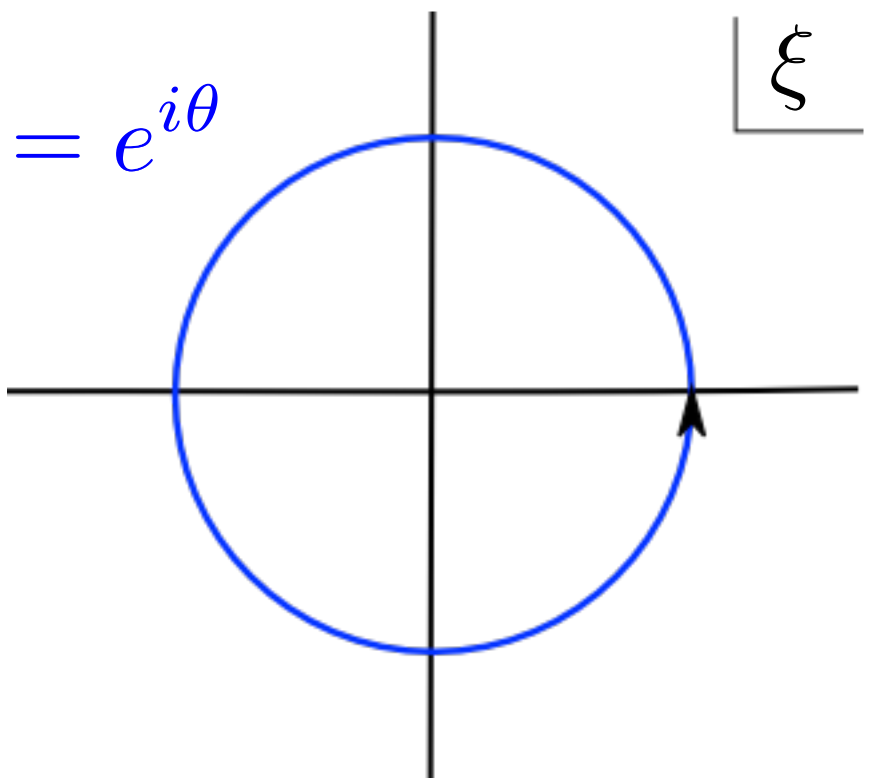

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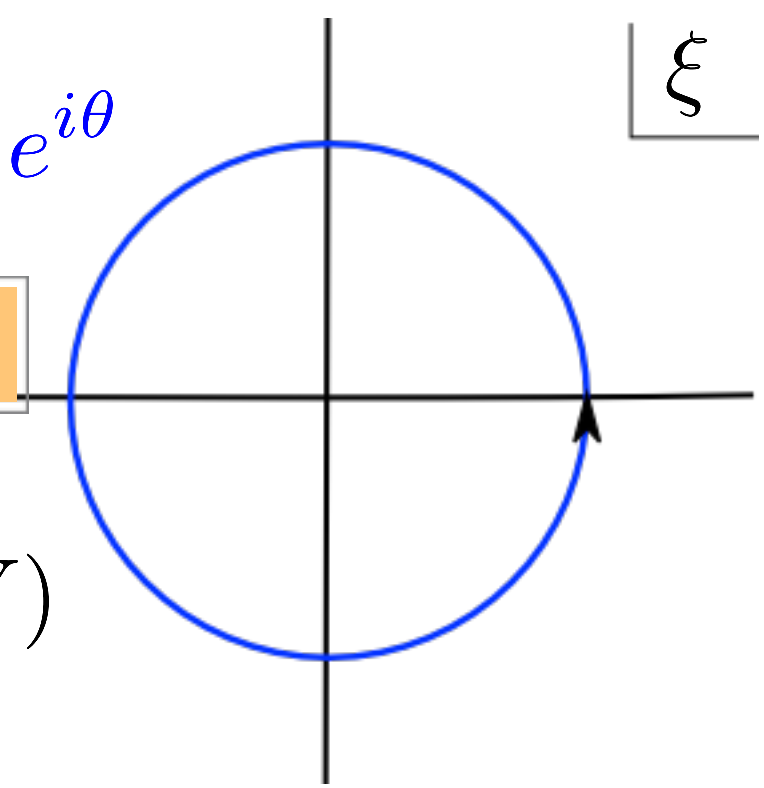

Fourier tr. in imaginary chemical potential!

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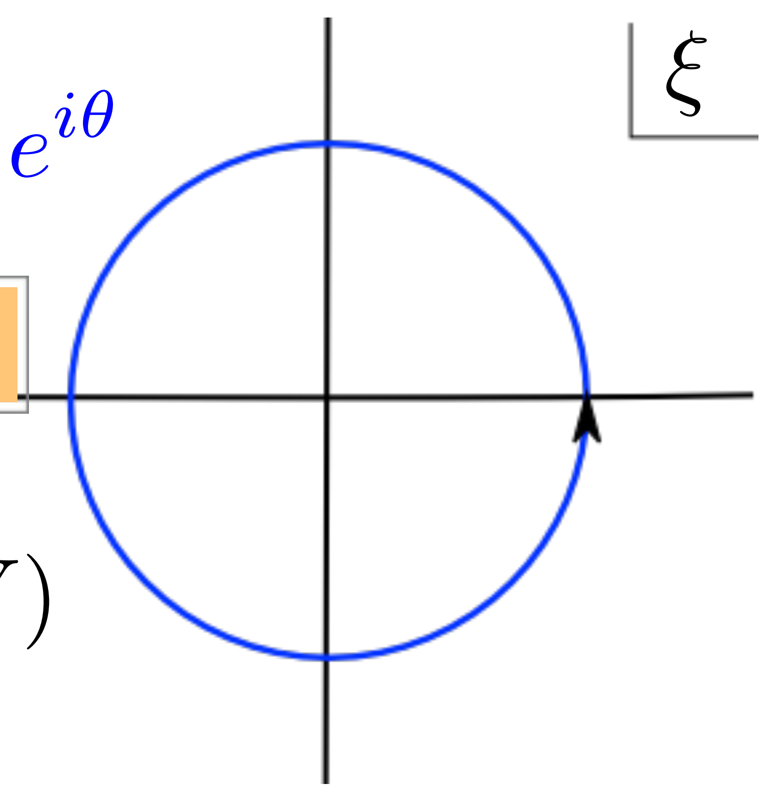

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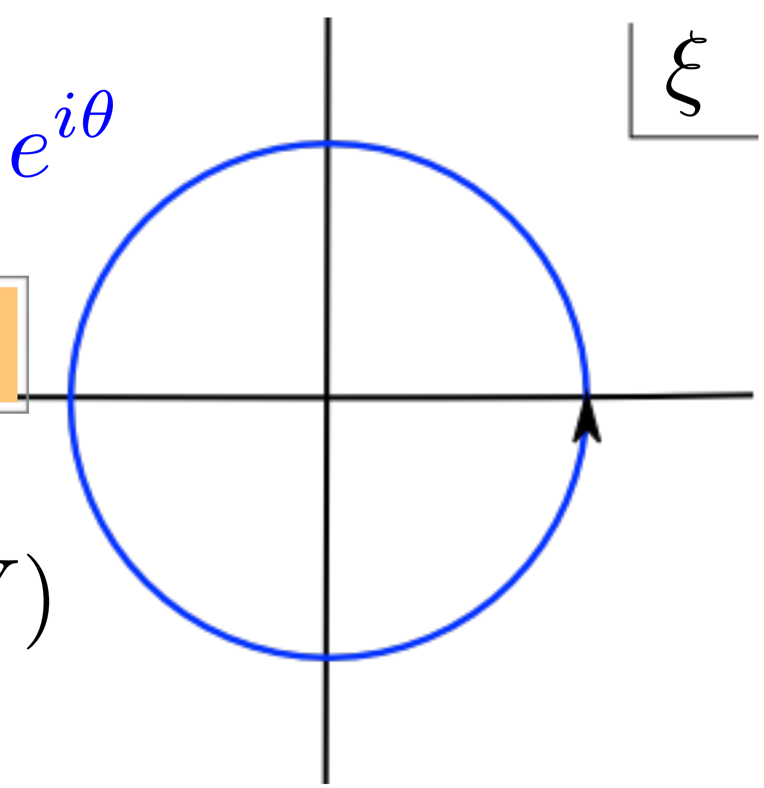

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Monte Carlo simulation

An implicit assumption

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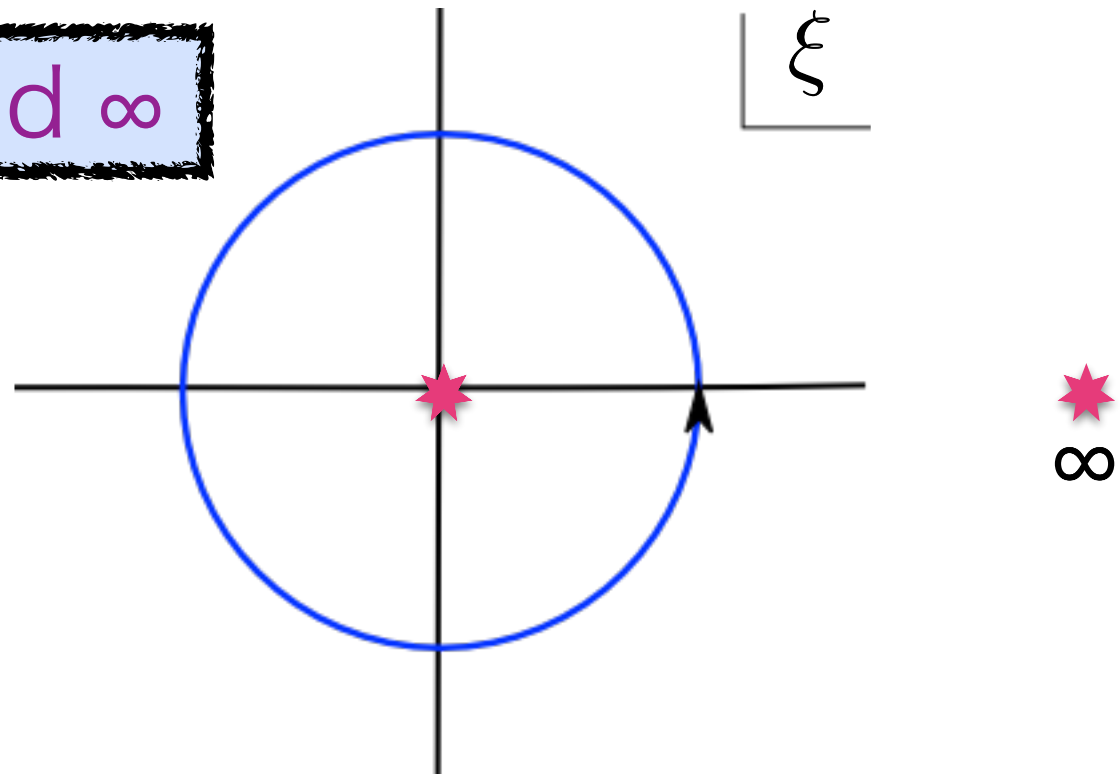
Singularities only at $\xi = 0$ and ∞



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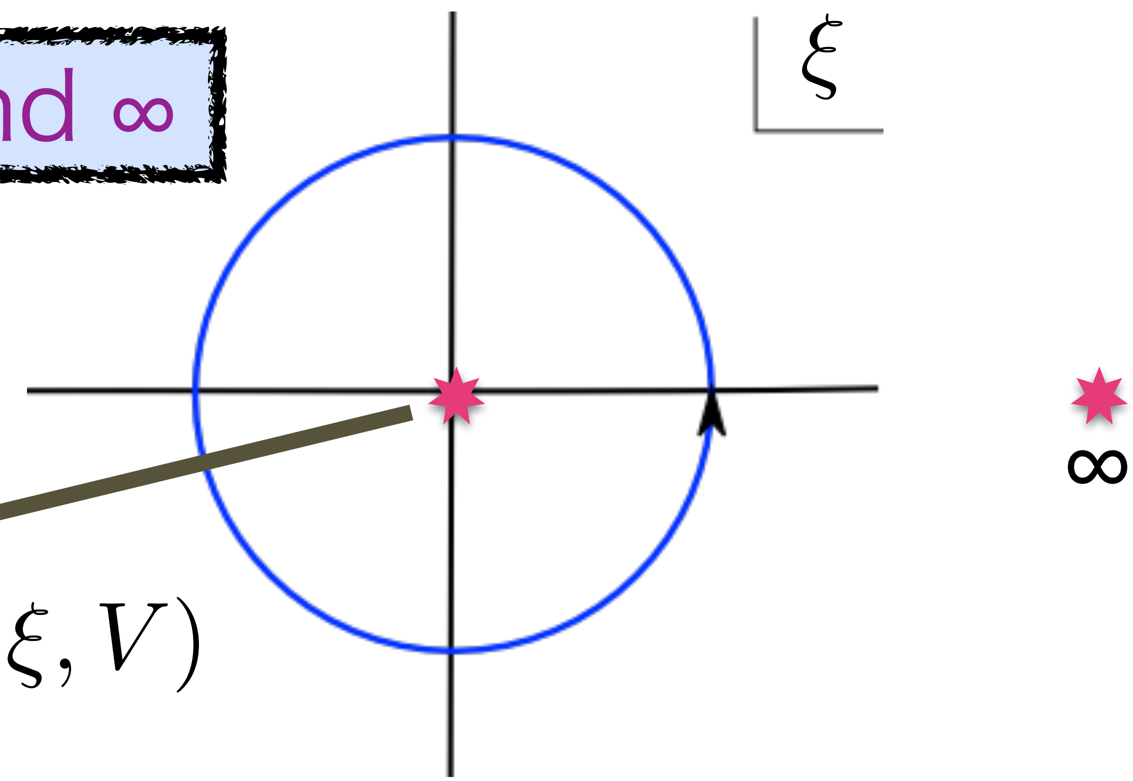

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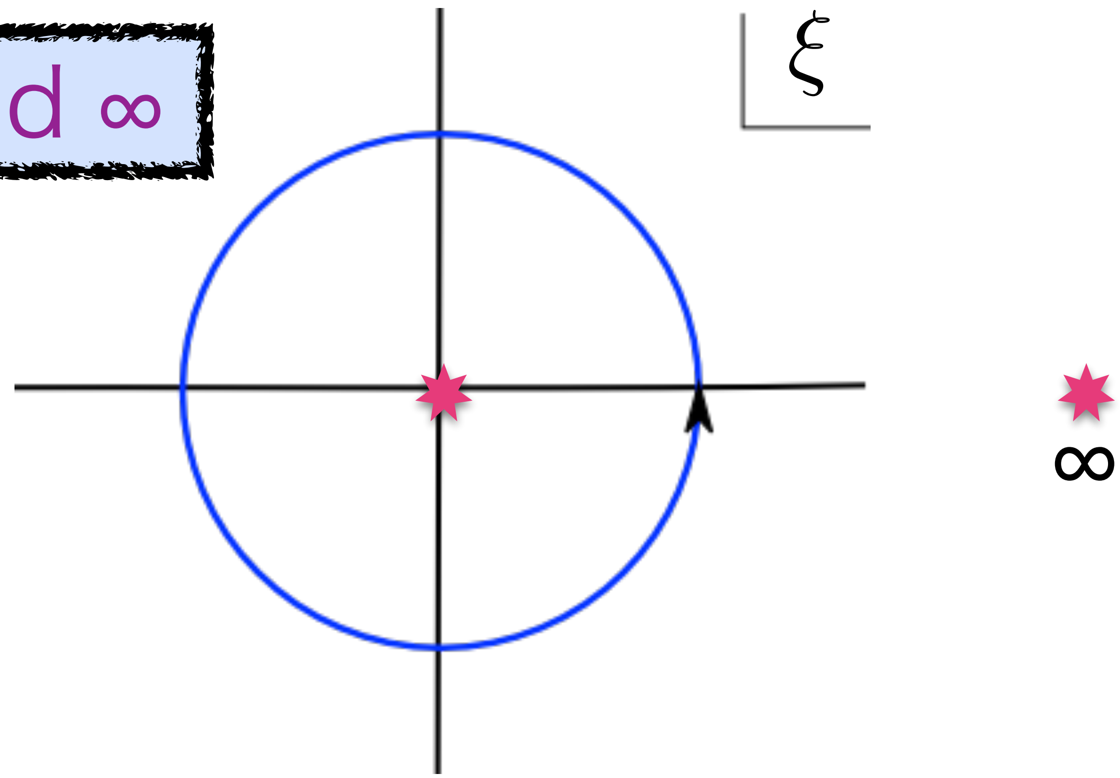

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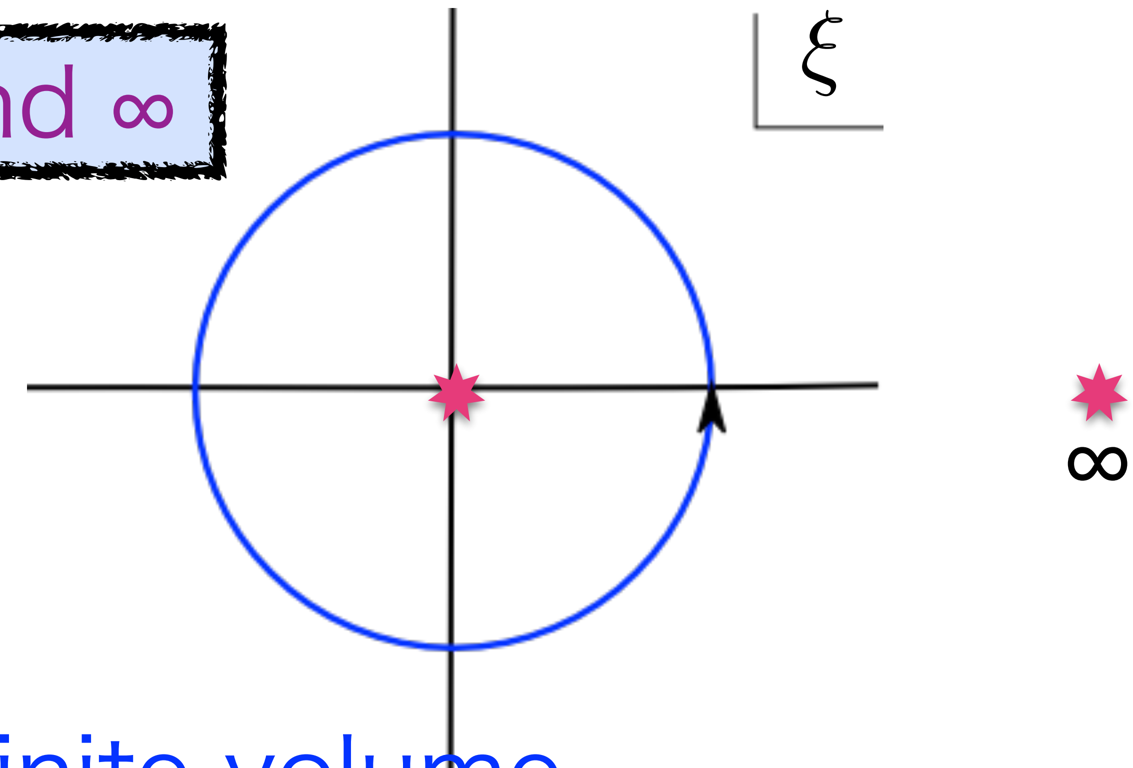

