Recent Progress in Conformal Bootstrap

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Part 1 : Bootstrap Minimal

Relevance of CFTs

- Why are conformal field theories (CFTs) so important?
- UV & IR limit of RG flows must be scale invariant.
- In most cases scale invariance together with unitarity & Lorentz invariance implies an enhanced symmetry, that is, conformal symmetry (see e.g. arXiv:1302.0884).
- Thus, CFTs are ubiquitous in theoretical (especially high-energy & condensed matter) physics.

Philosophy of the conformal bootstrap program

- **"Solve the CFTs from consistency conditions, without assuming Lagrangian"** (Ferrara-Gatto-Grillo '73, Polyakov '74)
- … Seems quite unwieldy because there is an infinite # of unknown parameters in CFTs and the consistency conditions are also infinite-dimensional.
- Is it just an empty dream?

Numerical conformal bootstrap
(Rattazzi, Rychkov, Tonni, Vichi, `08)

- It is realized that the CFT consistency condition alone allows us to delineate the following curve.

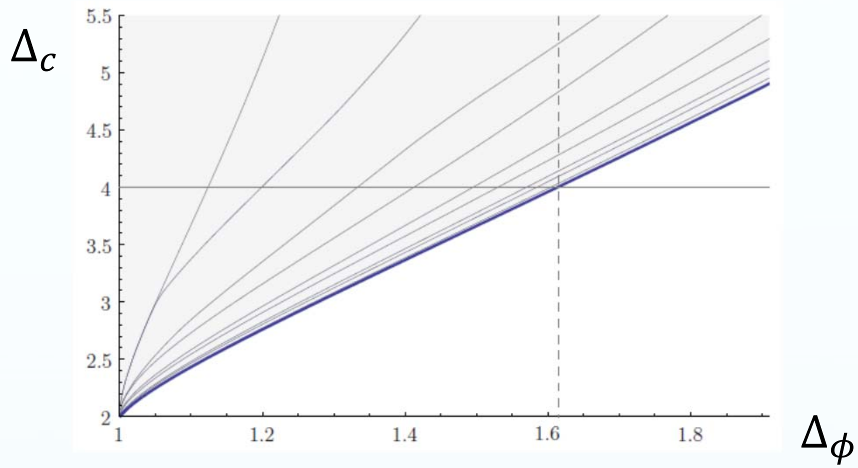
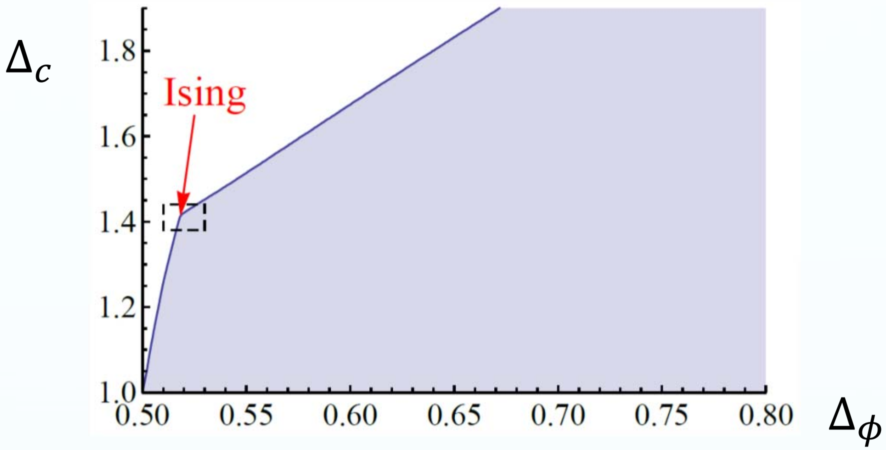


Figure from (Poland *et. al.,* *'11)*

- Meaning: for general 4d CFTs with dimension Δ_ϕ scalar ϕ, the $\phi \times \phi$ OPE must contain a scalar with dimension smaller than $\Delta_c(\Delta_\phi)$.

the 3d Ising "solution"

- Subsequently the analysis for $d = 3$ was made, resulting in:



- It developed a "kink" at the position where the 3d Ising model parameters sit. (El-showk *et. al.*, '12)
- It is quite mysterious why such a phenomenon take place.

Goal of Part I

- Tell you how we can obtain such non-trivial curves solely from CFT consistency conditions, without relying on Lagrangians!

Outline

1. Conformal kinematics
2. Conformal block decomposition of 4pt functions
3. The bootstrap equation and linear functional argument

1.1 Some conformal kinematics

The conformal invariance

- Conformal transformation means diffeomorphism which preserves the metric up to position dependent Weyl rescaling:

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x'^{\mu'}}\frac{\partial x^{\nu}}{\partial x'^{\nu'}}\, g_{\mu\nu} = \mathrm{e}^{\sigma(x)}\, g_{\mu\nu}$$

- It is easy to classify the solutions for the flat background. They are spanned by the usual Poincare transformation, dilatation, and special conformal transformation (SCT):

$$x \Rightarrow \frac{x + x^2 y}{1 + 2x \cdot y + x^2 y^2}$$

The state-operator correspondence

- We require that the states in the Hilbert space are in one-to-one with the local operators:
$$\phi(0)\,|0\rangle = |\phi\rangle$$
 (Note : If there's a Lagrangian, this can be derived.)

- Operators (states) which can be written as the derivatives (P^μ-action) of other local operators is called descendant operators (states). Otherwise they are called primary.

- The CFT Hilbert space are spanned by $\phi_1(0)\,|0\rangle$, $\partial^\mu\phi_1(0)\,|0\rangle = P^\mu|\phi_1\rangle$, $P^\mu P^\nu|\phi_1\rangle$, $P^\mu P^\nu P^\rho|\phi_1\rangle$, $\cdots$, $|\phi_2\rangle$, $P^\mu|\phi_2\rangle$,$P^\mu P^\nu|\phi_2\rangle$,$\cdots$, and so on.

Kinematical constraints from unitarity

- The scaling dimensions of operators cannot be arbitrary in unitary CFTs.
- In particular, primary operators (except for identity op.) must have its scaling dimension

$$\Delta_O \geq \begin{cases} \frac{d-2}{2} & (l=0) \\ l+d-2 & (l \neq 0) \end{cases}$$

, where l is the spin of O. Otherwise some descendant states acquire negative norm!

- But no upper bound in kinematical level.

Conformal 2 and 3pt functions

- Thanks to dilation, 2pt function of scale invariant QFT is determined to be
$$\langle O(x_1)O(x_2)\rangle = \frac{1}{(x_1-x_2)^{2\Delta_O}} \qquad (\Delta_O\text{: scaling dim. of } O)$$

- Thanks to the special conformal transf., in CFTs the 3pt functions is also determined by kinematics:
$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{\lambda_{O_1O_2O_3}}{(x_1-x_2)^{\Delta_1+\Delta_2-\Delta_3}(x_2-x_3)^{\Delta_2+\Delta_3-\Delta_1}(x_3-x_1)^{\Delta_3+\Delta_1-\Delta_2}}$$

  (First take x_3 to origin by translation, then x_1 to infinity by SCT and finally x_2 to (1,0,⋯,0) by rotation+ dilatation.)

- Here $\lambda_{O_1O_2O_3}$ refers the OPE coefficient,
$$O_1(x)O_2(0) \sim \frac{\lambda_{O_1O_2O_3}}{x^{\Delta_1+\Delta_2-\Delta_3}} O_3(0)$$

4pt function

- 4pt functions cannot be determined from kinematics alone.
- All we can do for $\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle$ is to

1. First take x_4 to the origin.
2. Then x_1 to the infinity by special conformal transf.
3. Then x_2 to $(1,0,\cdots,0)$ by dilation + rotation.
4. Finally x_3 to $(x,y,0,\cdots,0)$ by rotation (which fixes x_2).

- Thus the 4pt function is recovered from 2d-like "standard" configuration.

1.2 Conformal block decomposition of 4pt function

A lesson from elementary quantum mechanics

- 4pt correlators are encoded in some functions on 2d-plane, which is not fixed kinematically.
- But to some extent we can pursue, if one remembers the logic in elementary quantum mechanics, that is,

$$1 = \sum_{\psi:\text{all states}} |\psi\rangle\langle\psi|$$

- In CFTs the sum over states reads:

$$\sum_{\psi:\text{all states}} |\psi\rangle\langle\psi| = \sum_{\phi:\text{all primaries}} \sum_{\psi:\text{primaries of }\phi} |\psi\rangle\langle\psi|$$

"Conformal partial wave"

- Insert the complete set of states in the identical scalar 4pt function $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$

$$= \sum_{\psi:\text{all states}} \langle\phi(x_1)\phi(x_2)|\psi\rangle\langle\psi|\phi(x_3)\phi(x_4)\rangle$$

- Then organize them as a double-summation:

$$= \sum_{O:\text{all primaries}} \sum_{\substack{\psi:\text{ descendants} \\ \text{of } O}} \langle\phi(x_1)\phi(x_2)|\psi\rangle\langle\psi|\phi(x_3)\phi(x_4)\rangle$$

Matrix elements as 3pt function

- Fixing the primary O, evaluate the matrix elements like $\langle 0|\phi(x_1)\phi(x_2)|\psi\rangle$.
- Recall that $|\psi\rangle$ is of the form
$$P^{\mu_1}P^{\mu_2}\cdots P^{\mu_n}|O\rangle$$
and $|O\rangle = O(0)|0\rangle$ hence
$$P^{\mu_1}P^{\mu_2}\cdots P^{\mu_n}|\phi\rangle = \partial^{\mu_1}\partial^{\mu_2}\cdots\partial^{\mu_n}\phi(0)|0\,\rangle.$$
- Thus matrix elements are computed from the 3pt function,
$$\lim_{y\to 0}\partial_y^{\mu_1}\cdots\partial_y^{\mu_n}\langle 0|\phi(x_1)\phi(x_2)O(y)|0\rangle$$
Recall that 3pt function is determined kinematically up to OPE coefficients!