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at least for lattice QCD in finite volume

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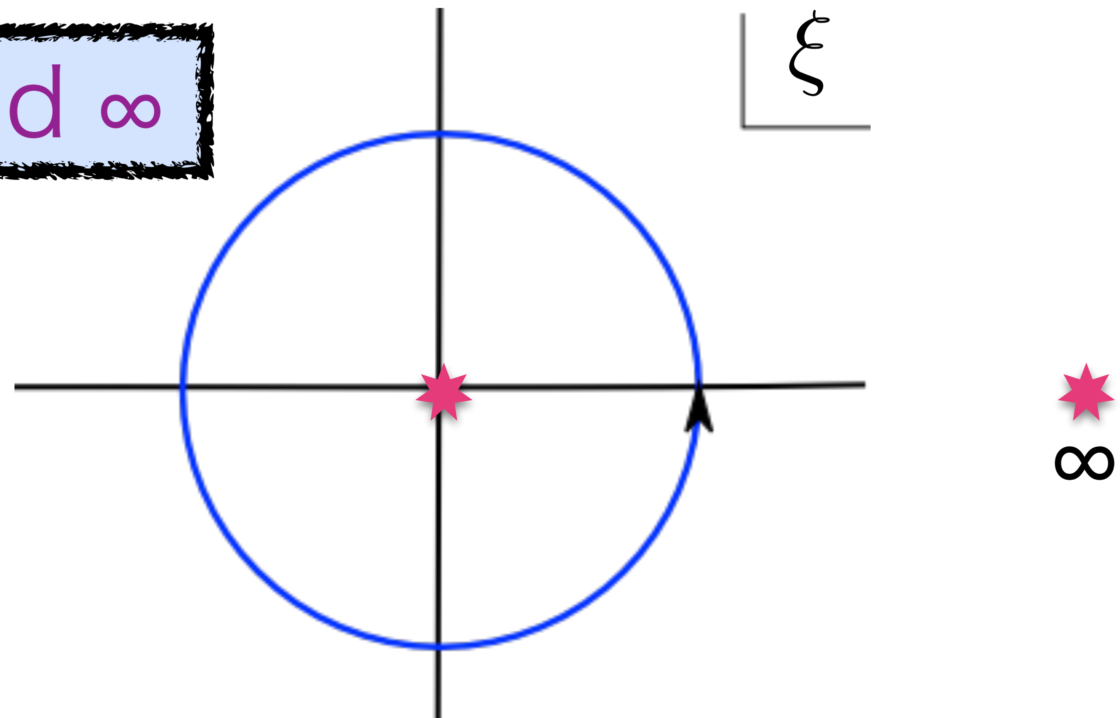

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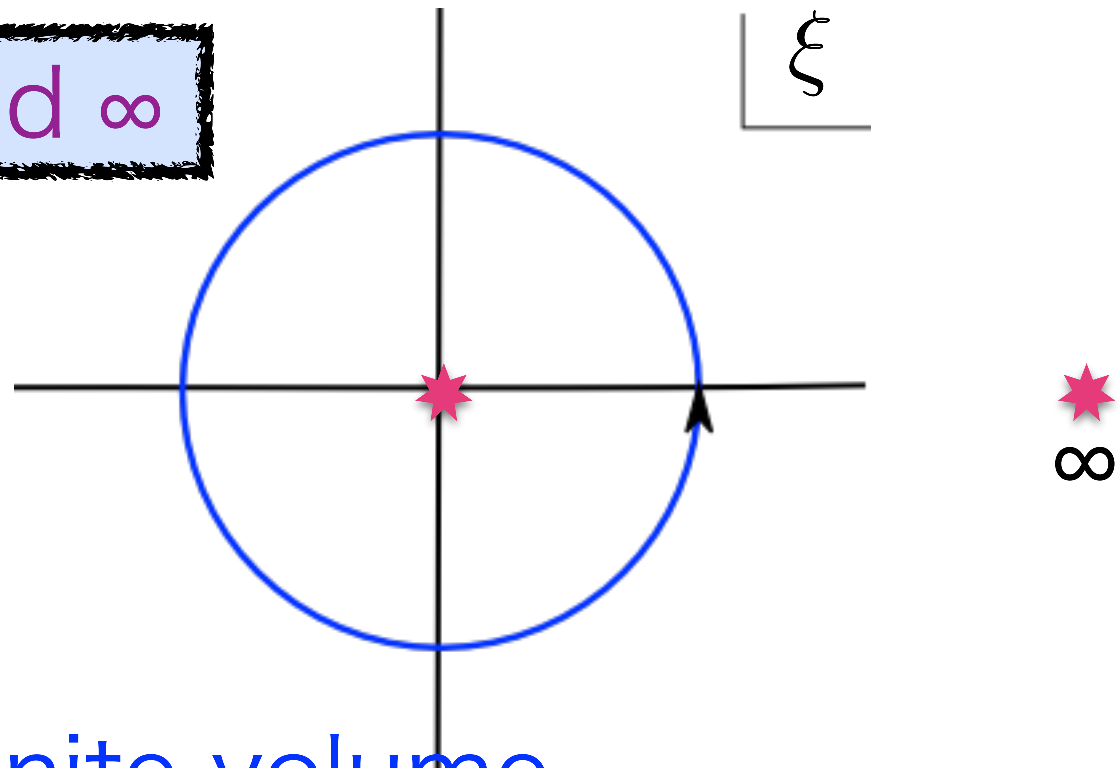

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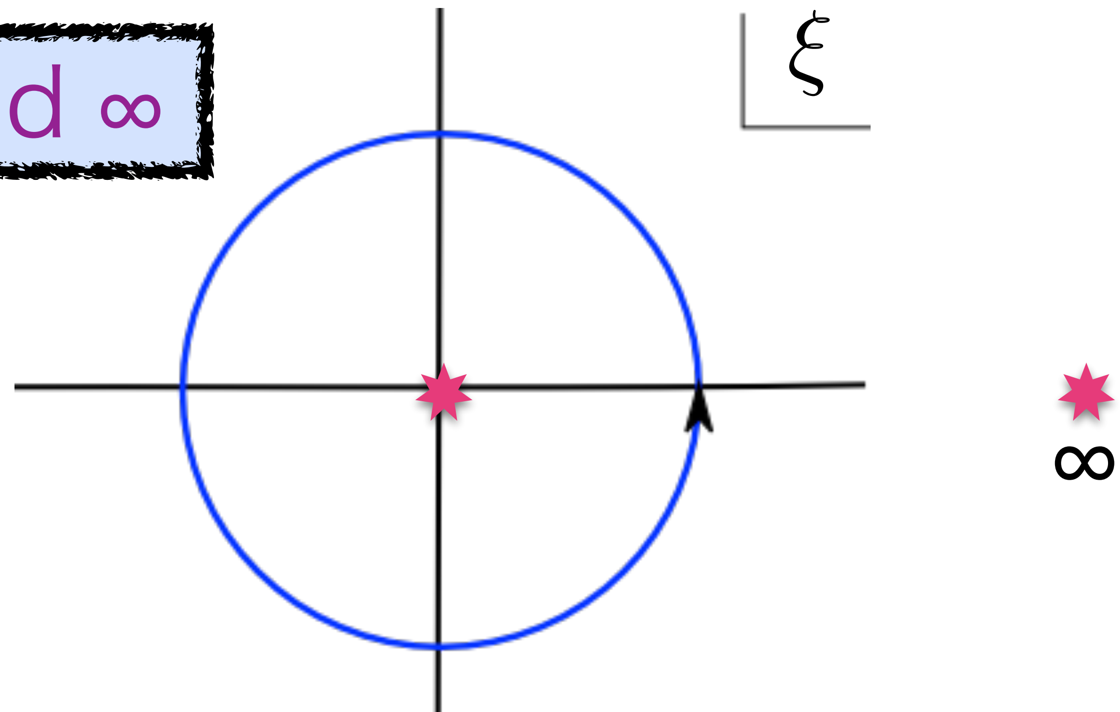

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How about the phase transition?

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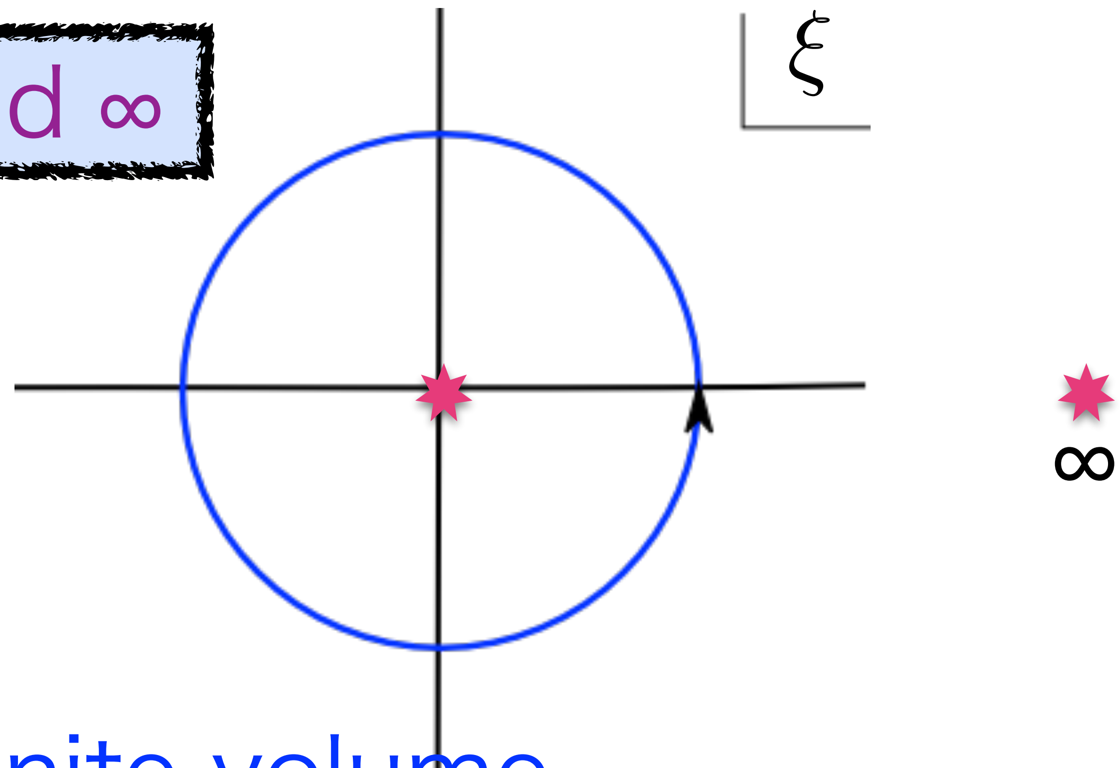

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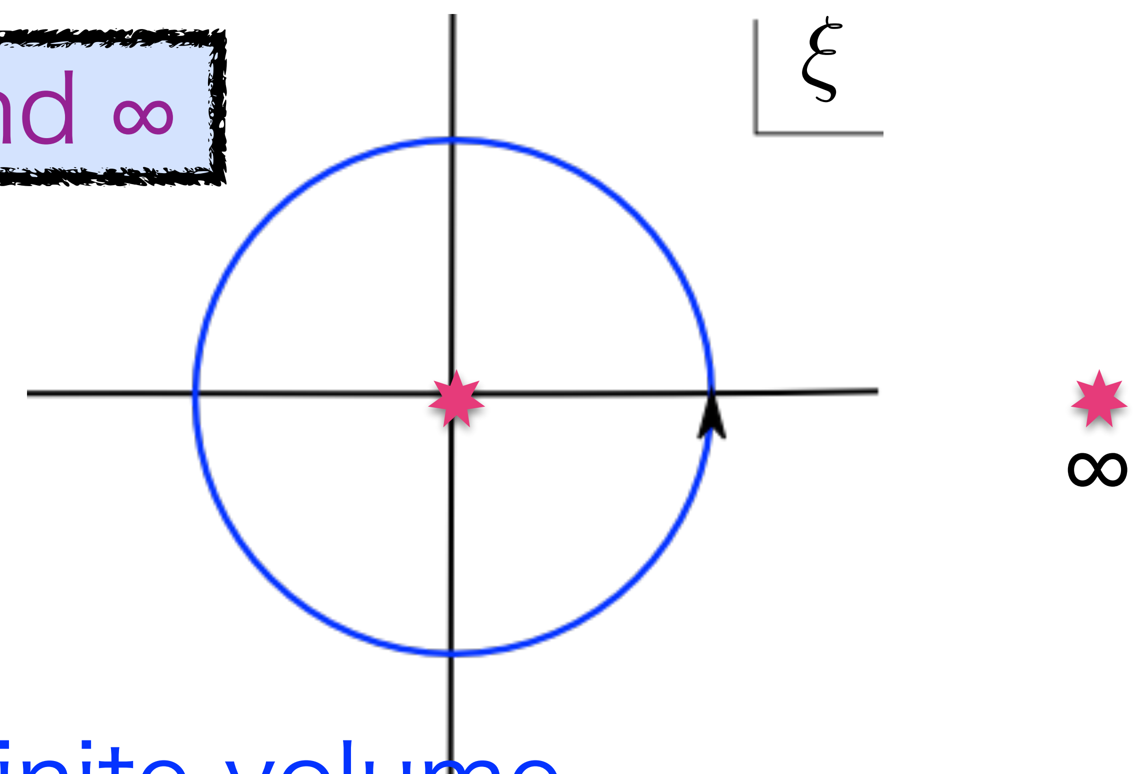

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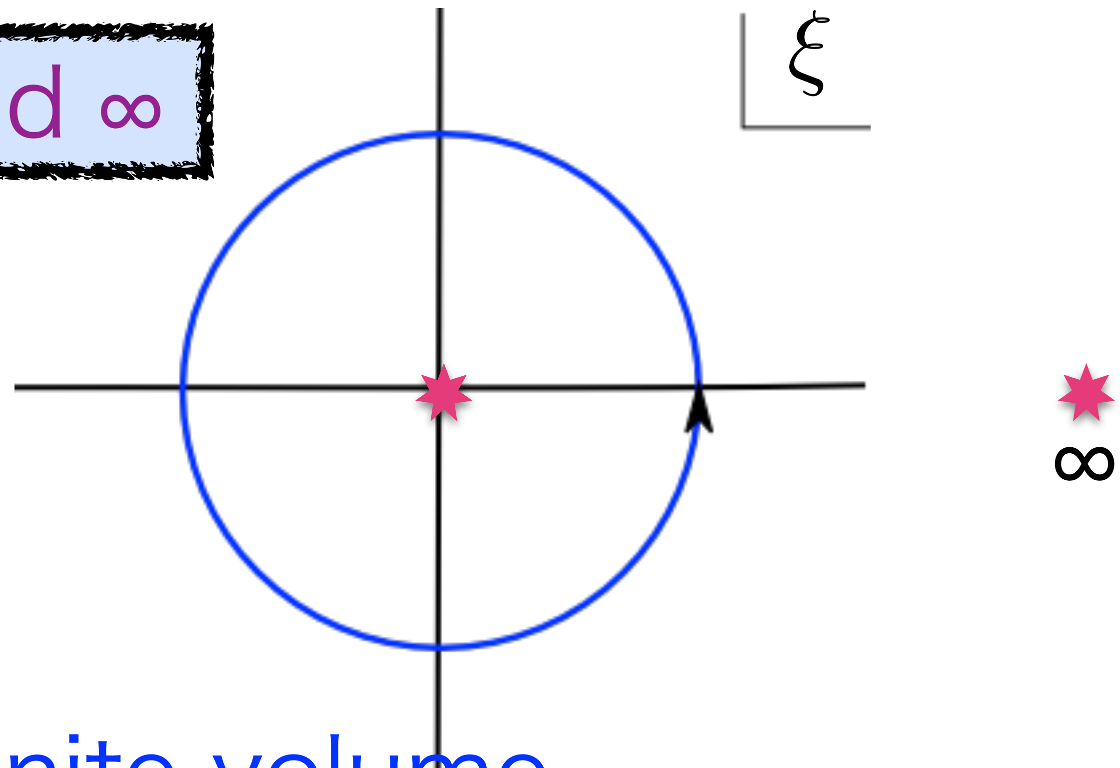
Lee-Yang zeros!

Analytic continuation is perfectly safe for ZG(ξ)!

Singularities only at ξ=0 and ∞



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$$Z_C\,(T, n, V) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} \underline{Z_G(T, e^{i\theta}, V)}$$

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There are two problems

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Partition function $Z_G(T,e^{i\theta},V)$ is numerically expensive

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Numerical Fourier transformation is difficult.

Frequent cancellation between plus-minus signs

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Easy way to solve these two!

Hopping parameter expansion

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Instability in Fourier tr. is solved

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If an analytic form of partition function $Z_G(\xi)$ is known

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Fugacity expansion of Dirac determinant Det $D(\xi)$

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Want to evaluate Dirac determinant cheaply!

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Lattice QCD Dirac operator

$$D_W(\mu) = 1 - \kappa Q_s + \kappa e^{\mu a} Q_4^+ + \kappa e^{-\mu a} Q_4^-$$

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$$\left(Q_\mu^+\right)_{nm} = (1 - \gamma_\mu)\, U_\mu(n) \delta_{m,n+\hat{\mu}}$$

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$$\left(Q_\mu^+\right)_{nm} = (1 - \gamma_\mu) U_\mu(n) \delta_{m,n+\hat{\mu}}$$

$$\left(Q_\mu^-\right)_{nm} = (1 + \gamma_\mu) U_\mu^\dagger(m) \delta_{m,n-\hat{\mu}}$$

Hopping parameter expansion

Instability in Fourier tr. is solved

↓

If an analytic form of partition function $Z_G(\xi)$ is known

Fugacity expansion of Dirac determinant Det D(ξ)

Want to evaluate Dirac determinant cheaply!