The conformal block

- Thus the partial sum can be rewritten,
$\sum_{\substack{\psi:\ \text{descendants} \\ \text{of } O}} \langle \phi(x_1)\phi(x_2)|\psi\rangle\langle\psi|\phi(x_3)\phi(x_4)\rangle$
$$= \lambda_{\phi\phi O}^2 \times \frac{g(z,\bar{z};\Delta_O,l_O,\Delta_\phi)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}$$

- Here z is the x_3 −coordinate in the "standard" configuration.

- The function g is fixed by CFT kinematics and called the "conformal block". (In old literature it is called "conformal partial wave")

The "s–channel" decomposition

- It turns out $g(z, \bar{z}; \Delta_O, \Delta_\phi)$ is independent of Δ_ϕ.
- The final form for 4pt function is

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{O:\mathrm{primaries}} \lambda_{\phi\phi O}^2 \, g(z, \bar{z}; \Delta_O, l_O)$$

- This expression is rapidly converging. (Pappadpulo, Espin, Rychkov, Rattazzi, '12)

1.3 The Bootstrap Equation and Linear functional Argument

Crossing symmetry

- Actually our conformal correlators behave properly only when they are “radial ordered”, i.e., operator ordering
$$\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)$$
means $|x_1| > |x_2| > |x_3| > |x_4|$.

- We can equally compute the 4pt function from another ordering, say $\phi(x_3)\phi(x_2)\phi(x_1)\phi(x_4)$, and perform the conformal block decomposition.

- These two expression must agree when both of them converge!

Conformal block in t−channel

- Correlator in the ordering $\phi(x_3)\phi(x_2)\phi(x_1)\phi(x_4)$ can be decomposed

$$\langle\phi(x_3)\phi(x_2)\phi(x_1)\phi(x_4)\rangle$$
$$= \frac{1}{x_{23}^{2\Delta_\phi} x_{14}^{2\Delta_\phi}} \sum_{O:\mathrm{primaries}} \lambda_{\phi\phi O}^2 \, g(1-z, 1-\bar{z}; \Delta_O, l_O)$$

- Note that,

1. Conformal block argument has been changed to $1-z$
2. OPE factor and the summation range are the same!

The bootstrap equation

- Thus we have, (adding prefactor $x_{13}^{2\Delta_\phi} x_{24}^{2\Delta_\phi}$,)

$$0 = \sum_{O:\text{primaries}} \lambda_{\phi\phi O}^2 \times F(z, \bar{z}\,; \Delta_O, l_O, \Delta_\phi)$$

, where

$$F(z, \bar{z}\colon \Delta_O, l_O, \Delta_\phi) = x_{13}^{2\Delta_\phi} x_{24}^{2\Delta_\phi} \left(\frac{g(z, \bar{z}\colon \Delta_O, l_O)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} - \frac{g(1-z, 1-\bar{z}\colon \Delta_O, l_O)}{x_{32}^{2\Delta_\phi} x_{14}^{2\Delta_\phi}} \right)$$

- Thus in CFT, infinitely many unknowns are constrained by infinite-dimensional constraint.

Linear functional argument –1

- Fix Δ_ϕ and assume there is a linear functional
$$\Lambda : (\text{functions on } z - \text{plane}) \to \mathbb{R}$$
and some "hypothetical gap" Δ_c, with the following property:
- $$\begin{cases} \Lambda\left(F\left(z,\bar{z},\Delta,l,\Delta_\phi\right)\right) \geq 0 \ \ (\text{for } \Delta \geq l+d-2, \ \ \text{if } \ l \neq 0) \\ \Lambda\left(F\left(z,\bar{z},\Delta,l,\Delta_\phi\right)\right) \geq 0 \ \ (\text{for } \Delta \geq \Delta_c, \ \ \text{if } l = 0) \\ \Lambda\left(F\left(z,\bar{z},0,0,\Delta_\phi\right)\right) \geq 0 \end{cases}$$
- Then apply this functional to the bootstrap equation,
$$0 = \sum_{O:\text{primaries}} \lambda^2_{\phi\phi O} \times F(z,\bar{z}\,;\Delta_O, l_O, \Delta_\phi)$$

Linear functional argument –2

- $0 = \Lambda(0) = \sum_O \lambda_{\phi\phi O}^2 \Lambda(F(z, \bar{z}; \Delta_O, l_O))$

$$= \sum_{\substack{O: \\ l_O \neq 0}} \lambda_{\phi\phi O}^2 \Lambda\Big(F\big(z, \bar{z}; \Delta_O, l_O, \Delta_\phi\big)\Big) + \sum_{\substack{O: \\ l_O = 0, \Delta_O \geq \Delta_c}} \lambda_{\phi\phi O}^2 \Lambda\Big(F\big(z, \bar{z}; \Delta_O, l_O, \Delta_\phi\big)\Big)$$

$$+ \sum_{\substack{O: \\ l_O = 0, \frac{d-2}{2} < \Delta_O < \Delta_c}} \lambda_{\phi\phi O}^2 \Lambda\Big(F\big(z, \bar{z}; \Delta_O, l_O, \Delta_\phi\big)\Big) + \Lambda(F(z, \bar{z}; 0,0, \Delta_\phi))$$

- The first, second and the last term is positive by the assumption!
- ➔ Contribution of the third term must be present!
- ***There must be an operator with*** $\frac{d-2}{2} < \Delta_O < \Delta_c$***!!***

Difficulties

- Thus finding such Λ and Δ_c has quite important meaning.
- The smaller the value Δ_c, the stronger the bound.
- How can we find such a linear functional? There are two difficulties.

1. The space of all the linear functionals is ∞−dimensional.
2. Also ∞ly many inequalities must be checked.

Truncating the difficulties 1.

- Though not ideal, we can restrict our attentions within finite dimensional subspace.

- Adopt the ansatz

$$\Lambda(*) = \sum_{m,n\geq 0}^{m+n\leq N_{\max}} c_{m,n} \partial_z^m \partial_{\bar{z}}^n (*) \Big|_{z=\frac{1}{2}}$$

, where $N_{\max}$ is some cutoff and search for $c_{m,n}$.

- As you take $N_{\max} \to \infty$, we expect we can approach the ideal choice for Δ_c.

Truncating the difficulties 2.

- The difficulty 2. is actually harder to over come.
- An obvious way is to discretize: i.e., check the inequalities
$$\Lambda\left(F\left(z,\bar{z},\Delta,l,\Delta_\phi\right)\right) \geq 0$$
for $\Delta = \text{(unitarity bound)} + \varepsilon * i,$ starting from $i = 0, 1, \cdots$ until $\varepsilon * i$ becomes sufficiently large.
- Since conformal block take universal form as $l \to \infty$, check this for $l \leq 30$ is enough.
- Note: now there is a sophisticated way of avoiding this ugly discretization.

Reduction to the linear programming

- Now the problem is to find real numbers $\{c_{m,n}\}_{m,n\geq 0}^{m+n\leq N_{\max}}$ satisfying inequality

$$\sum_{m,n\geq 0}^{m+n\leq N_{\max}} c_{m,n}\left(\partial^m \bar{\partial}^n F\left(z,\bar{z},\Delta,l,\Delta_\phi\right)\Big|_{z=\frac{1}{2}}\right) \geq 0$$

  for with finite # of (Δ, l) pairs.

- ➔Linear programming! The computers can address the problem much better.
- Repeating the analysis for each Δ_ϕ we obtain $\Delta_c(\Delta_\phi)$.