Lattice QCD Dirac operator

$$D_W(\mu) = 1 - \kappa Q_s + \kappa e^{\mu a} Q_4^+ + \kappa e^{-\mu a} Q_4^-$$

expansion in $\kappa = \dfrac{1}{2(ma+4)}$ ⇔ expansion in $e^{\pm \mu a}$

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Expand $\mathrm{TrLog} D_W(\mu)$

$$\mathrm{Log}\,(I - \kappa Q) = -\sum_n \frac{\kappa^n Q^n}{n}$$

Fugacity expansion of Dirac determinant Det D(ξ)

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⇒ Fugacity expansion of Dirac determinant Det D(ξ)

Meng et al. (Kentucky)

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Winding number expansion

$$\mathrm{Tr}\mathrm{Log}D_W(\mu)$$

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quark hopping need to make a loop for $\mathrm{Tr}Q^n$

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Non-zero winding in T direction

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→ non-trivial μ dependence

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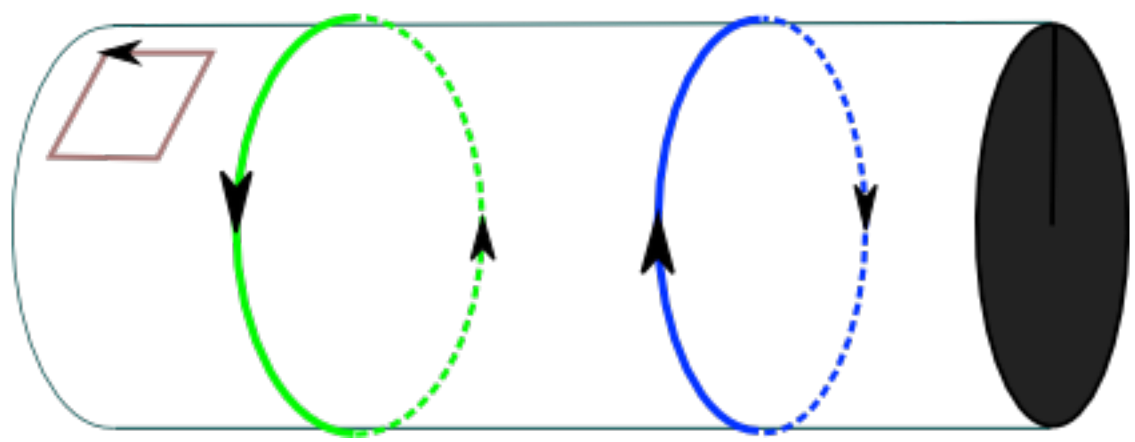

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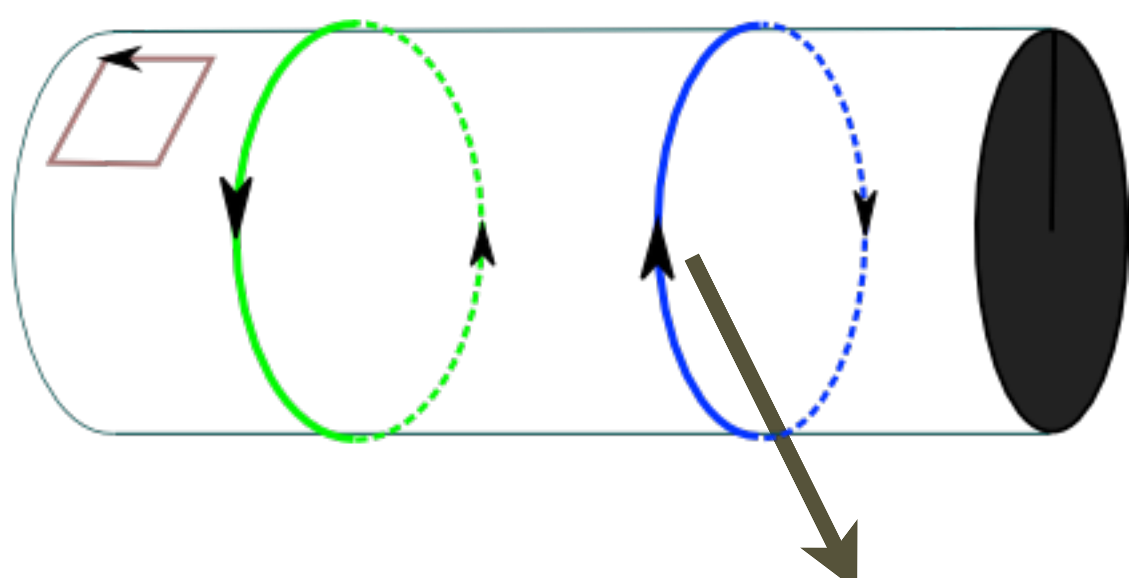

$$(e^{\mu a})^{N_t} = e^{\mu a N_t} = e^{\mu/T} = \xi$$

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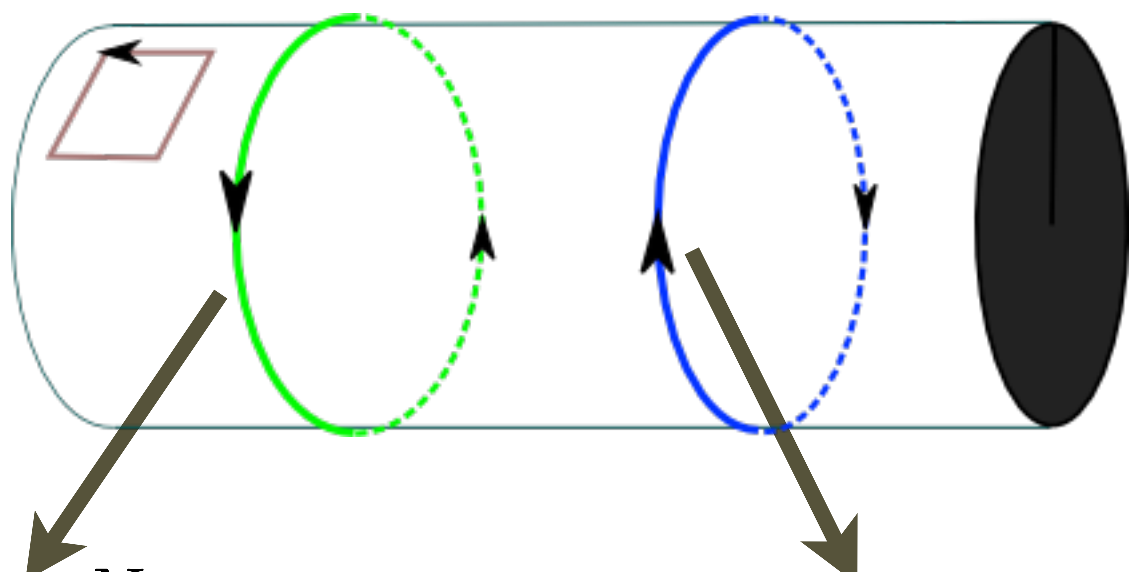

$$(e^{-\mu a})^{N_t} = \xi^{-1} \qquad (e^{\mu a})^{N_t} = e^{\mu a N_t} = e^{\mu/T} = \xi$$

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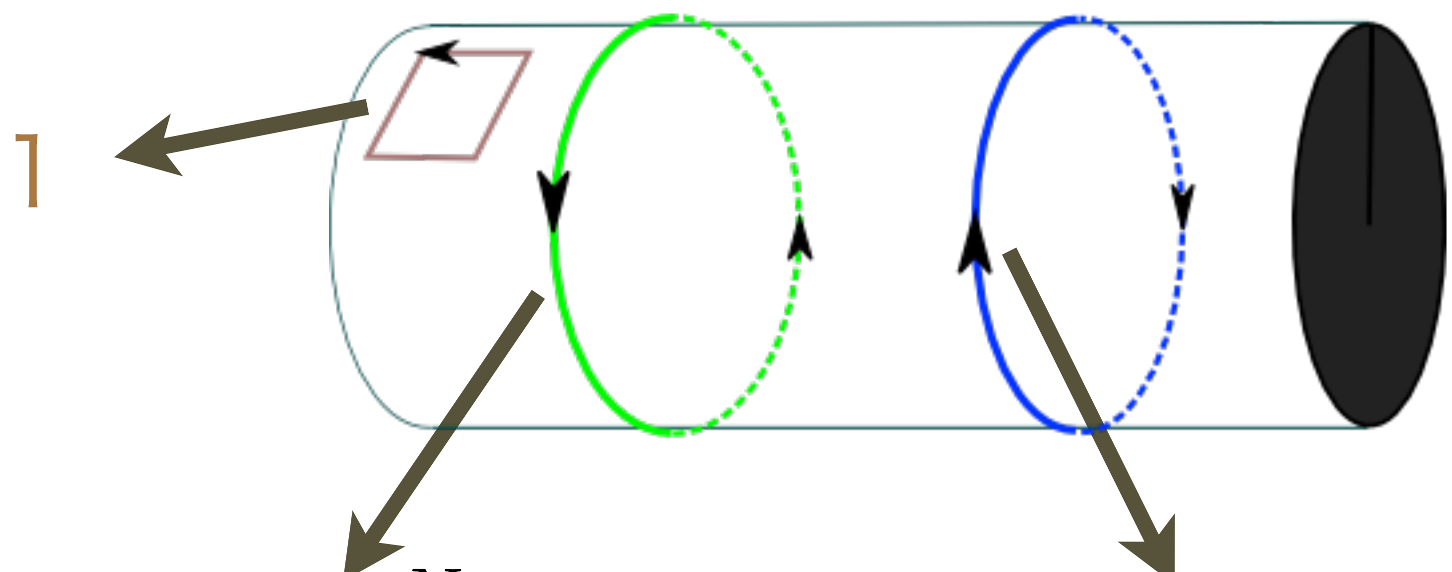

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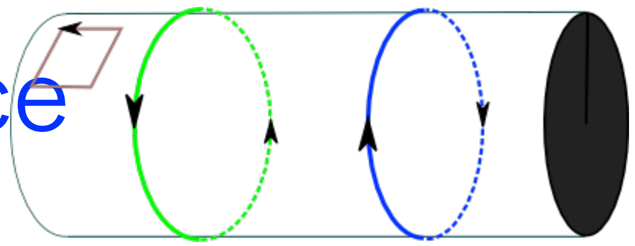

Count a number of windings for each $\mathrm{Tr}Q^n$

Winding number expansion

$$\mathrm{Tr}\mathrm{Log}D_W(\mu)$$

quark hopping need to make a loop for $\mathrm{Tr}Q^n$

Non-zero winding in T direction

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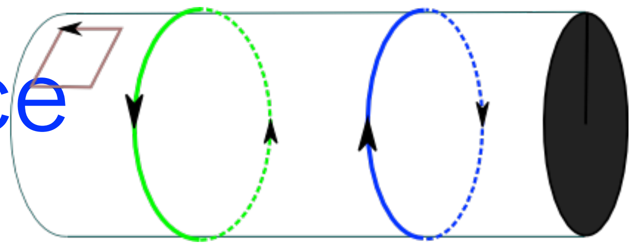

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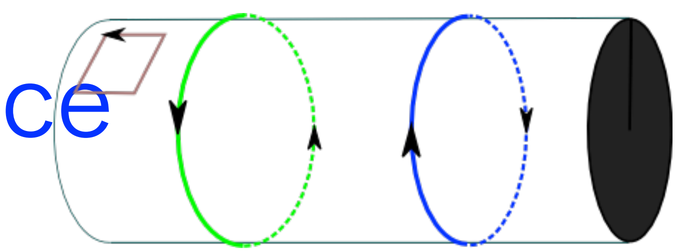

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resummation →

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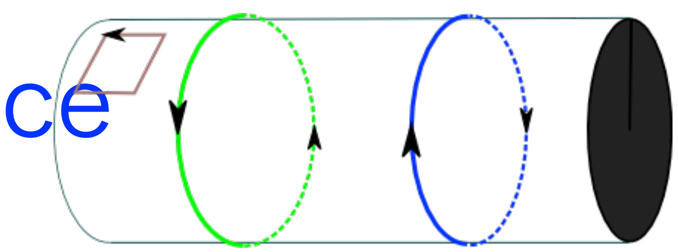

Count a number of windings for each $\mathrm{Tr}Q^n$

$$\mathrm{Tr}\mathrm{Log}\left(I-\kappa Q\right)=-\sum_{n=1}^{\infty}\frac{\kappa^n}{n}\mathrm{Tr}Q^n$$

resummation →
$$=\sum_{N=-\infty}^{\infty}W_N\xi^N$$

Kentucky '08

Canonical partition function Zc(n)

Kentucky '08

Grand partition fn. $Z_G(\mu)$ ← re-weighting

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hopping parameter exp.

$$= \left\langle \frac{\exp\left(\sum_{k=-\infty}^{\infty} W_k \xi^k\right)}{\mathrm{Det} D_W(\mu_0)} \right\rangle_0 Z_G(\mu_0)$$

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hopping parameter exp.

$$\sum_{n=-\infty}^{\infty} Z_C(n)\xi^n = \left\langle \frac{\exp\left(\sum_{k=-\infty}^{\infty} W_k \xi^k\right)}{\mathrm{Det} D_W(\mu_0)} \right\rangle_0 Z_G(\mu_0)$$

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Fourier Transformation

$$Z_C(n) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} \left\langle \frac{\exp\left(\sum_{k=-\infty}^{\infty} W_k e^{ik\theta}\right)}{\mathrm{Det} D_W(\mu_0)} \right\rangle_0$$

Plan of the talk

✔ 1. Introduction

✔ 2. Hopping parameter expansion

3. Numerical setup

4. Canonical partition function Zn

5. Hadronic observables

6. Conclusion

Numerical setup

★ Iwasaki gauge action

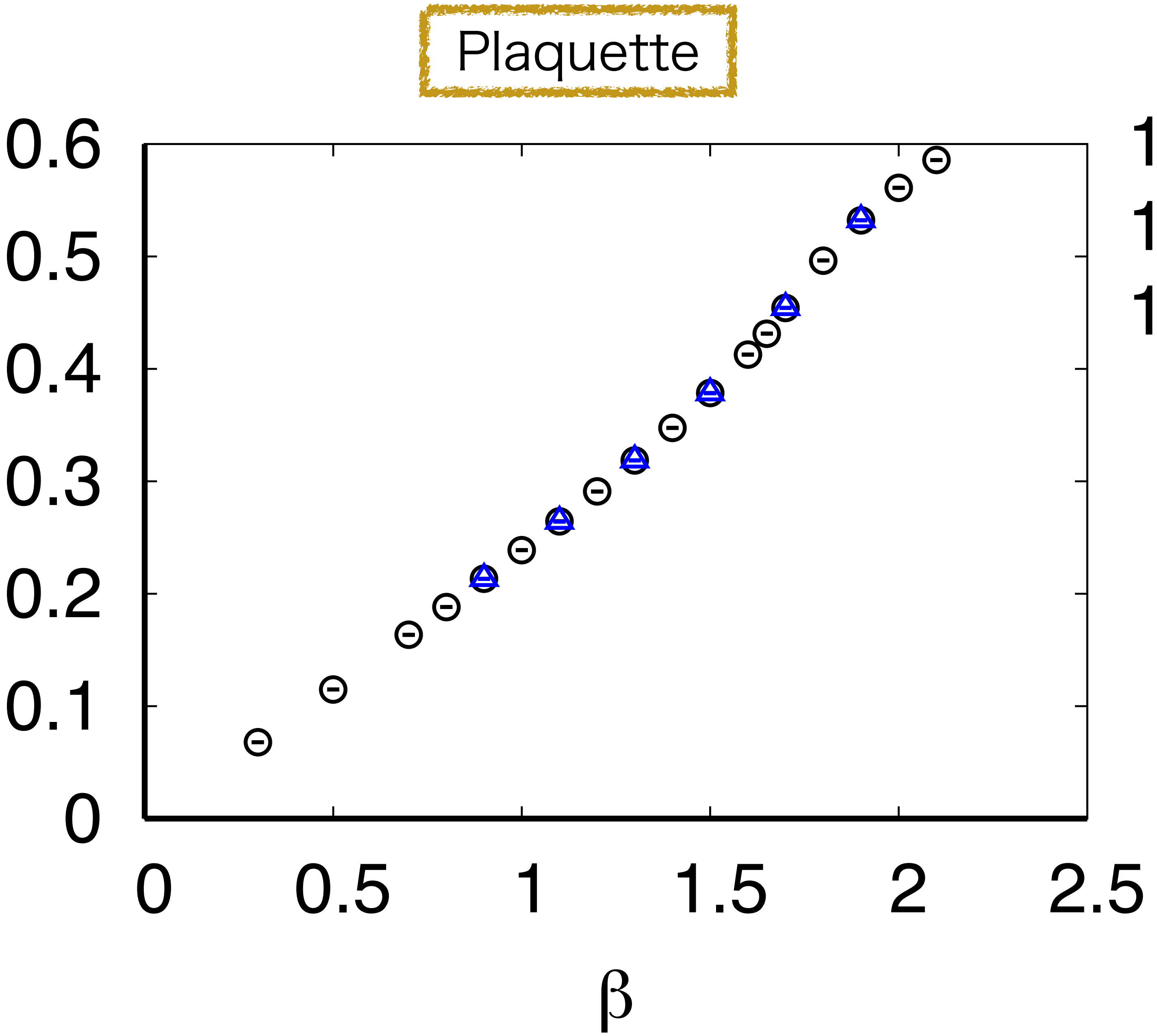
★ Clover fermion Nf=2

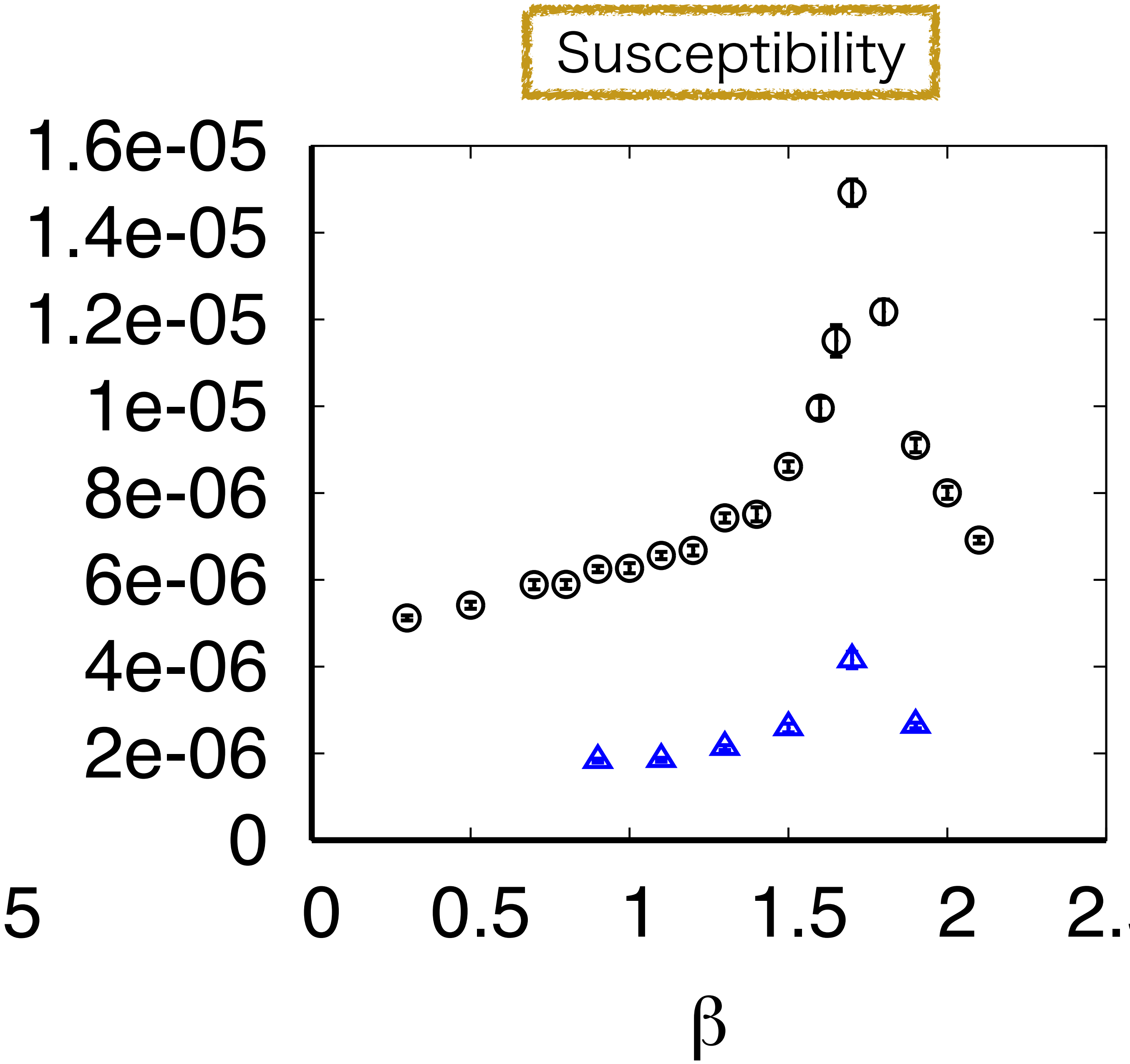
- APE stout smeared gauge link $c_{\mathrm{SW}} = 1.1$

★ Box sizes $8^3 \times 4$ $12^3 \times 4$

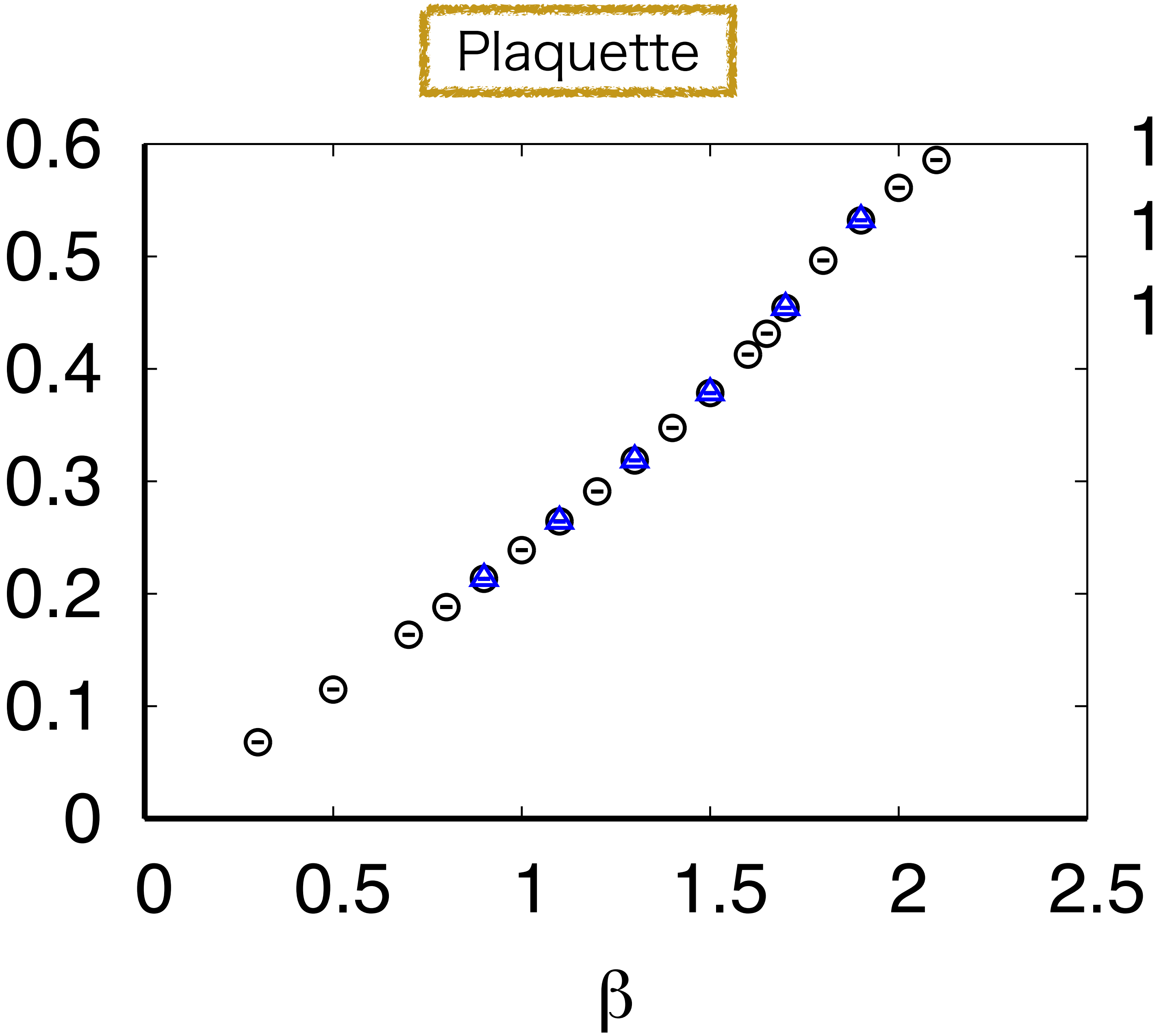
β	T/Tc	κ	mπ/mρ
0.9	0.67	0.137	0.8978(55)
1.1	0.69	0.133	0.9038(56)
1.3	0.72	0.138	0.809(12)
1.5	0.78	0.136	0.756(13)
1.7	1	0.129	0.770(13)
1.9	1.46	0.125	0.714(15)
2.1	3.22	0.122	0.836(47)

Plaquette

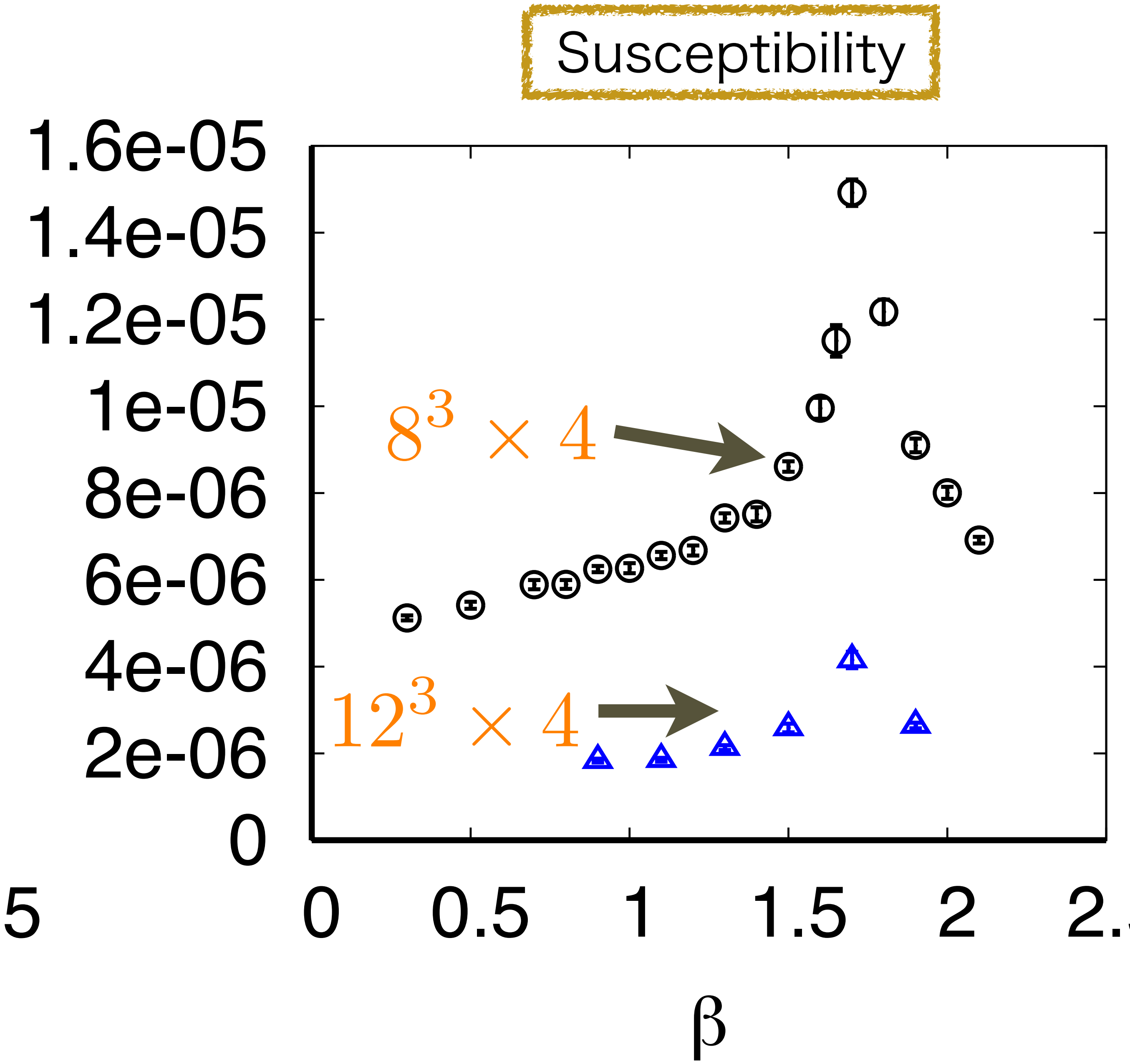

Plaquette
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
β


Susceptibility
1.6e-05
1.4e-05
1.2e-05
1e-05
8e-06
6e-06
4e-06
2e-06
0
0
0.5
1
1.5
2
β

Plaquette

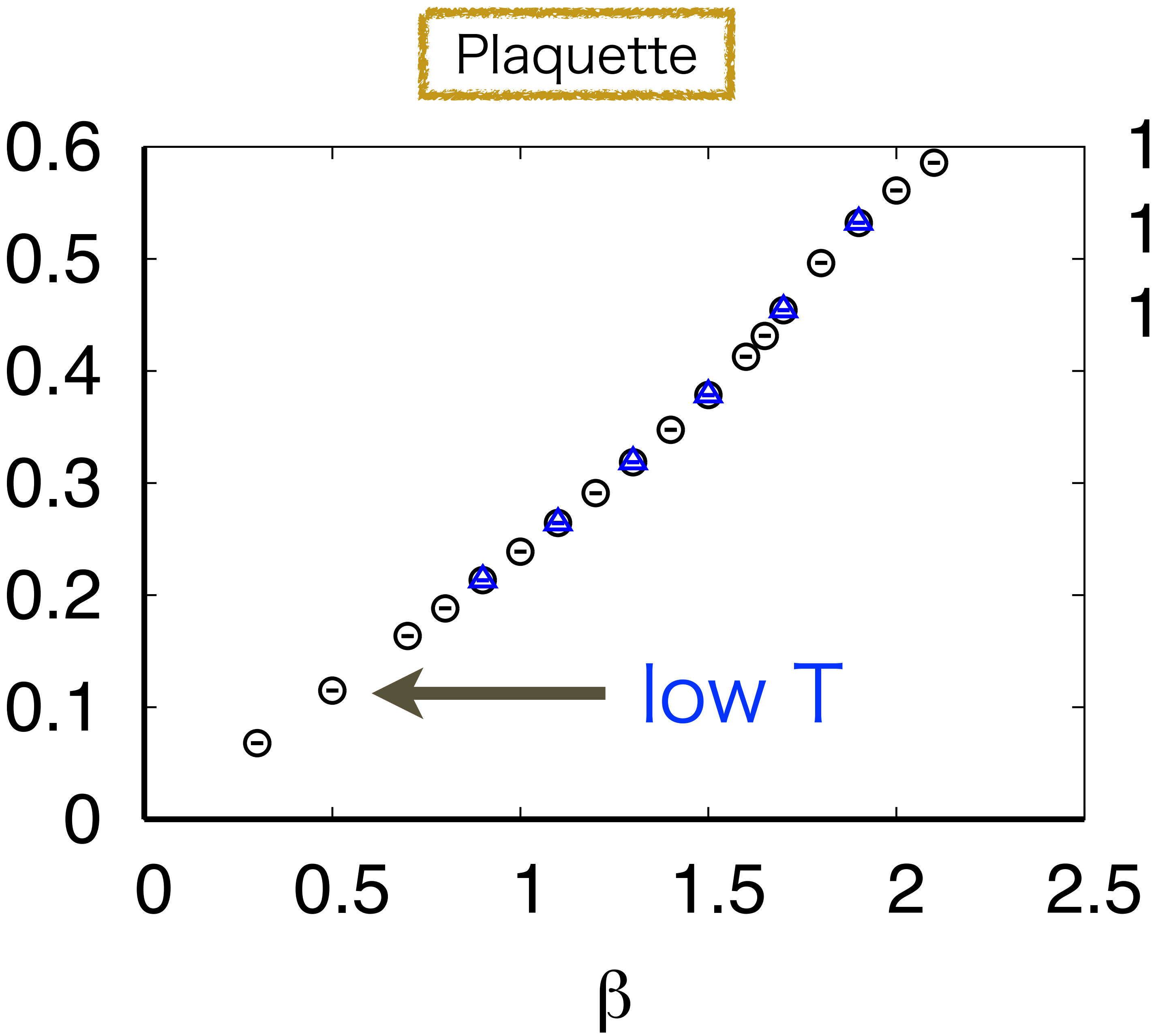


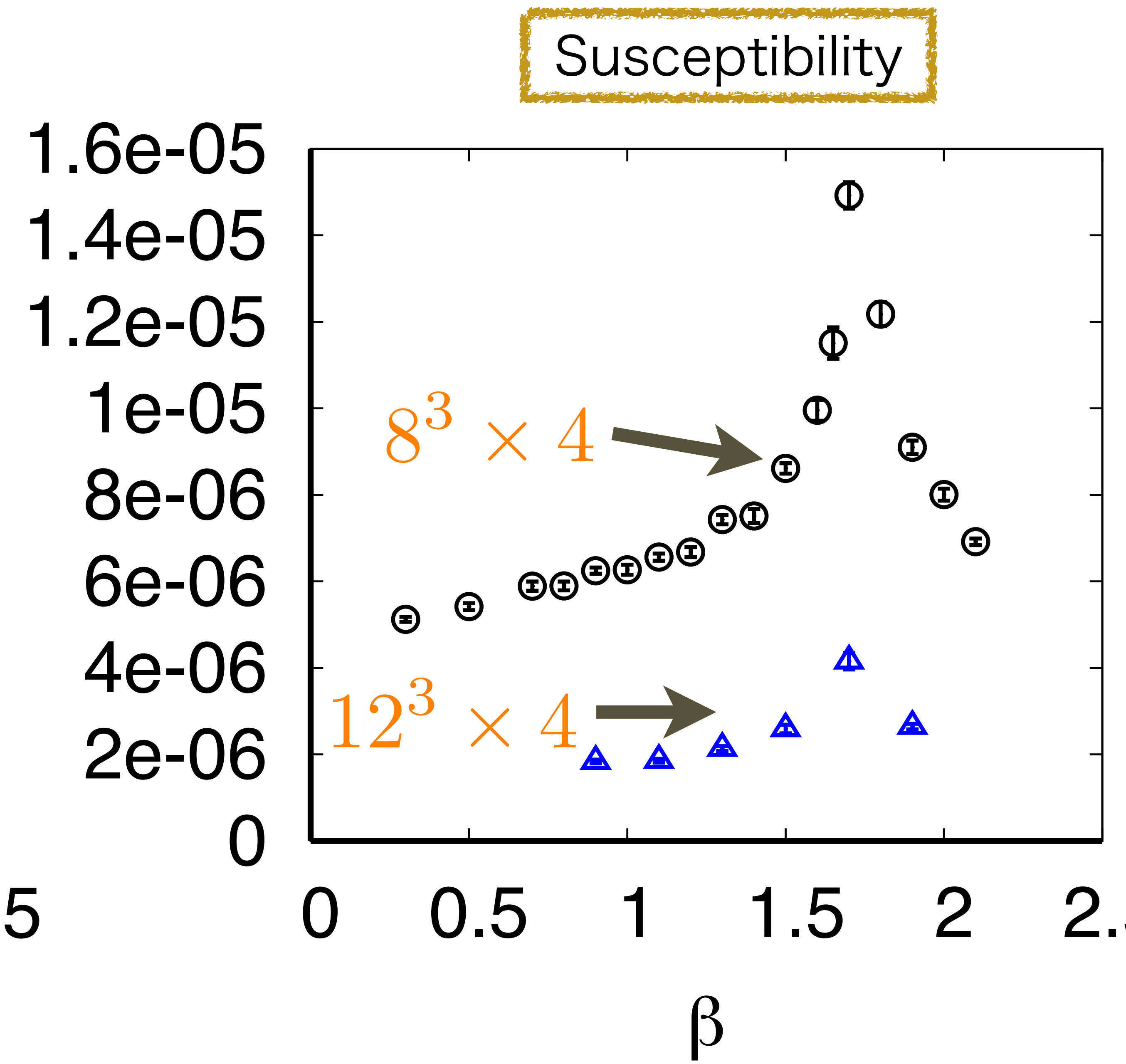
Plaquette
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
β



Susceptibility
1.6e-05
1.4e-05
1.2e-05
1e-05
8e-06
6e-06
4e-06
2e-06
0
$8^3 \times 4$
$12^3 \times 4$
0
0.5
1
1.5
2
β