Numerical output in 4d
(Rattazzi-Rychkov-Tonni-Vichi, '08)

- The output of the above procedure with various $N_{\max}$

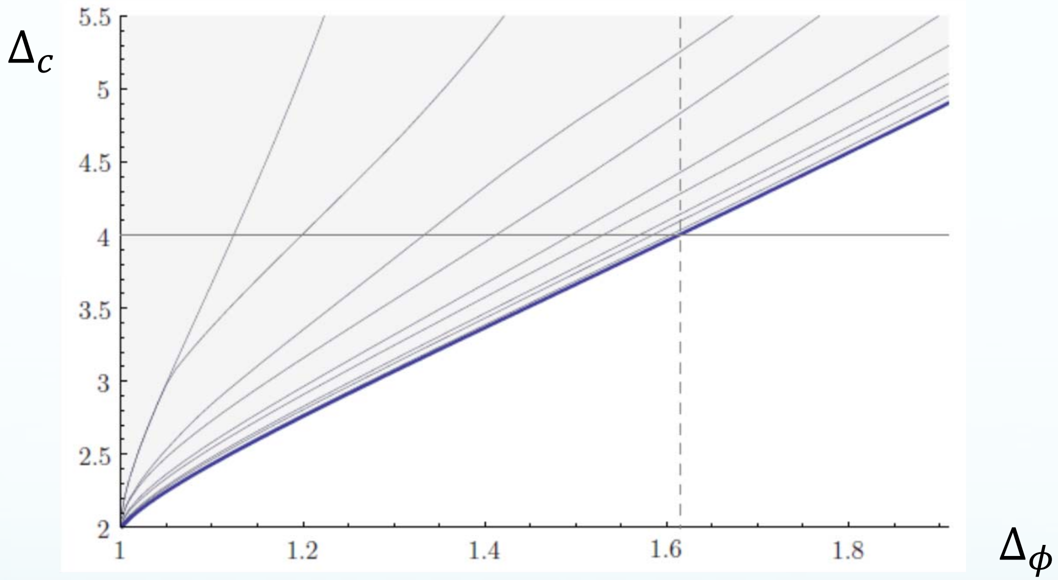
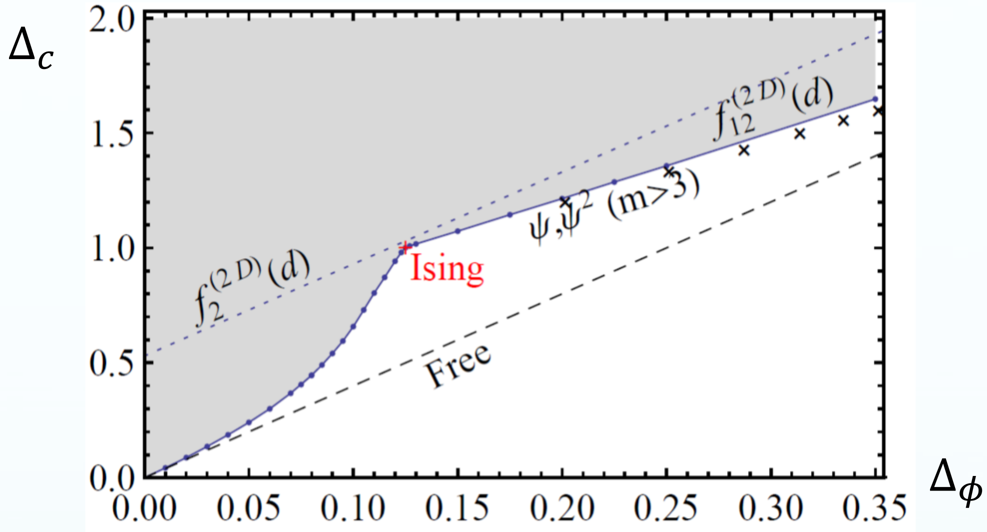


Figure from (Poland *et.al.*, '11)

- The convergence w.r.t. $N_{\max}$ is not so bad.

Numerical output in 2d (Rychkov-Vichi, '09)

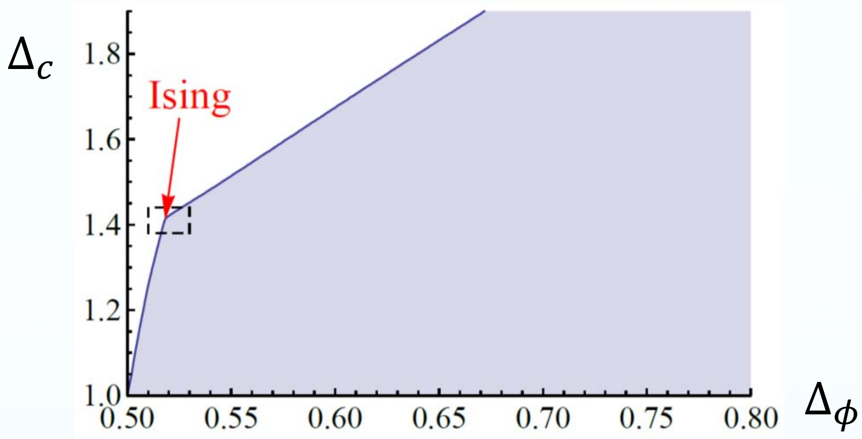
- For $d = 2$, the same procedure gives us



- This encouraged Slava Rychkov and collaborators to analyze the same problem in $d = 3$, though there was no good algorithm to compute conformal block for $d \neq 2,4$ at that time. (They had to develop!)

3d Ising "solution"
(El-showk, *et. al.*, '12)

- The output for 3d then turned out to be:



- Although still far from the "solution", the 3d Ising model is cornered by the bootstrap method.

Previously found kinks

- Wilson–Fisher fixed points in the $2 < d < 4$ interval
- $d = 3$ $O(n)$–vector models (to be explained below)
- $d = 5$ $O(n)$–universality class
- $d = 3$ $N = 1,2$ super–Ising model
- $d = 3$ $U(2)_1 \times U(1)_{-1}$– ABJ model (with $N = 8$ SUSY)

For a much more exciting example,
stay here for Part 2!

Summary

- CFTs are much more tightly constrained than the ordinary QFTs due to conformal invariance & unitarity& crossing relations.
- The bootstrap equation together with the linear functional method gives a powerful bound, sometimes showing “kink” at the actual CFT(e.g. Ising) location, but the fundamental reason is mysterious…

Part 2 : Applications to $O(n) \times O(m)$ CFTs

Based on, arXiv:1404.0489, 1407.6195 with Yu Nakayama

2.1 Introduction:
The $O(n) \times O(m)$ wonderland

Pisarski–Wilczek Argument for the Chiral Phase Transition

- 2–flavor QCD classically has the symmetry,
  $$SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A$$
  but the last factor $U(1)_A$ is explicitly broken by anomaly.

- Consider the chiral symmetry breaking transition. There the effective DOF comprises of mesons, neutral under $U(1)_B$.

- After the thermal compactification the effective action is that of 3d $SU(2) \times SU(2) \simeq O(4)$–symmetric sigma model.

- The transition is either 1st order or 2nd order with $O(4)$–universality class…

$U(1)_A$–controversies

- But the anomaly is less visible at higher temperature.
- It is quite controversial to what extent the anomaly effect is weaken around the chiral phase transition.
- (Aoki–Fukaya–Taniguchi, '12) argued it is invisible (at the level of effective σ–model) above the critical temperature.
- If it is the case, the relevant σ–model is

$$SU(2)_L \times SU(2)_R \times U(1)_A \simeq O(4) \times O(2)$$

–symmetric Landau–Ginzburg model.

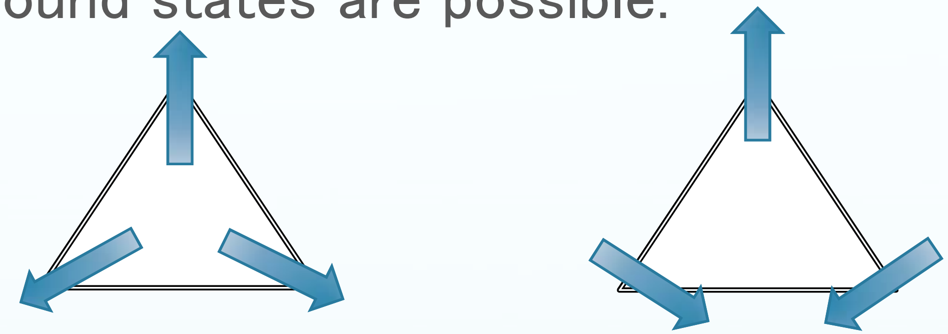
Controversies over Controversies

- Even if the anomaly-restoration scenario is true, we still have controversies: the presence of IR-stable fixed points in the $O(4) \times O(2)$-LG model is so hard to investigate!

Reference	Method	Result
Pisarski-Wilczek '81	1-loop	No
Wetterich, '97	Functional RG(LPA)	No
Calabrese *et. al.* , '03 (A cond-mat paper!)	5-loop + resummation	Yes
Calabrese-*Paruccini,* '04	5-loop + resummation	No
Fukushima *et .al.,* '10	Functional RG (∞ dim LPA)	No
Grahl '14	Functional RG (LPA')	Depends heavily on how you truncate. No critical exponent agreed with the 5-loop result.

Another Realization

- $O(n) \times O(m)$–LG model offers a big business also for condensed–matter theorists: imagine n–component anti–ferromagnetic spin systems on triangular lattice. When $n = 2$, two ground states are possible:



- The effective field theory at criticality is described by $O(n) \times O(2)$ LGW model (Kawamura `85).
- Presence of IR fixed points is again controversial!