Plaquette


Plaquette
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
β
low T


Susceptibility
1.6e-05
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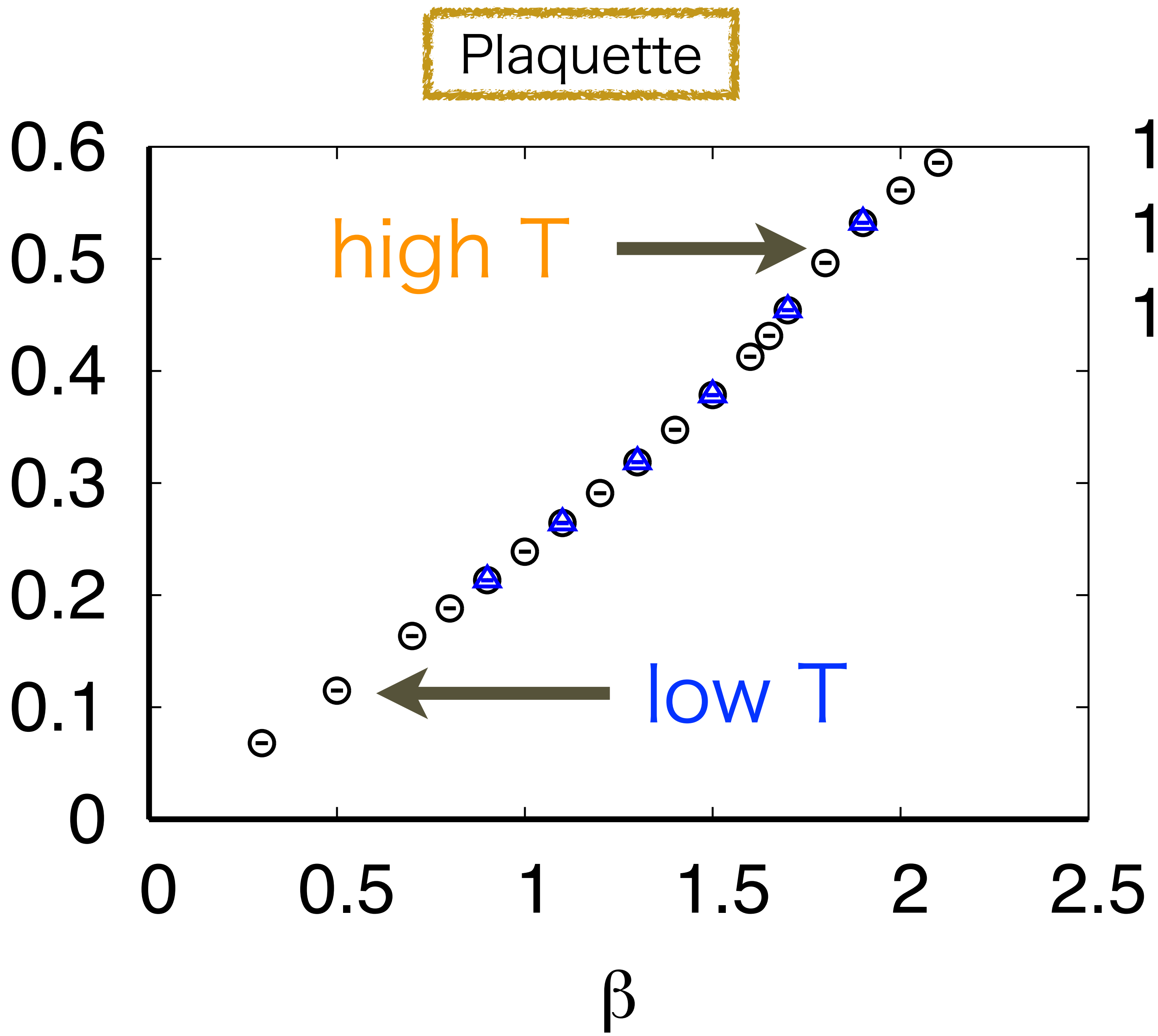
1.2e-05
1e-05
8e-06
6e-06
4e-06
2e-06
0
0
0.5
1
1.5
2
β
$8^3 \times 4$
$12^3 \times 4$

Plaquette


Plaquette
0.6
0.5
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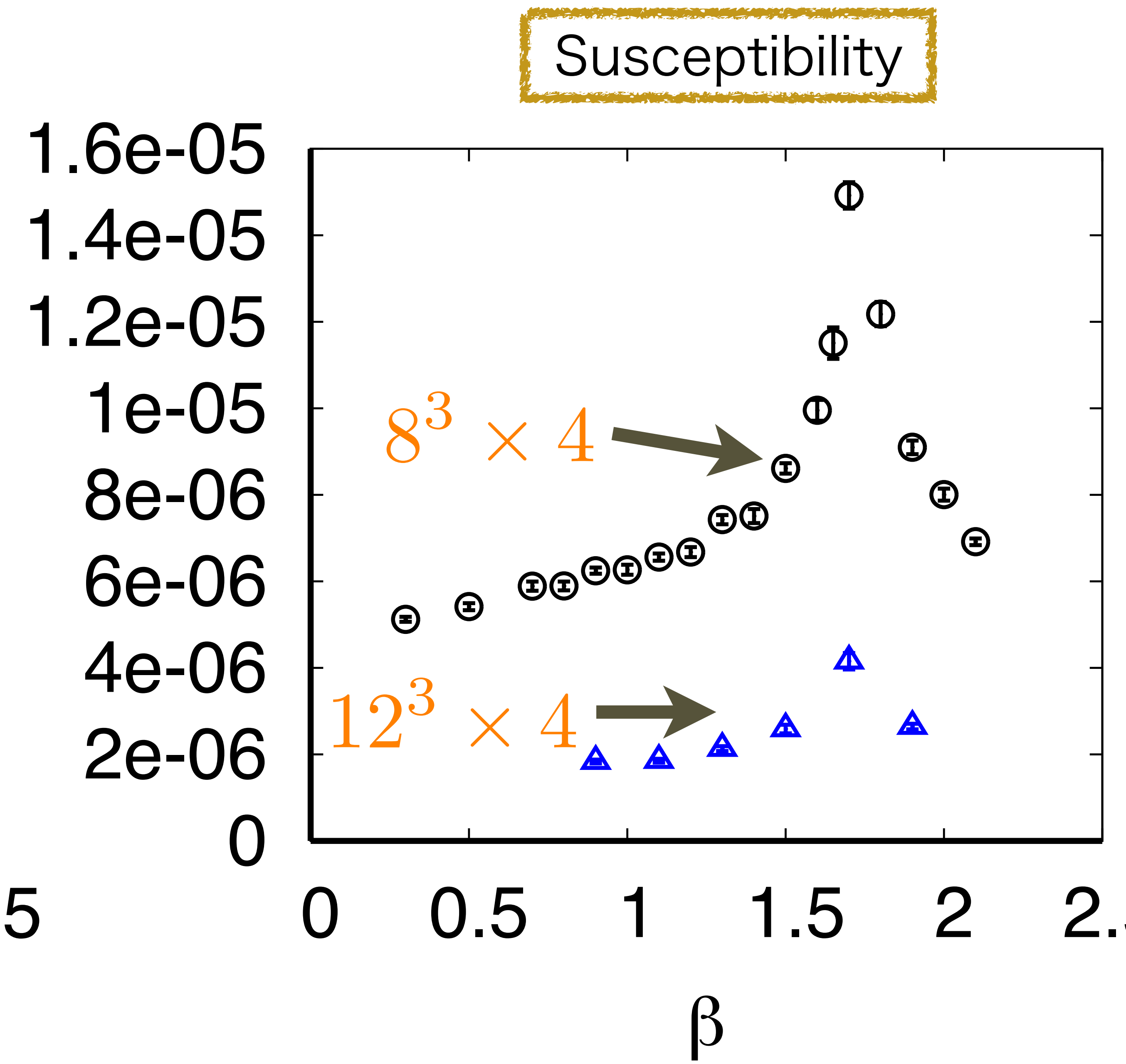
1.5
2
2.5
β
high T
low T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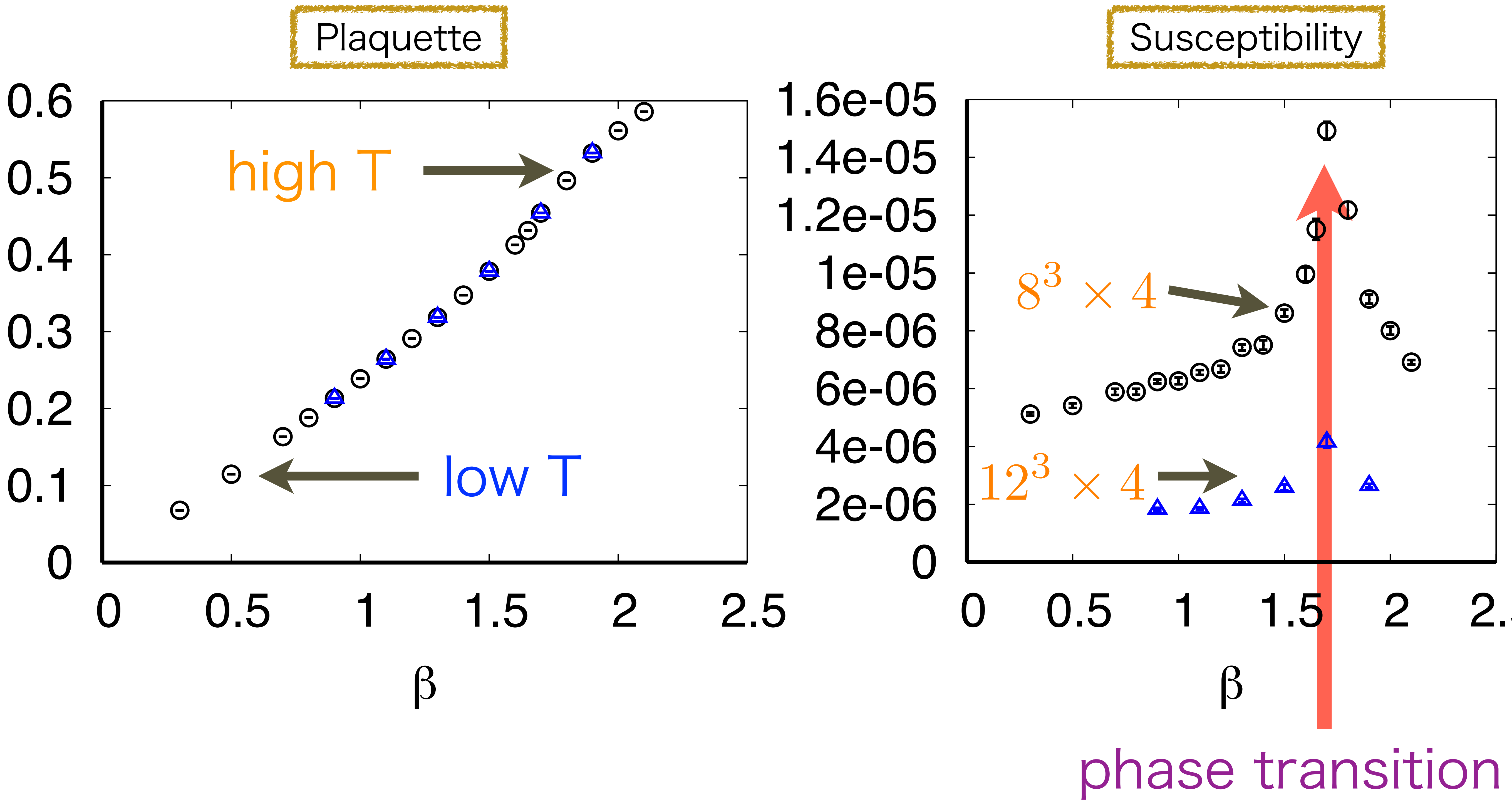
β
$8^3 \times 4$
$12^3 \times 4$

Plaquette


Plaquette
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0.5
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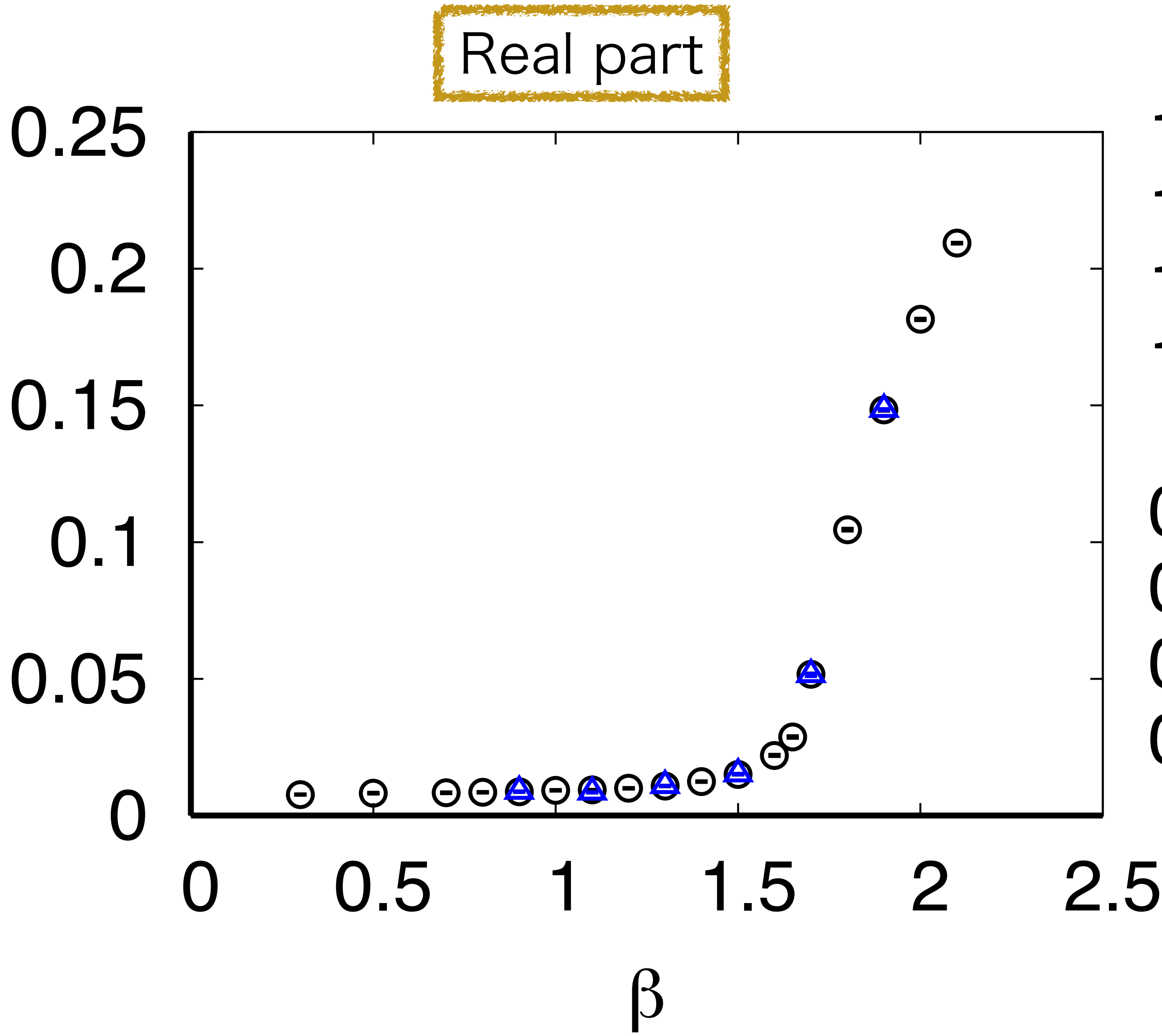
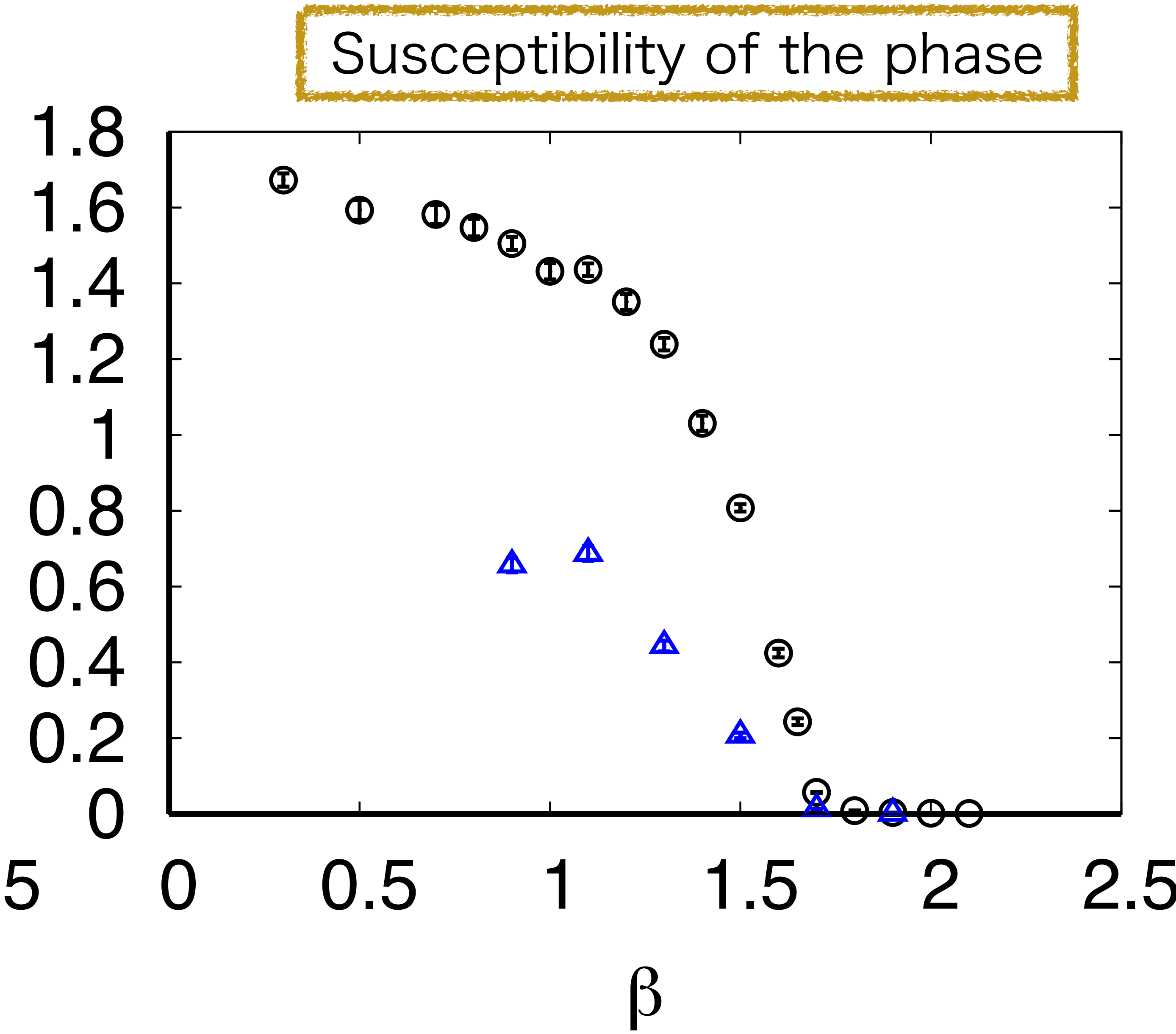
1.5
2
2.5
β
high T
low T
Susceptibility
1.6e-05
1.4e-05
1.2e-05
1e-05
8e-06
6e-06
4e-06
2e-06
0
0
0.5
1
1.5
2
β
$8^3 \times 4$
$12^3 \times 4$
phase transition