What is problematic in the perturbative RG methods?

- The fixed points are quite unusual: they exist only in the region $d \sim 3$ (and not in d= $4-\epsilon$) and small n.
- Perturbative results can be altered when one considers even higher loop series. As an example, consider $O(3)$–LG model, with an IR (Heisenberg) fixed point. At 3–loop order the fixed point vanishes! Only at 4–loop order is it restored…
- Dependence on the resummation parameters is also criticized.

What is problematic in the functional RG method?

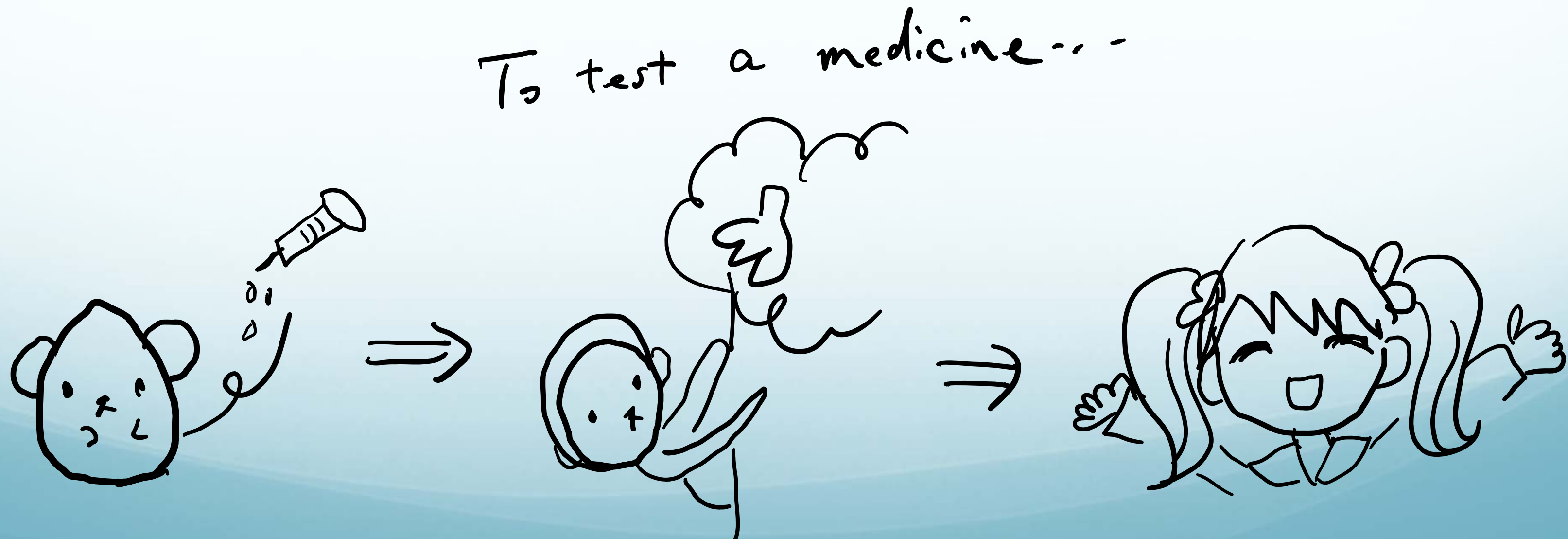
- Functional RG equation is 1-loop exact!
- But is formulated in the infinite-dimensional space of all possible interaction term.
- We have to truncate this space to some convenient subspace. (E.g. LPA)
- The truncation procedure cannot be justified. Indeed it is reported that the results (existence of fixed point) depend heavily on the order of truncation (Grahl, '14.)

Milestone

- If $O(n) \times O(m)$ −LG model has an IR FP, there we must have an $O(n) \times O(m)$ −symmetric CFT there.
- We can derive the bootstrap constraints in these cases as well (by a generalized method to be explained.)
- Can we observe the kink as in the 3d Ising (no continuous global symmetry) case?

Outline

1. Introduction : The $O(n) \times O(m)$ wonderland
2. Mouse : $O(n)$
3. Monkey : $O(n) \times O(3)$ with $n \gg 3$
4. Human : $O(n) \times O(2)$ with $n = 3, 4$



2.2 Mouse: $O(n)$

$O(n)$–LG model

- Before applying the conformal bootstrap to $O(n) \times O(m)$ models, we briefly review how it proceeds for $O(n)$ models.
- Recall that $O(n)$–symmetric Landau–Ginzburg model or $O(n)$ –vector model, we have a scalar field transforming in a vector (v) rep. of $O(n)$, ϕ_i.
- Consider a 4pt correlation function :

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4) \rangle$$

- Then we perform conformal block decomposition as well.

Structured conformal block decomposition

- In order for $\langle 0|\,\phi_i(x_1)\phi_j(x_2)|\psi\rangle$ to survive, ψ has to transform in some irreducible rep contained in $\mathrm{v}\otimes\mathrm{v}$.
- Thus ψ is either scalar (S) or traceless-symmetric(T) or anti-symmetric (A).
- The final form of the decomposition contains Kronecker deltas because

$$\langle 0|\phi_i\phi_j|\mathrm{S}\rangle \propto \delta_{ij}$$

$$\langle 0|\phi_i\phi_j|\mathrm{T}\!: mn\rangle \propto \left(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}-\frac{2}{N}\delta_{ij}\delta_{mn}\right)$$

$$\langle 0|\phi_i\phi_j|\mathrm{A}\!: mn\rangle \propto \left(-\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}\right)$$

Vectorial Bootstrap equation

- Here we require the crossing symmetry w.r.t simultaneous exchange $x_1 \leftrightarrow x_3$, $i \leftrightarrow k$.
- We have to match each coefficient of independent δ.
- Three constraints. The bootstrap equation looks like

$$\sum_{O:\mathrm{S}} \begin{pmatrix} 0 \\ -H_O \\ F_O \end{pmatrix} + \sum_{O:\mathrm{T}} \begin{pmatrix} F_O \\ \left(1+\frac{2}{N}\right) H_O \\ \left(1-\frac{2}{N}\right) F_O \end{pmatrix} + \sum_{O:\mathrm{A}} \begin{pmatrix} F_O \\ -H_O \\ -F_O \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Linear functional argument

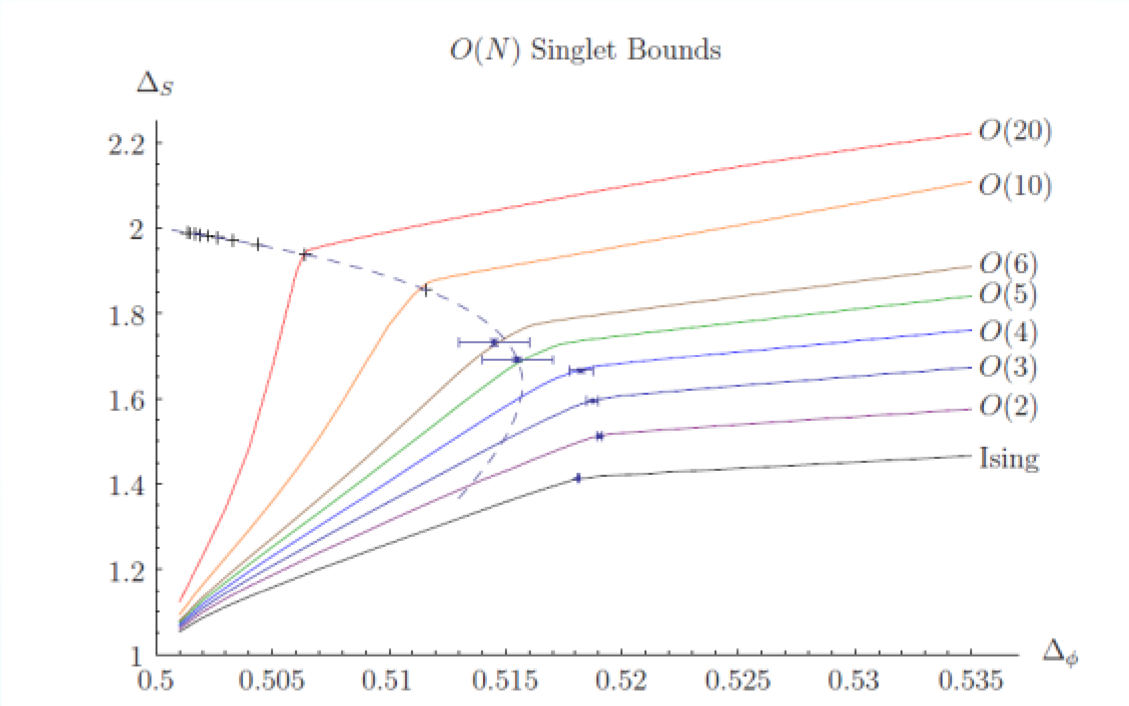
- Thus to generalize the linear functional argument we search for

  $$\Lambda: (3-\text{component vector valued function}) \rightarrow \mathbb{R}$$

  satisfying positivity condition for S,T,A sectors separately.