Polyakov loop





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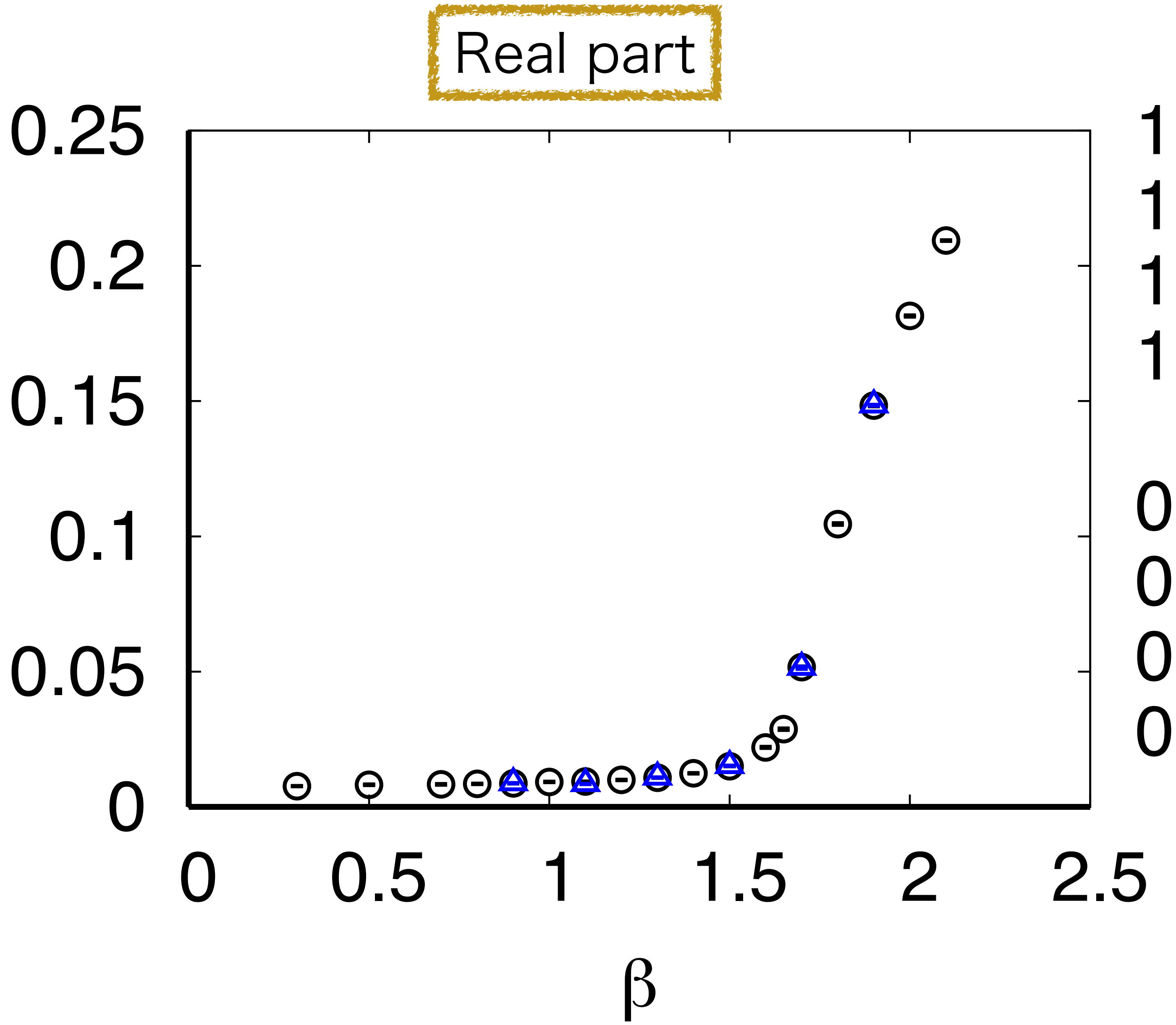
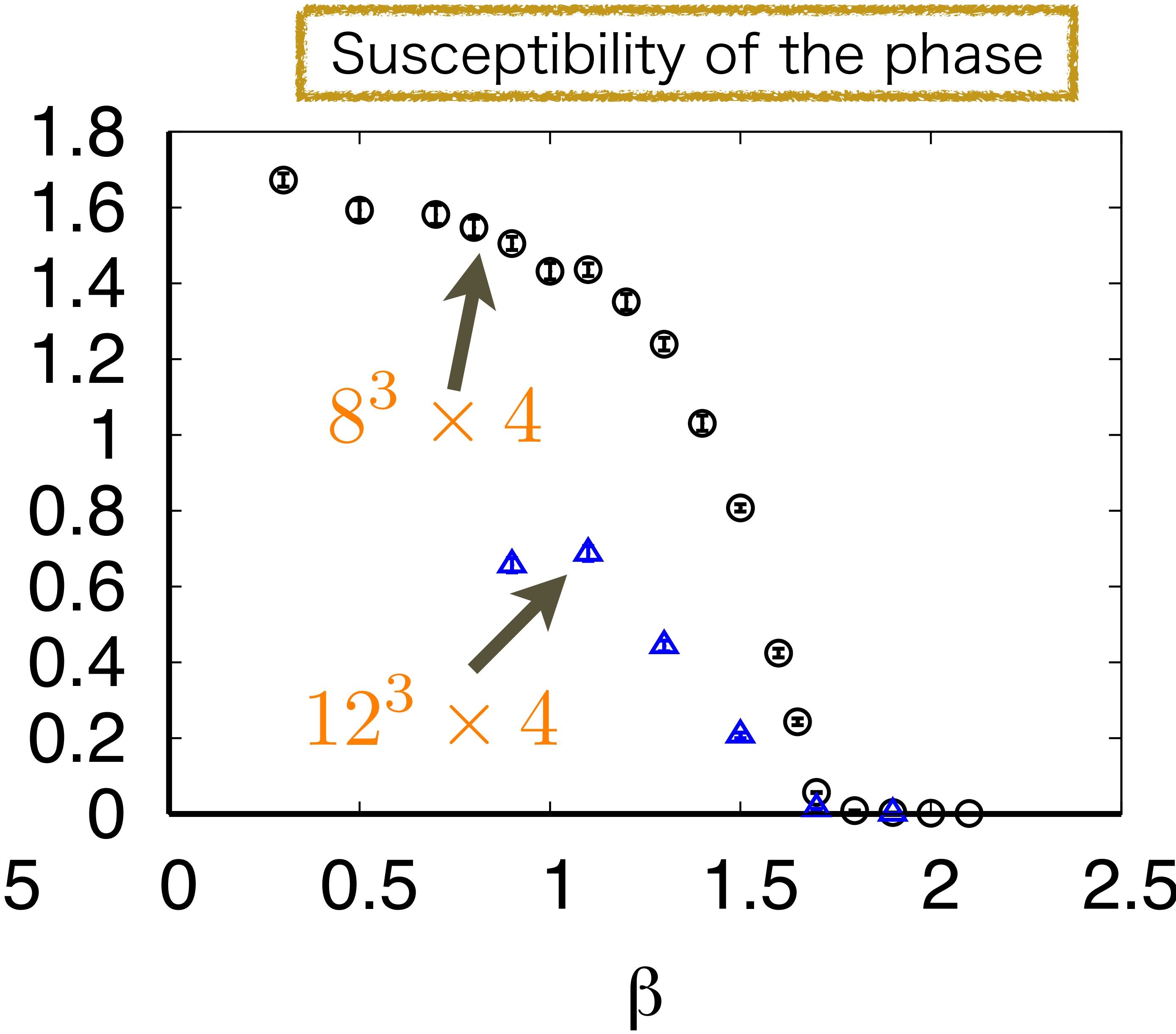
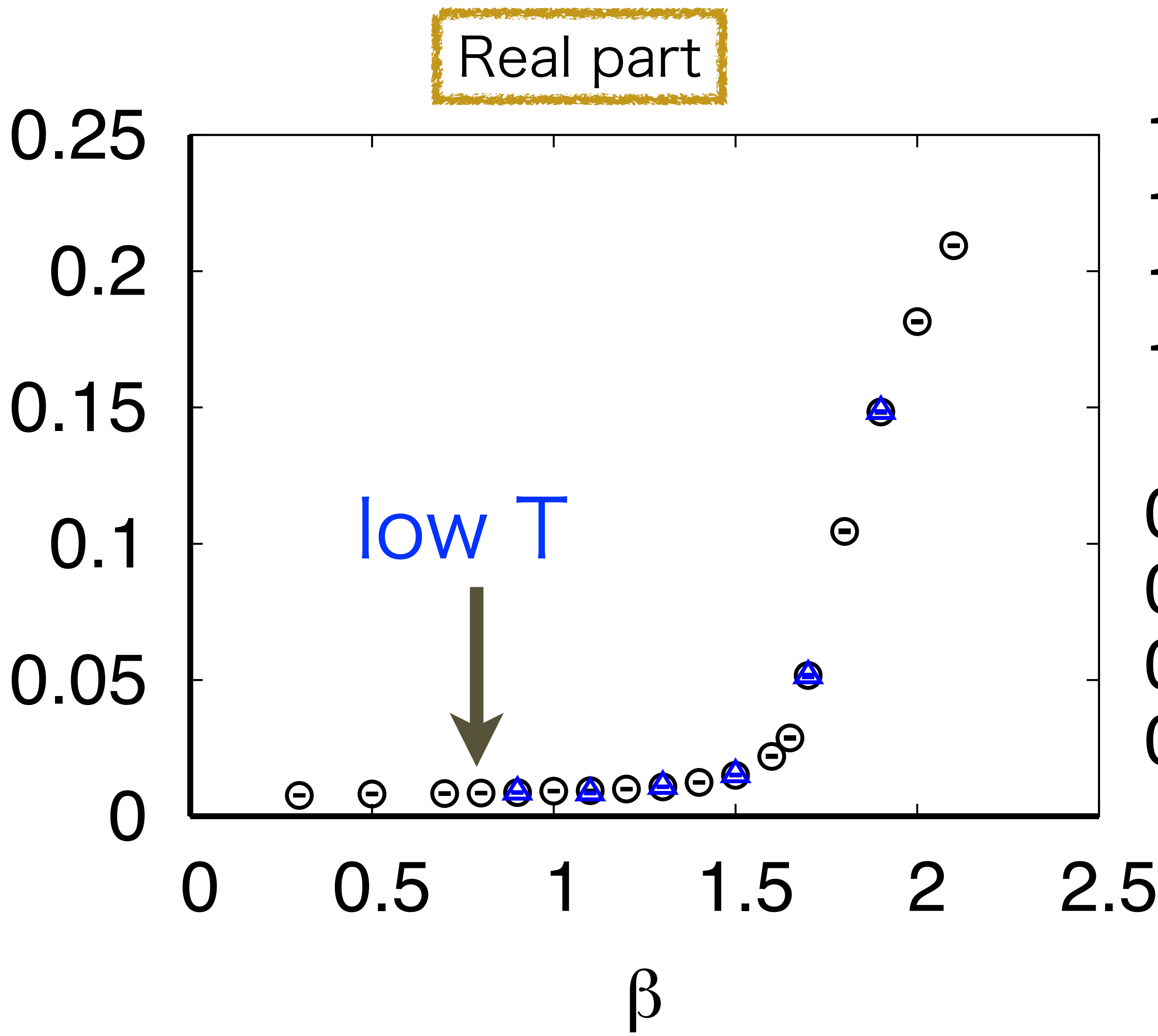
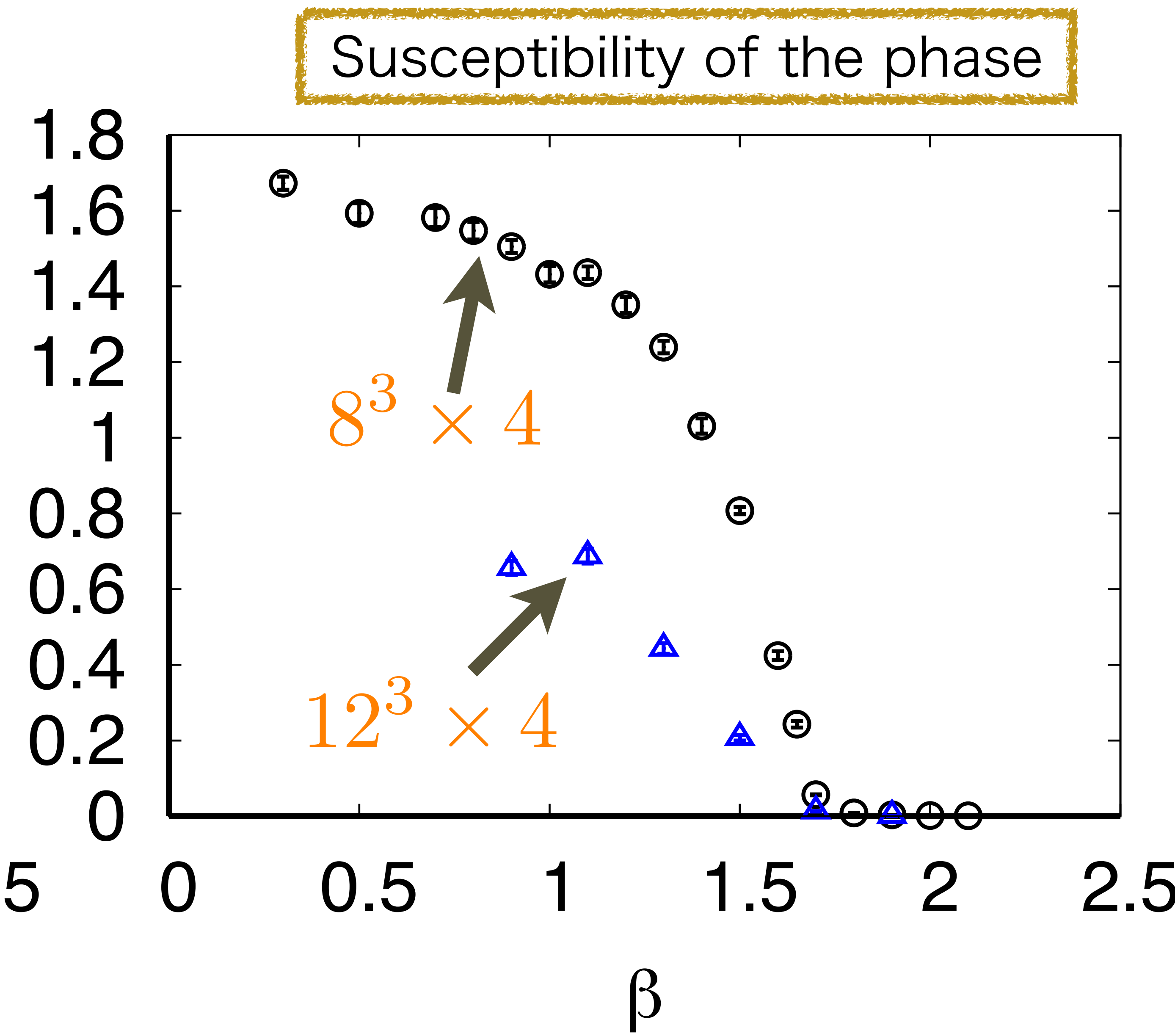