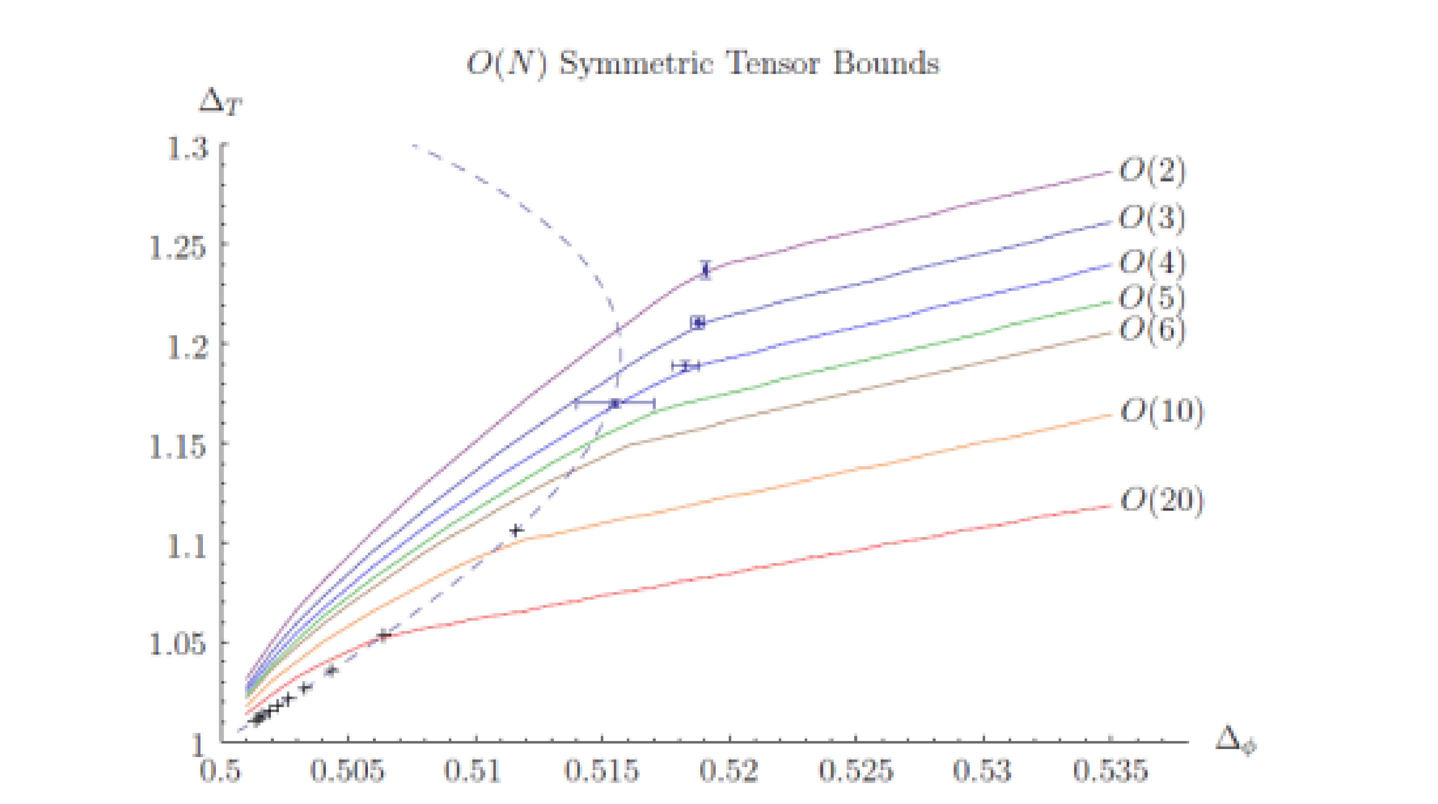
- According to in which sector (S,T,A) we assume hypothetical gap Δ_c, we can independently bound the lowest dimension of operators.

S −scalar
(Kos, Poland, Simons-Duffin, `13)



- The precise meaning: ∃ a scalar operator in the S-rep in $\phi_i \times \phi_j$ OPE with dimension below $\Delta_{c,S,N}(\Delta_\phi)$.

$d = 3\ O(N)$ bounds for T −scalar



Lessons

- Kinks corresponding to $O(N)$–LG models show up.
- The operator dimension bound can be derived equally well for every global symmetry sector.
- In this $O(n)$–case, the S and T bounds point out the single model (but there's no guarantee for this).

2.3 Monkey: $O(n) \times O(3)$ with $n \gg 3$

Based on arXiv.1404.0489

More serious experiment with $O(n) \times O(3)$

- The situation in $O(n) \times O(2)$ with $n = 3{,}4$ looked like swampland…
- We decided to work in the case where the RG-theoretical studies are solid.
- Thus we started with the $O(n) \times O(3)$-model with $n \gg 3$. In the $n \to \infty$ limit the model is solvable! We have $1/n$ -expansion.
- Still dynamically rich: the presence of conformal window (more precisely "conformal half-line") is predicted.

Lagrangian description of the $O(n) \times O(m)$–LG model

- Consider a Lagrangian formed from scalar field,

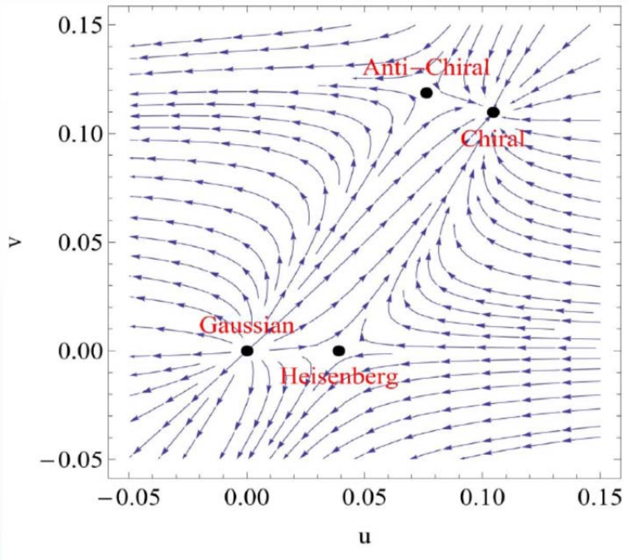
$$\mathcal{L} = \sum_{i,a} \partial_\mu \phi_{ia} \partial_\mu \phi_{ia} + u \left(\sum_{i,a} \phi_{ia}\phi_{ia} \right)^2$$
$$+ v \sum_{i,j,a,b} (\phi_{ia}\phi_{ja}\phi_{jb}\phi_{ib} - \phi_{ia}\phi_{ia}\phi_{jb}\phi_{jb})$$

  Here the indices run over $i, j = 1, \ldots, n, \;\; a, b = 1, \ldots, m$, i.e., ϕ transforms as a bifundamental of $O(n) \times O(m)$.

- The term proportional to u is actually $O(nm)$ – invariant. When $v \neq 0$, $O(n\, m) \to O(n) \times O(m)$ explicitly.

RG prediction

- When n is sufficiently large, the RG flow looks like



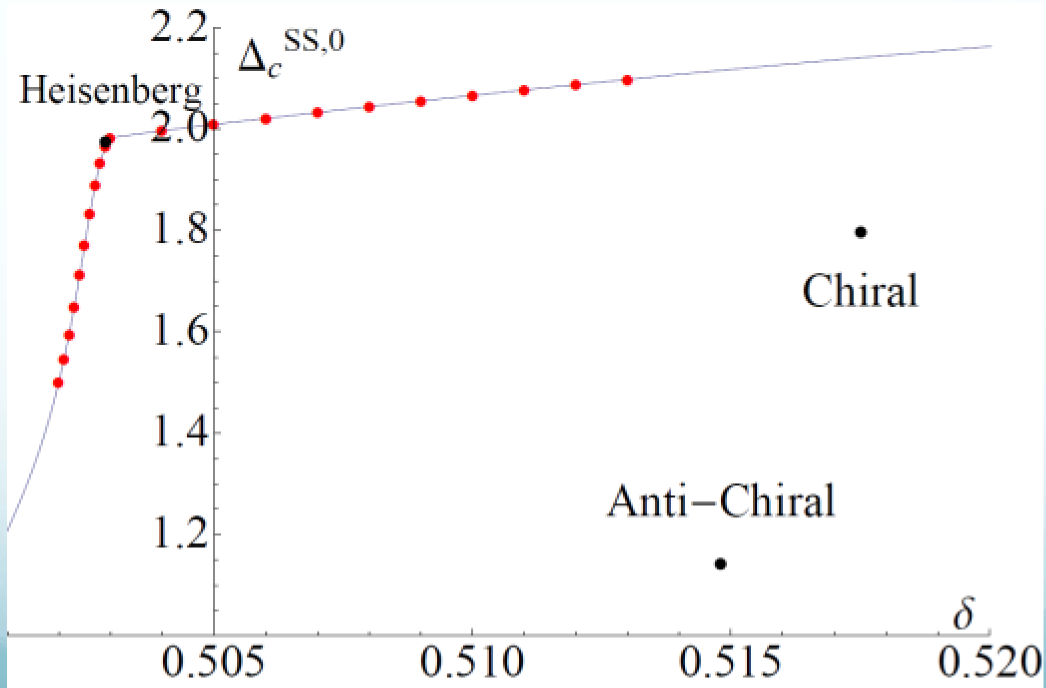
- 2 more fixed points. The (un)stable one is called the (anti−)chiral fixed point.

The bootstrap equation: too lengthy to repeat

- Can we observe these additional fixed points??
- To obtain the bootstrap constraints we consider
$$\langle \phi_{ia}(x_1)\phi_{jb}(x_2)\phi_{kc}(x_3)\phi_{ld}(x_4) \rangle$$
- => There are 9 global symmetry sectors, denoted by
$$\mathrm{SS, ST, SA, TS, TT, TA, AS, AT, AA}$$
- The bootstrap equation is huge. Consequently the computation is ~100 times heavier than the Ising case.

Bounds for SS spin 0 operator in $O(15) \times O(3)$ model

- Our first sample is $O(15) \times O(3)$ model, where the presence of non-Heisenberg FPs (called "chiral" and "anti-chiral") is undoubtable.
- The bound for SS, spin 0 operator is shown by red dots:



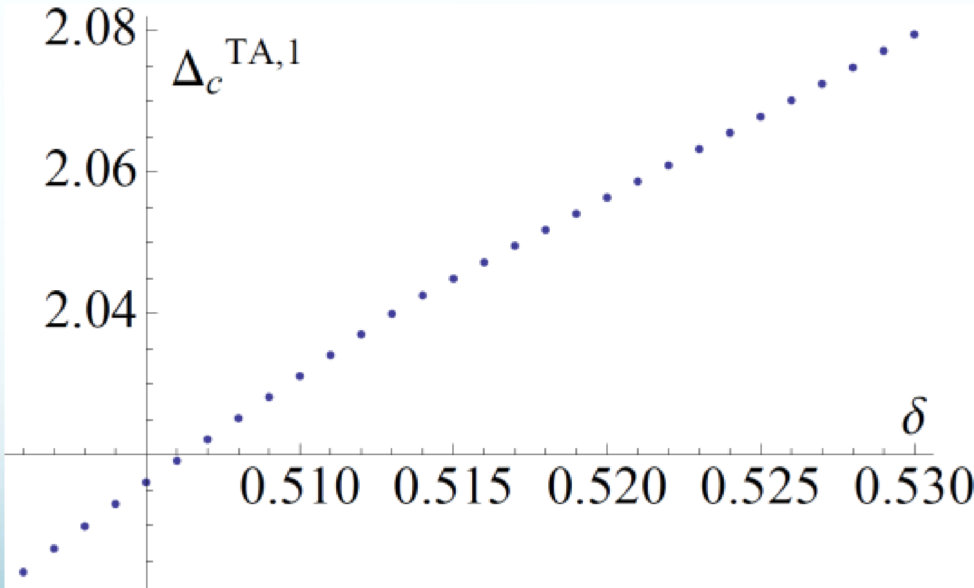
Symmetry enhancement

- Within the precision the bound is identical to that of O(45).
- Such "symmetry enhancement" has been reported for the 4d SU(N)/SO(2N). Is it a general mathematical statement?
- large N prediction for the additional FPs are well-below the bounds. There are two aspects:

1. ☺ The upper bounds are satisfied and consistent!
2. 😢 We cannot observe any symptom of these fixed points from this computation. Can't we "solve" them??

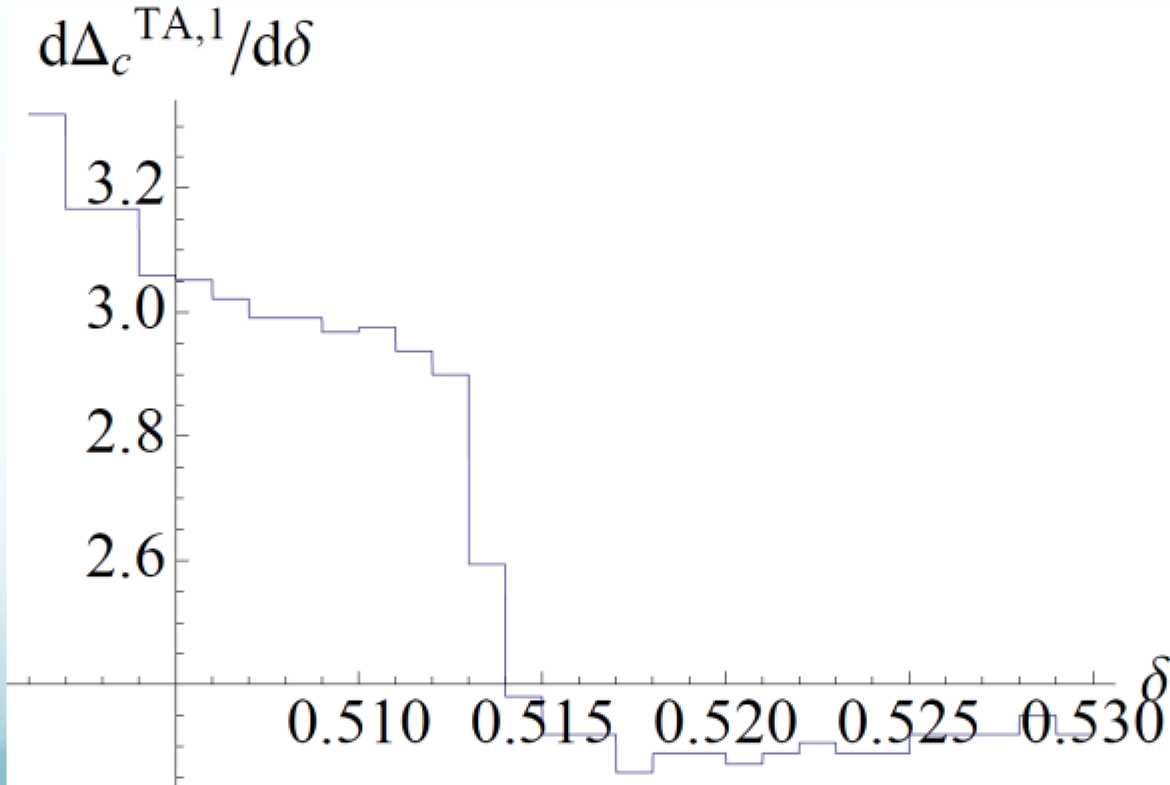
Salvation : Bounds for spin 1 operator in TA sector

- Then we computed the dimension bounds for spin 1 operator in TA representation. Note that such operator has dimension exactly 2 at $O(nm)$ Heisenberg fixed points but not when $O(nm)$ is broken to $O(n) \times O(m)$.



“Kink” in the bound

- When differentiated, it becomes apparent that the slope changes around $\delta \cong 0.515$.



Spectral study

- (El-Showk, Paulos `12) has shown that once a CFT saturate this kind of bounds, spectrum contained in $\phi_I \times \phi_J$ can be uniquely reproduced from the bootstrap output.

- Our result: $$(\Delta_\phi, \Delta^{SS}) = (0.515, 1.16)$$

  Note: although this CFT saturate $\Delta_{c,TA}(\delta)$, it may not do so for the bound in the other sector like $\Delta_{c,SS}(\delta)$!