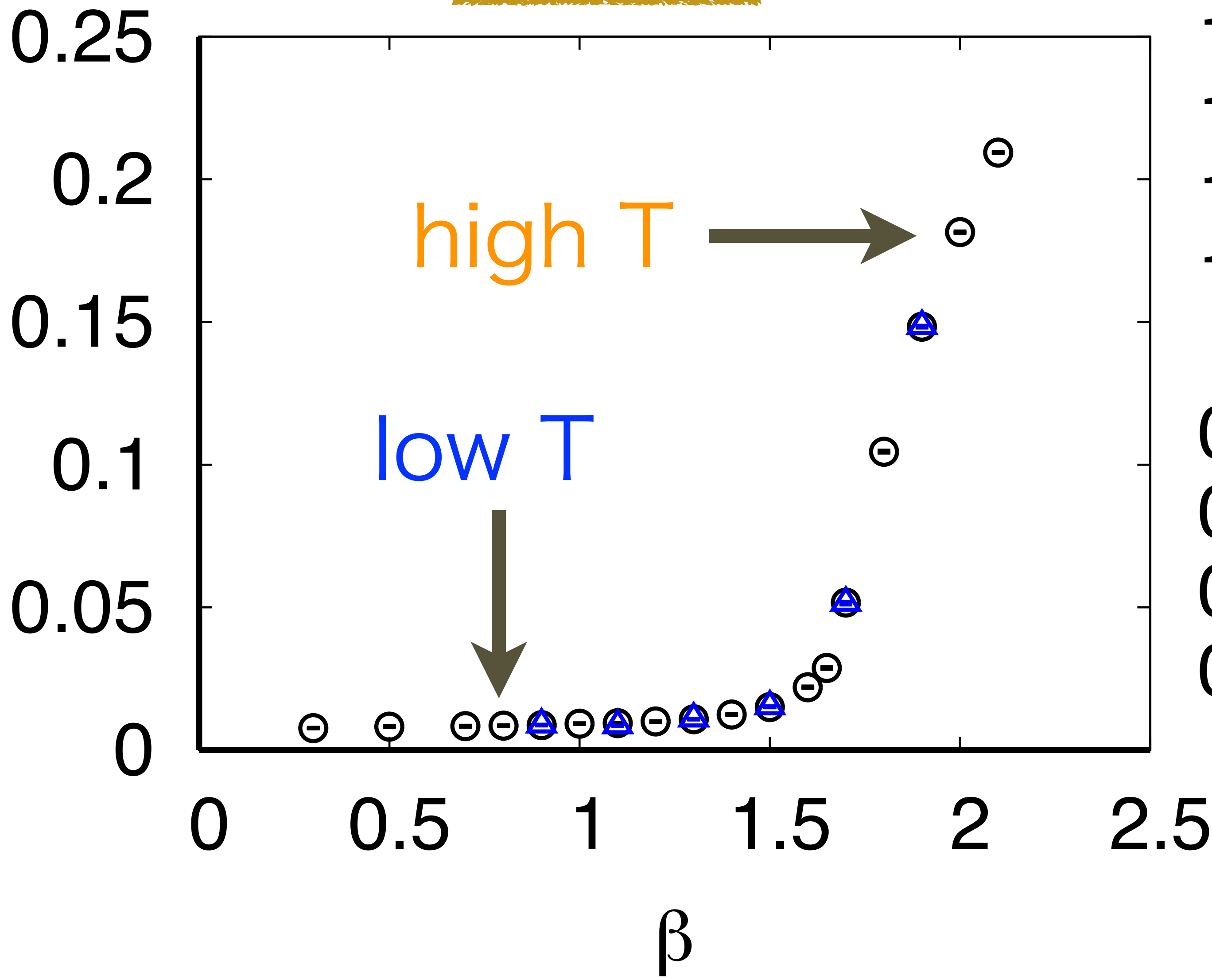
Polyakov loop



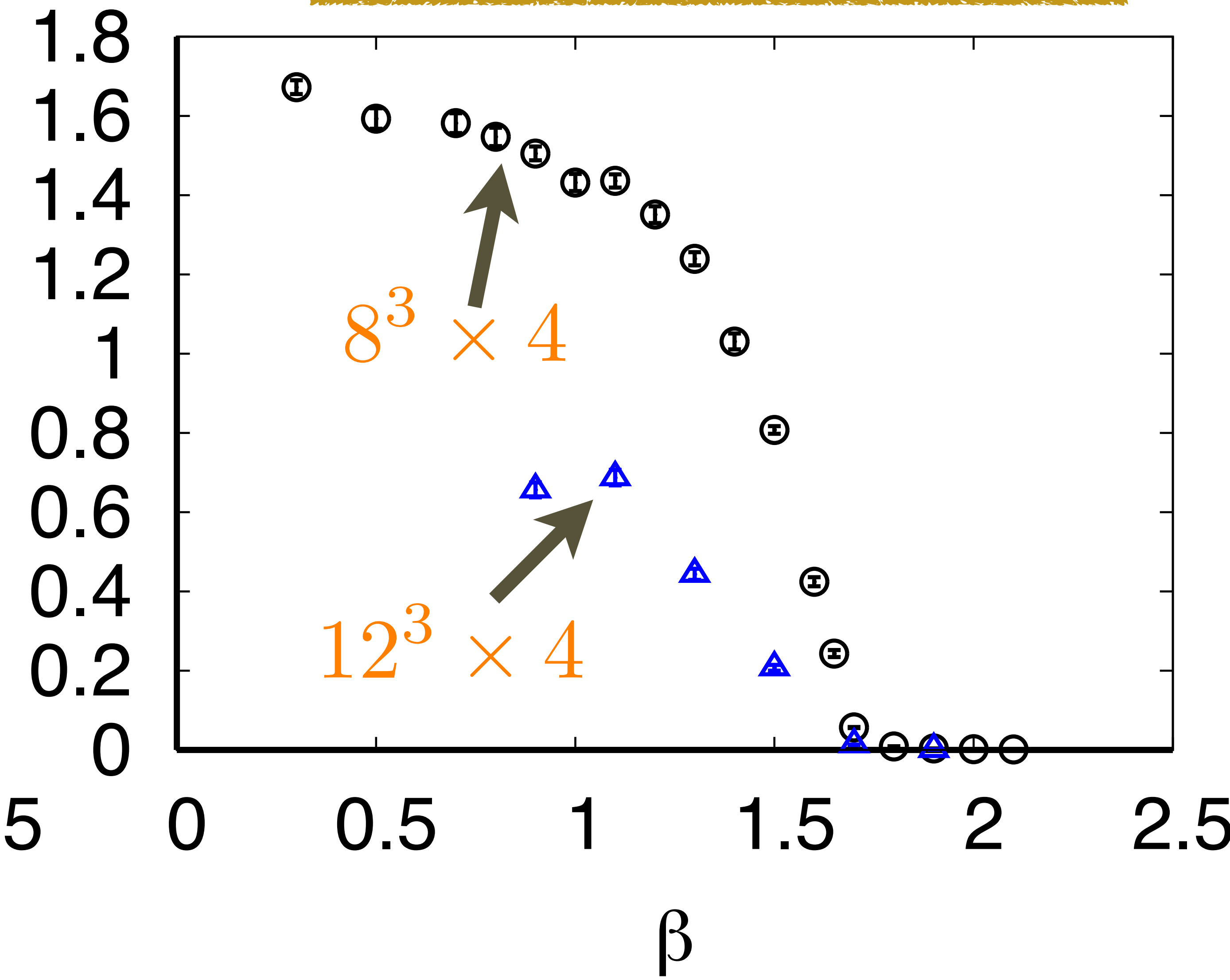


Polyakov loop

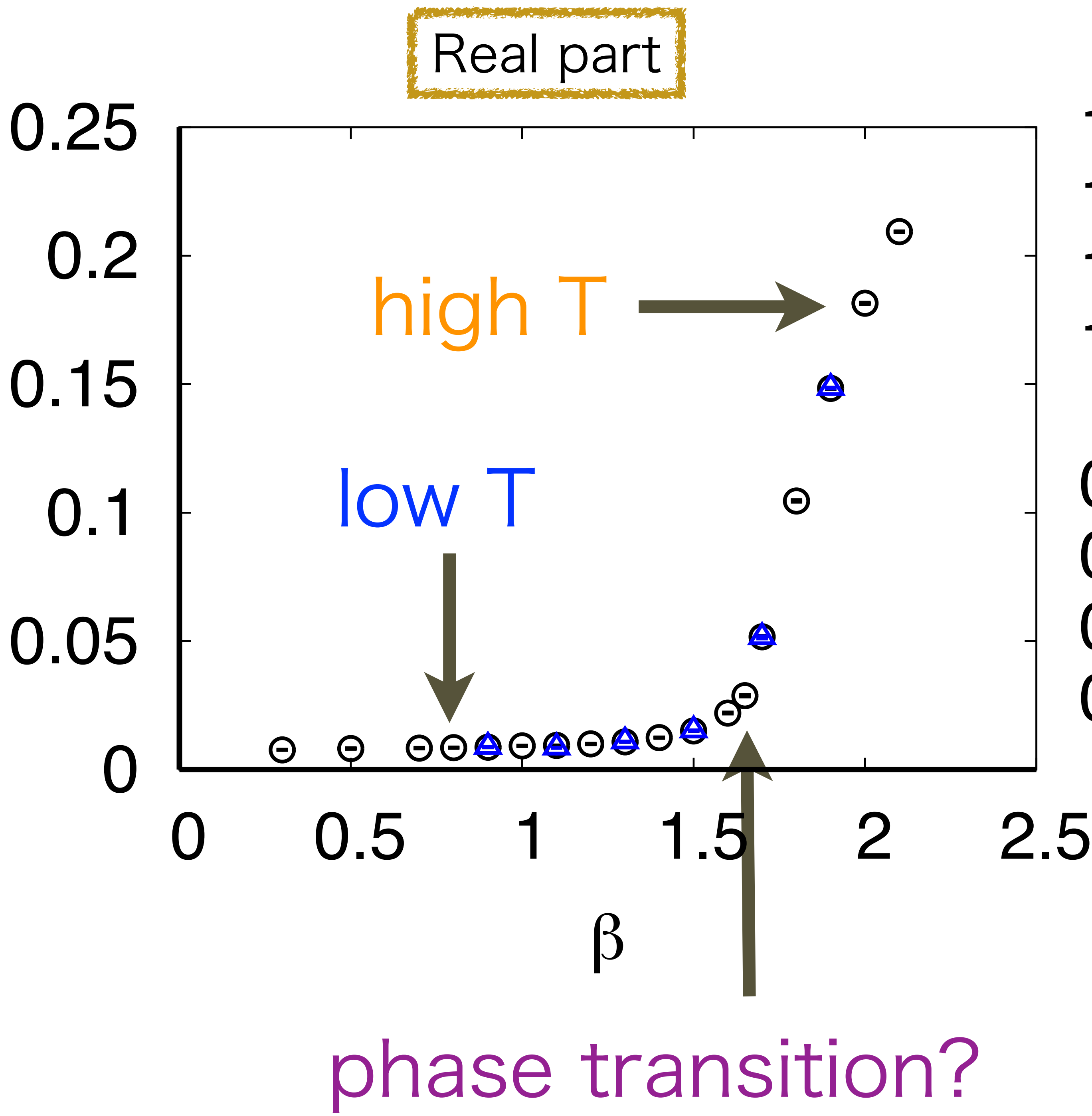
Real part

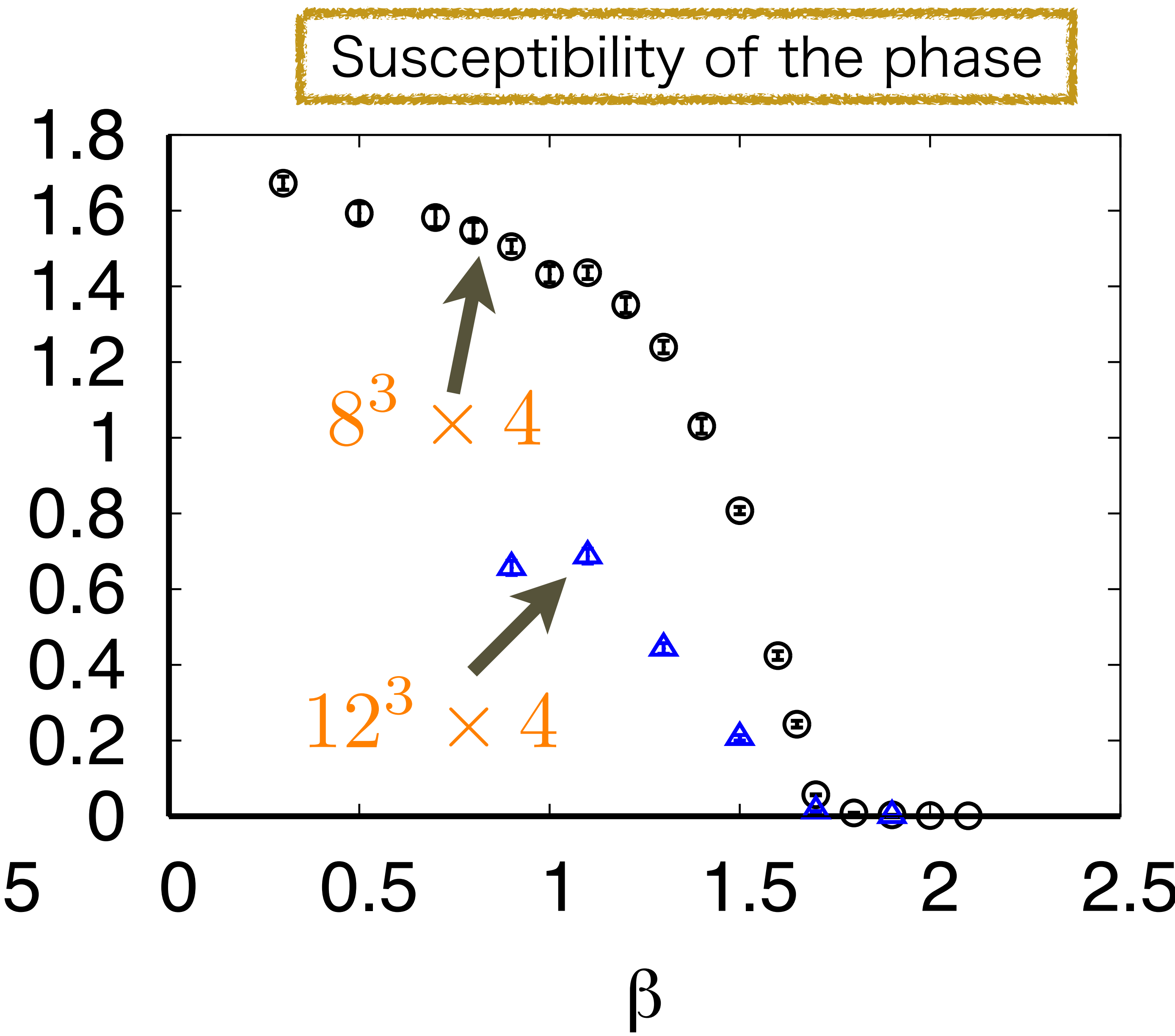


Susceptibility of the phase



Polyakov loop





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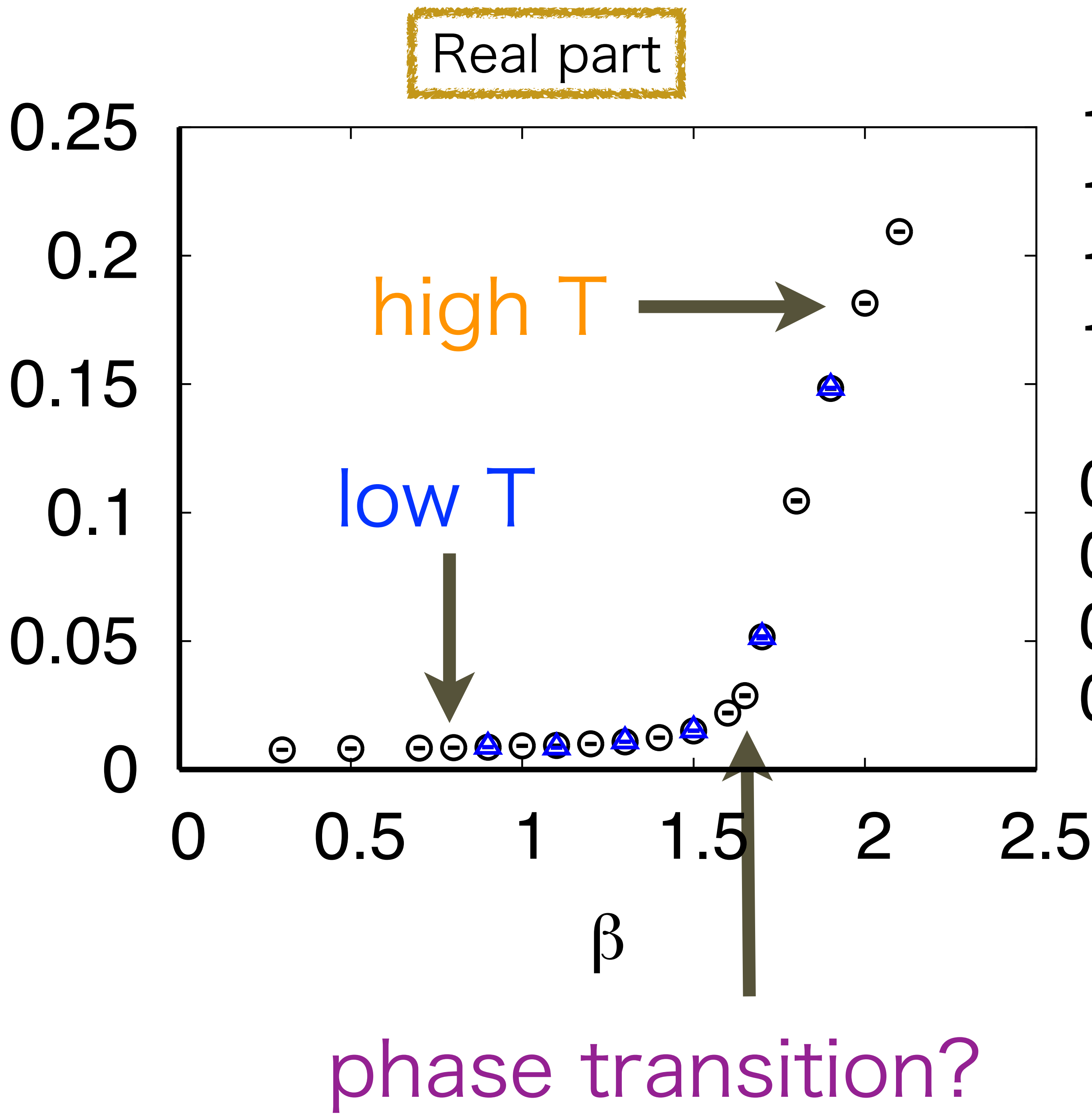
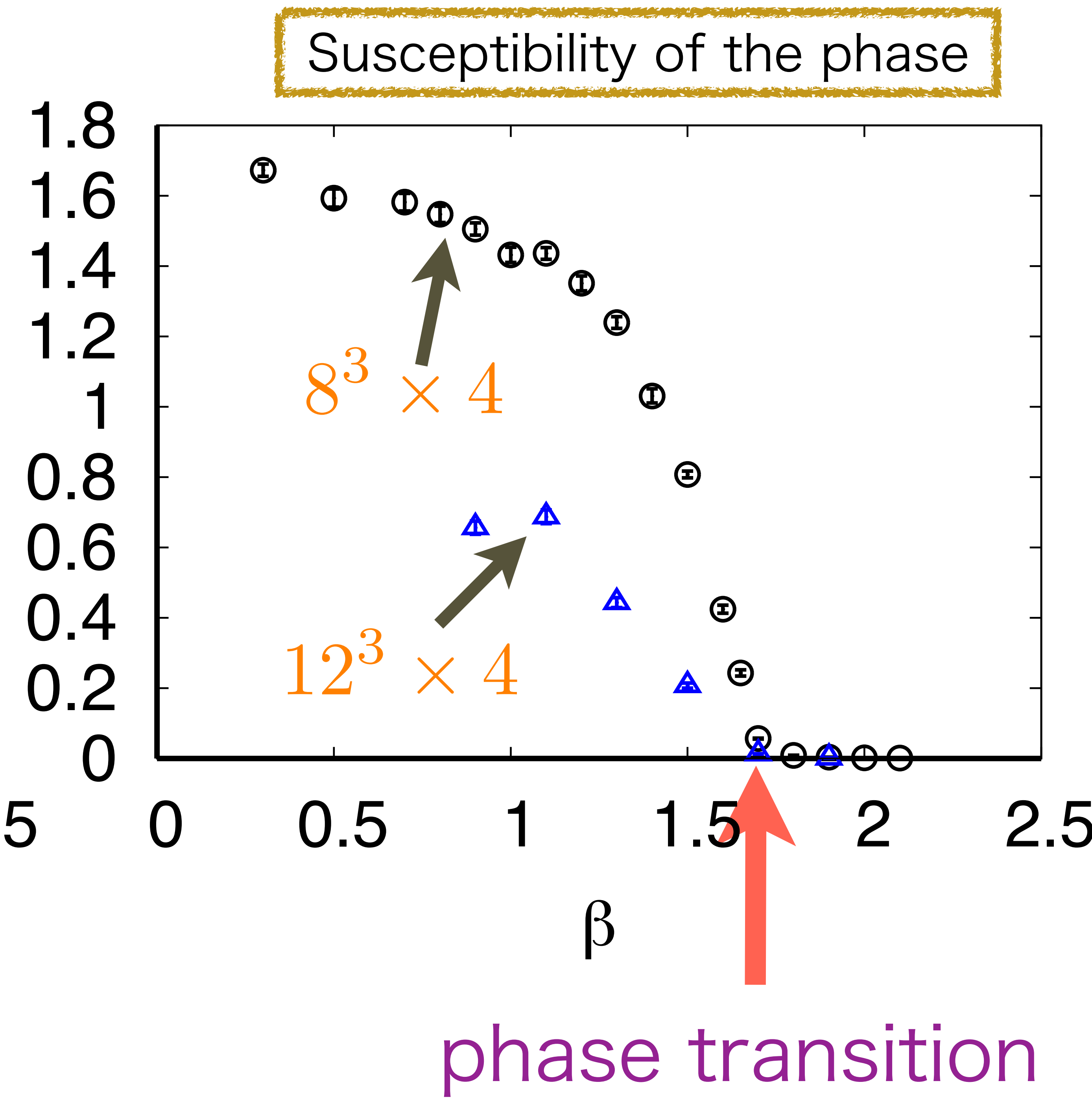




Plan of the talk

✔ 1. Introduction

✔ 2. Hopping parameter expansion

✔ 3. Numerical setup

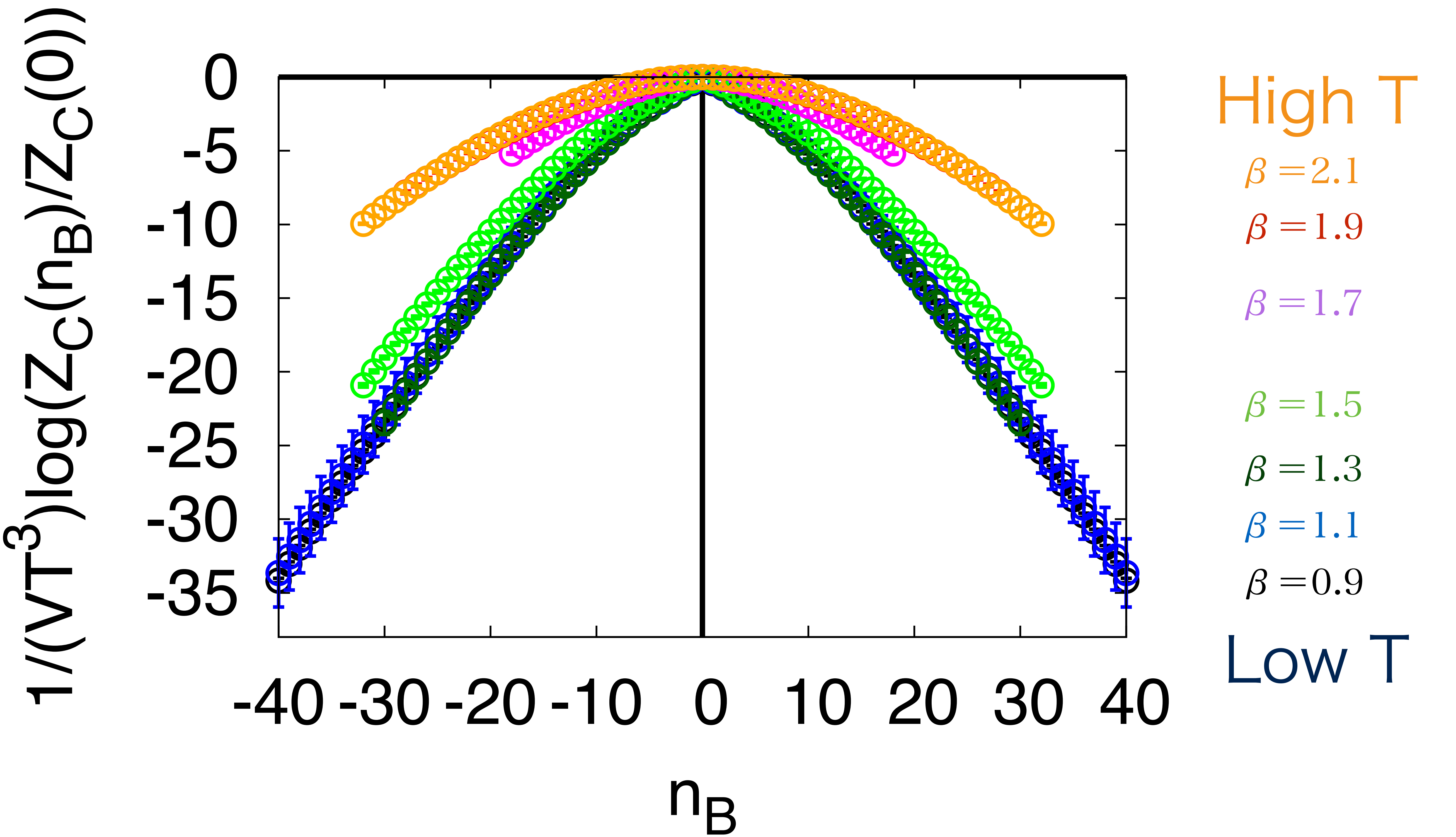
4. Canonical partition function Zn

5. Hadronic observables

6. Conclusion

Canonical |Zc(n)|

canonical partition fn. $Z_C\,(T,n,V) = |Z_C(\beta,n)|e^{i\theta(\beta,n)}$



Where can we apply HPE?