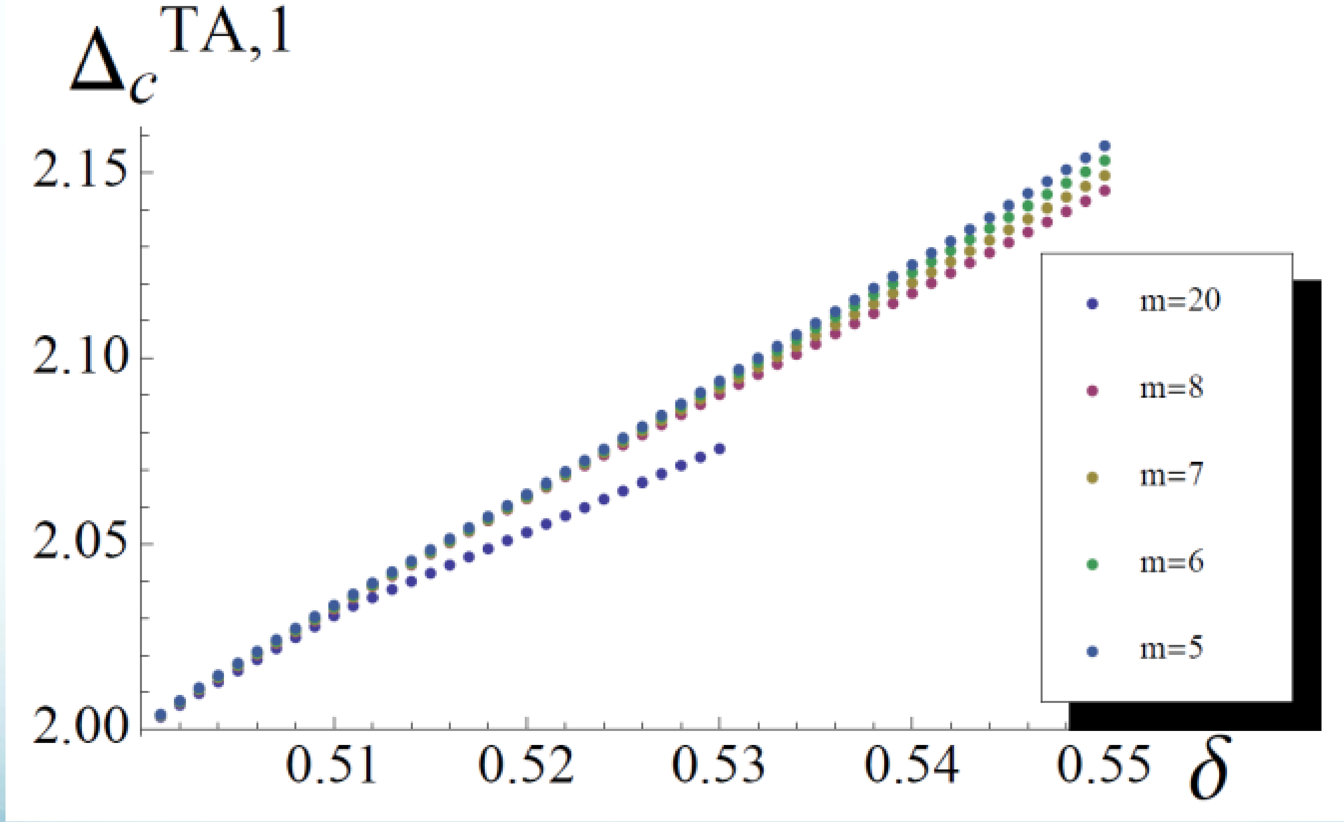
- The $1/n$ – prediction for "anti-chiral" fixed point :

$$(\Delta_\phi, \Delta^{SS}) = (0.5148, 1.142)$$

=>anti-chiral fixed point is observed!

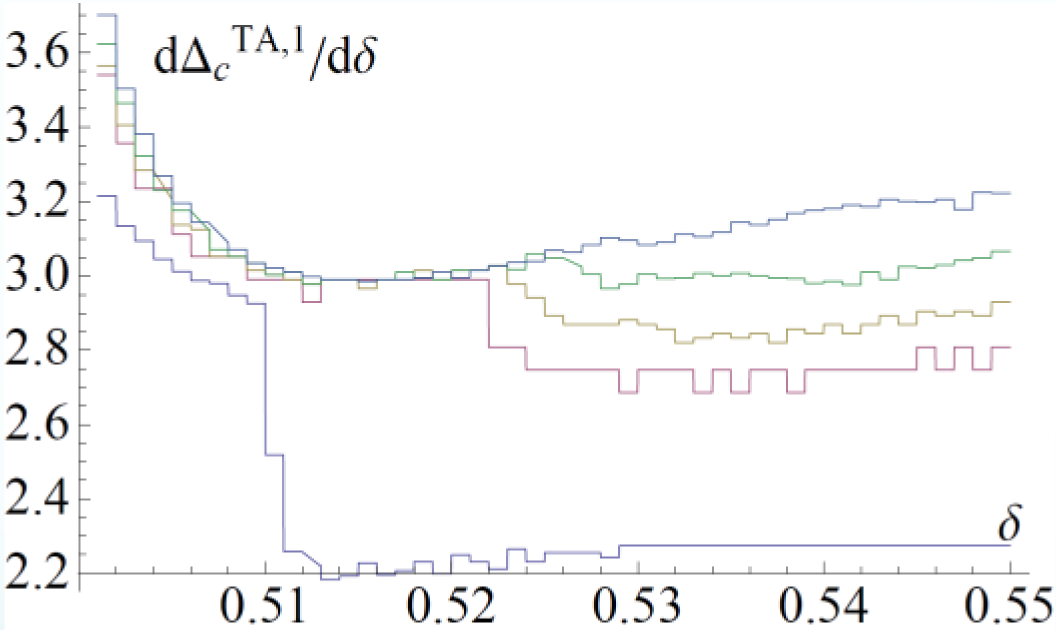
$O(n) \times O(3)$ family

- Varying n, the bounds $\Delta_{c,TA}(\delta)$ changes its form like



Slope change disappearance

- Around $n = 6 \sim 7$, the kink in $\Delta_{c,TA}(\delta)$ disappears.



- According to large n analysis, such a fixed point disappears at $n = 7.3$.

Summary for $O(n) \times O(3)$

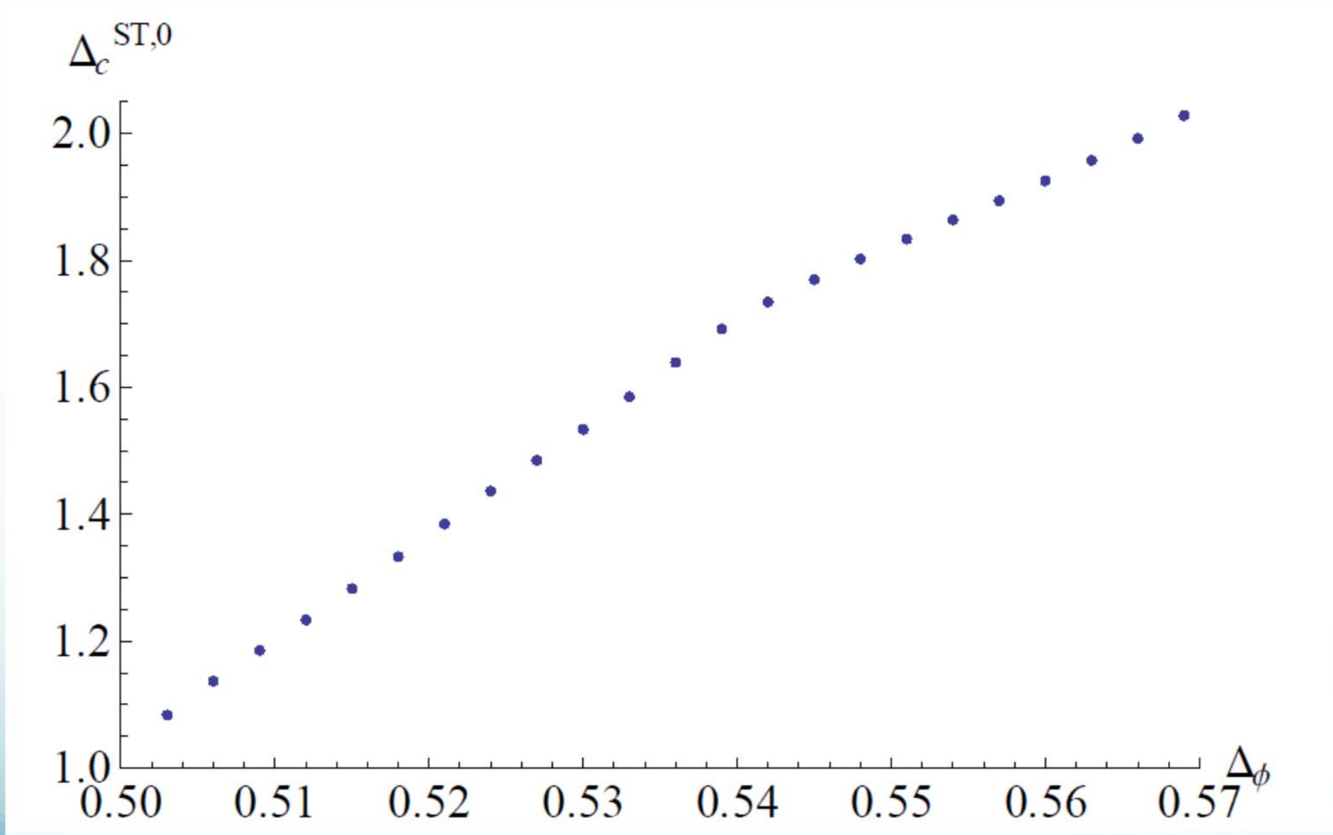
- We examined operator dimension bounds for $O(15) \times O(3)$ model in various global symmetry sector and found that the one in TA sector is saturated by the anti-chiral fixed point. => It is "solvable" as in the $d = 3$ Ising!
- For smaller values of n, the kink present in spin 1 TA sector bounds of $O(n) \times O(3)$ model disappears.
＝＞Might be a reflection of the conformal window.
- This is the first example where we can observe multiple interacting CFTs in single bootstrap eq.
- Conclusion: Everything is consistent with the bootstrap!

2.4 Human : $O(n) \times O(2)$

Based on: arXiv:1407.6195

$O(3) \times O(2)$: a signal of frustrated magnet transitions

- For $O(3) \times O(2)$, the bound for ST sector look like:



The spectra agree!