Convergence radius

$$\sum_{n=-\infty}^{\infty} Z_n \xi^n = Z_0 + Z_1 \xi + Z_2 \xi^2 + \cdots$$

$$+ Z_{-1} \xi^{-1} + Z_{-2} \xi^{-2} + \cdots$$

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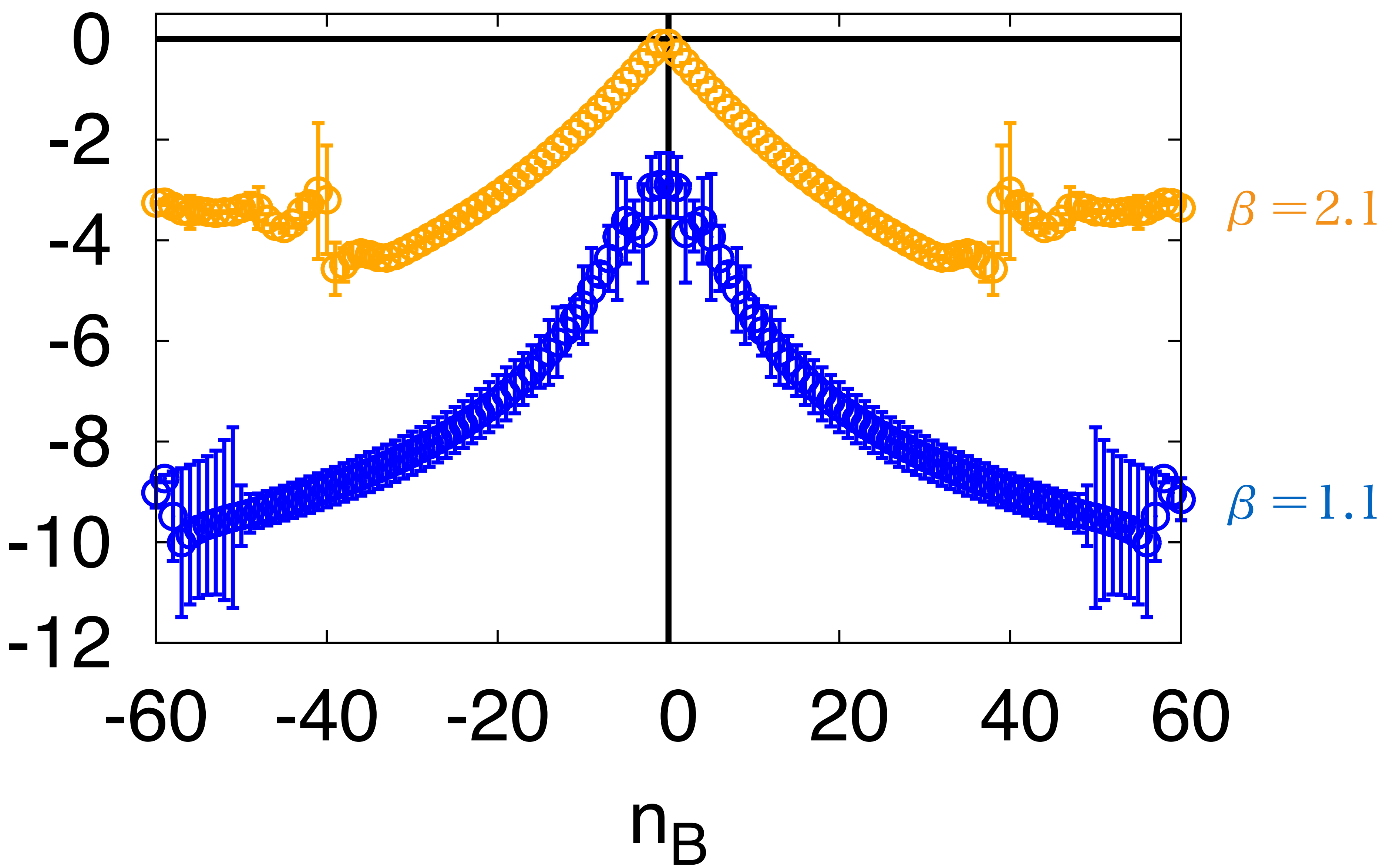
$$\sum_{n=-\infty}^{\infty} Z_n \xi^n = Z_0 + Z_1\xi + Z_2\xi^2 + \cdots + Z_{-1}\xi^{-1} + Z_{-2}\xi^{-2} + \cdots$$

$$\log\left|\frac{Z_{n+1}}{Z_n}\right|$$



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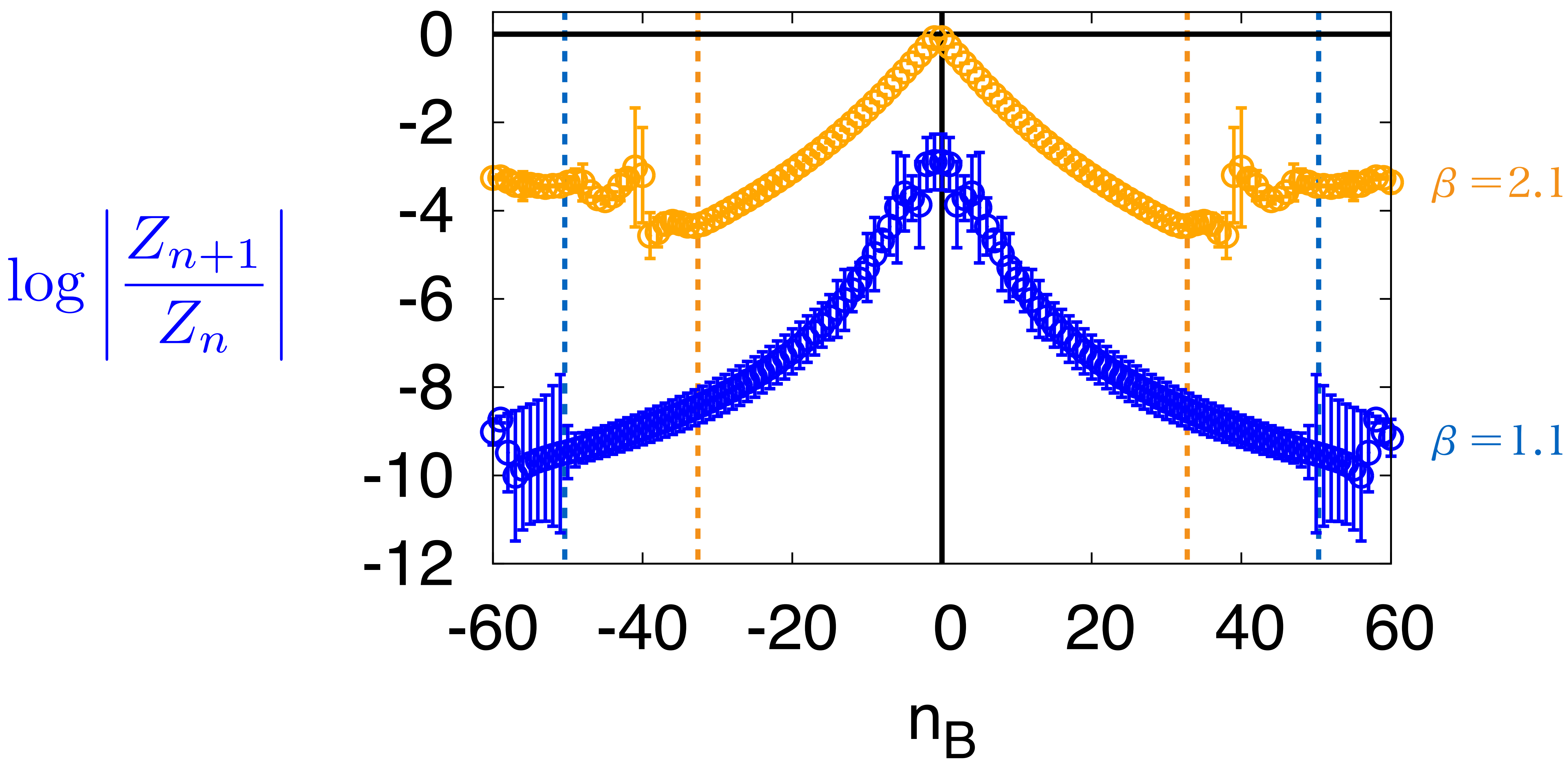
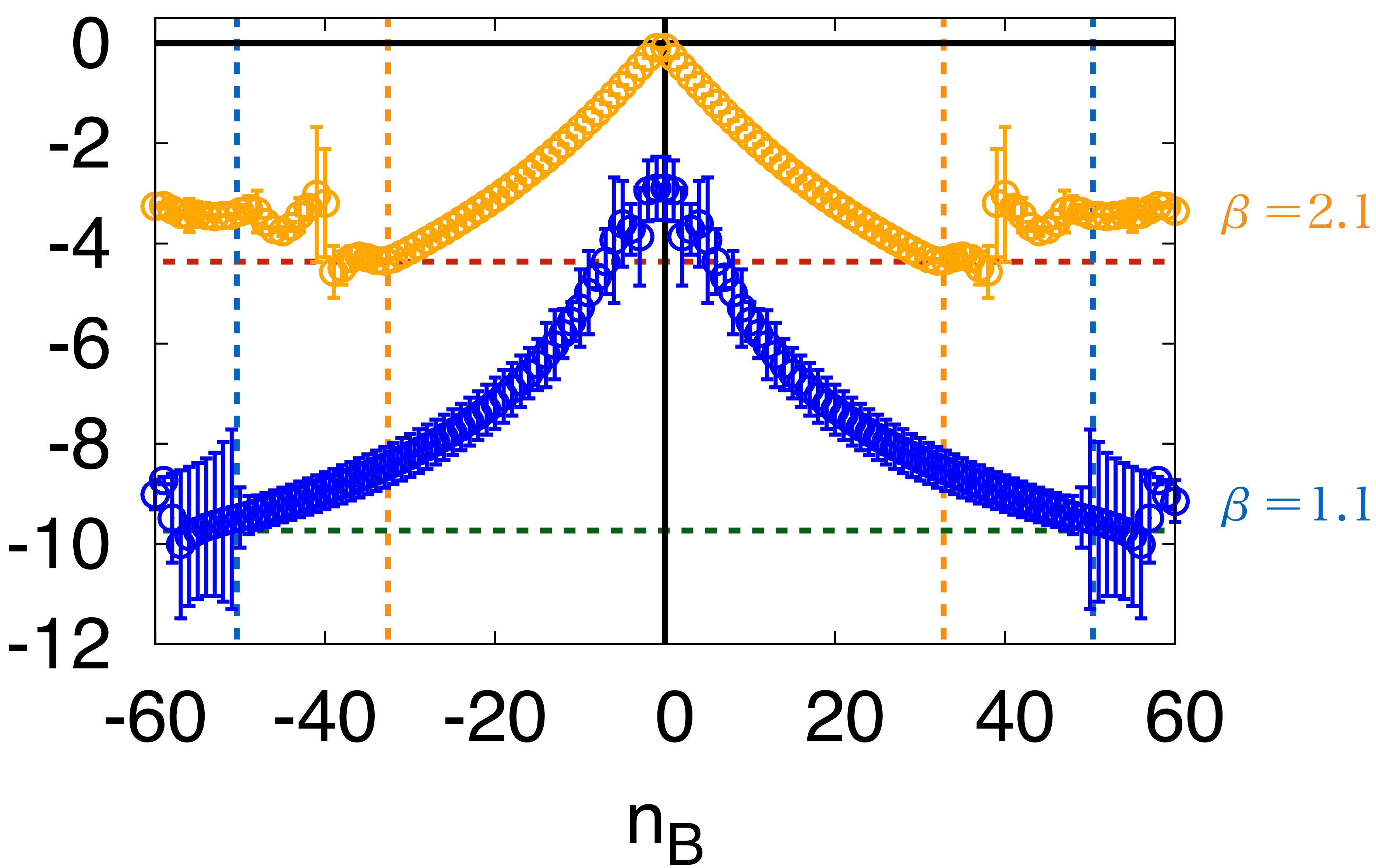


Where can we apply HPE?

Convergence radius

$$\sum_{n=-\infty}^{\infty} Z_n \xi^n = Z_0 + Z_1\xi + Z_2\xi^2 + \cdots + Z_{-1}\xi^{-1} + Z_{-2}\xi^{-2} + \cdots$$

$$\log\left|\frac{Z_{n+1}}{Z_n}\right| \rightarrow \log\left|\frac{1}{\xi_c}\right|$$



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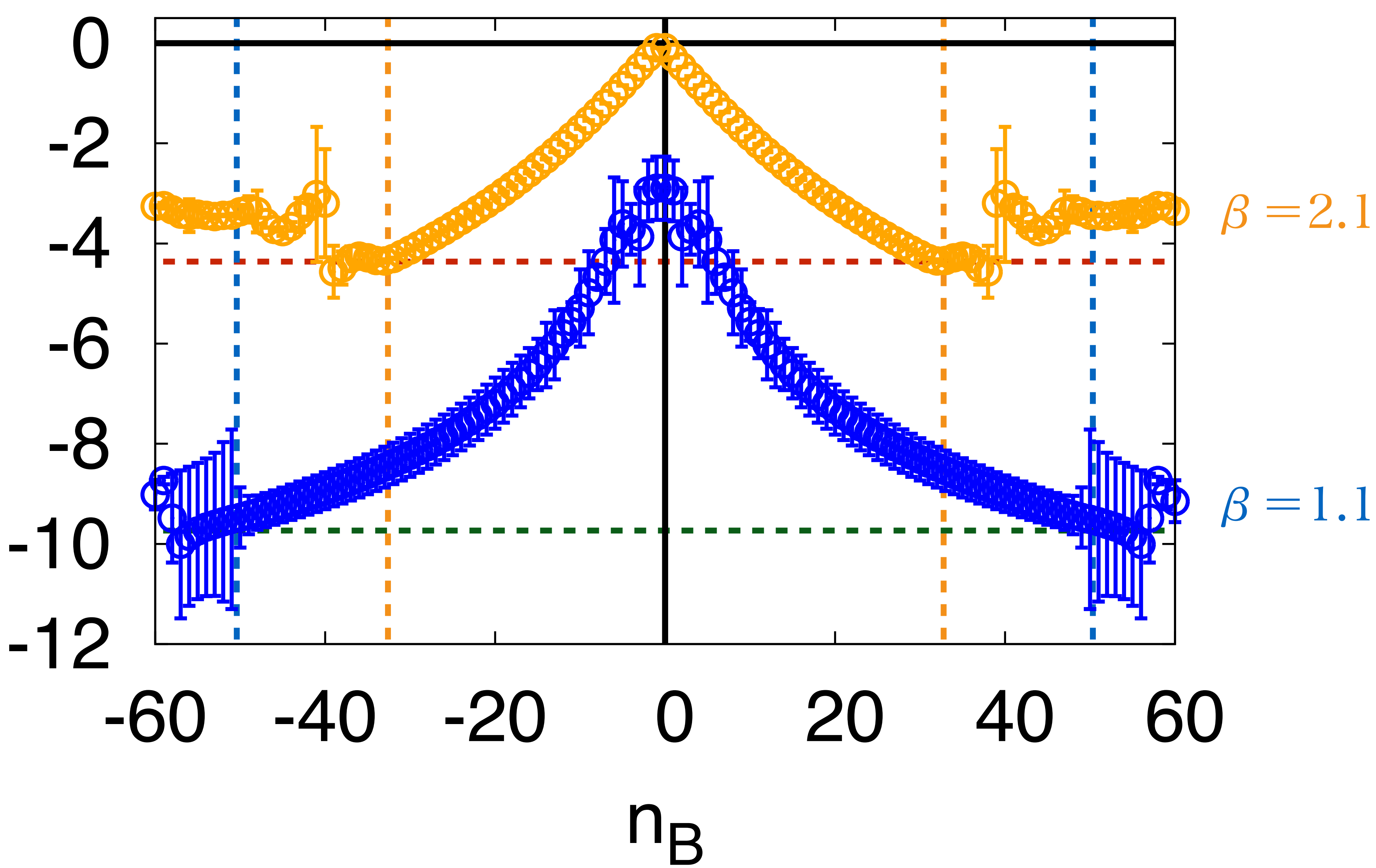
$\log\left|\frac{Z_{n+1}}{Z_n}\right|$

$\log\left|\frac{1}{\xi_c}\right|$



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