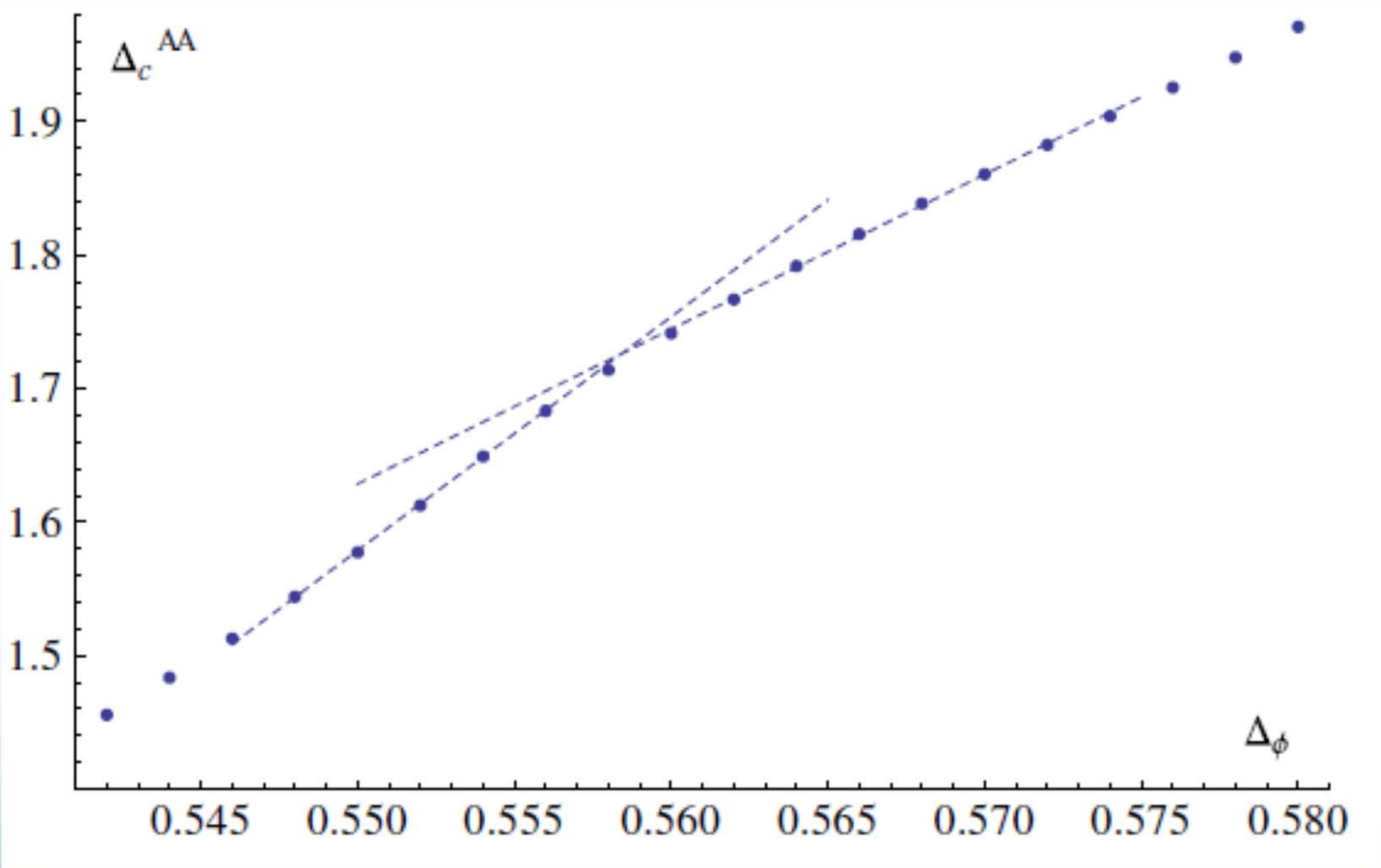
	Δ_ϕ	Δ_{SS}	Δ_{ST}	Δ_{TS}	Δ_{TT}	Δ_{AA}
bootstrap	0.539(3)	1.42(4)	1.69(6)	1.39(3)	1.113(3)	0.89(2)
$\overline{MS}$	0.54(2)	1.41(12)	1.79(9)	1.46(8)	1.04(11)	0.75(12)
MZM	0.55(1)	1.18 (10)	1.91(5)	1.49(3)	1.01(4)	0.65(13)

- The spectra read off around the kink and the higher order $\overline{MS}$ results agree within systematic errors.

- **Most natural explanation: the fixed point actually exists!**

- According to the perturbative analysis, this is IR-stable.

$O(4) \times O(2)$: Signal of the chiral phase transition CFT



The spectral agreement

	Δ_ϕ	Δ_{SS}	Δ_{ST}	Δ_{TS}	Δ_{TT}	Δ_{AA}
bootstrap	0.558(4)	1.52(5)	0.82(2)	1.045(3)	1.26(1)	1.71(6)
$\overline{\text{MS}}$	0.56(3)	1.68(17)	1.0(3)	1.10(15)	1.35(10)	1.9(1)
MZM	0.56(1)	1.59(14)	0.95(15)	1.25(10)	1.34(5)	1.90(15)

- Again they agree and we conjecture that the FP exists.
- IR stable according to the perturbative results.

Summary & Discussions

- Despite various criticism, resumed perturbative RG seems to be robust from the comparison with the bootstrap.
- In particular certain frustrated Heisenberg models, i.e., $O(3) \times O(2)$ LGW model can transit continuously.
- Even when $U(1)_A$ is restored, 2–flavor QCD chiral phase transition could be of second order!! We predicted the critical exponents most precisely.

Theoretical backup needed?

- Our working hypothesis “kink => CFT” has not been rigorously proven even for the simplest cases. At this stage our results are phenomenological.
- The deeper understanding of the bootstrap program would provide the complete answer.

Thank you!

- The legend of bootstrap, applied to modern controversy…

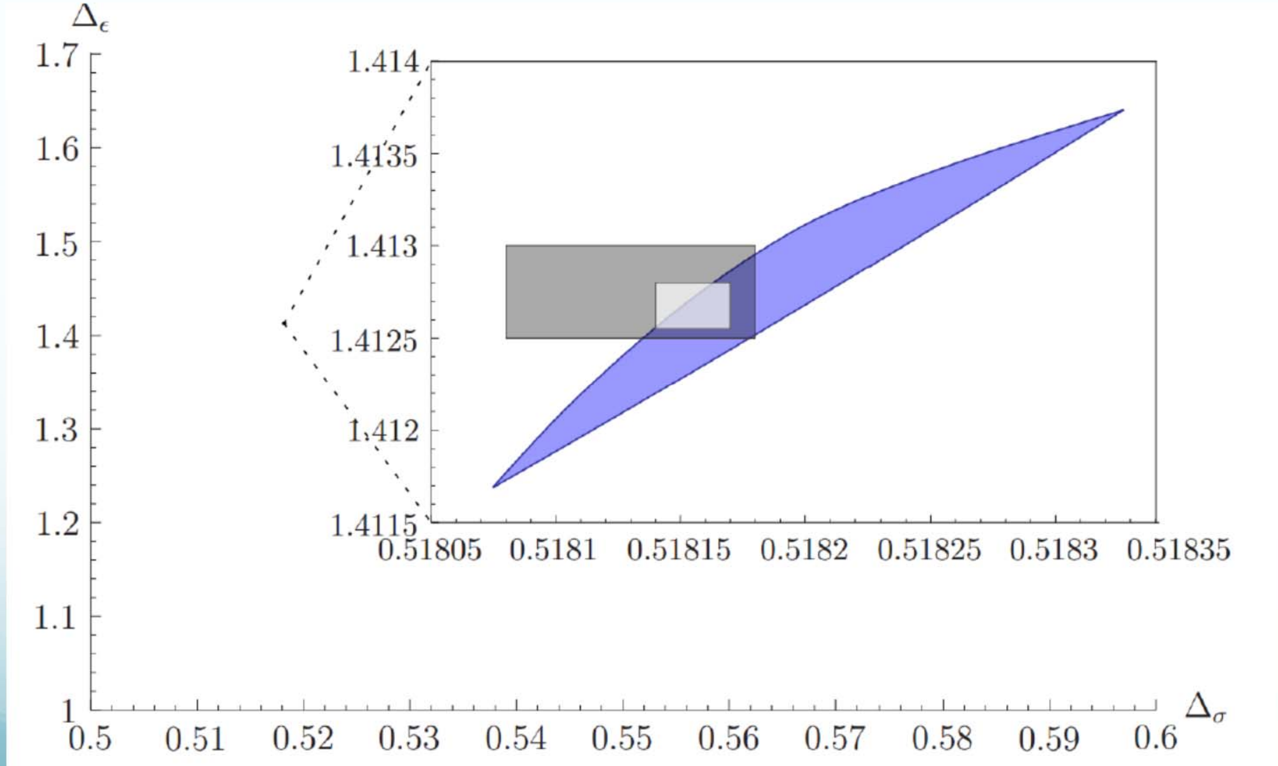
Part 3: Future Prospects

Mixed Correlator study

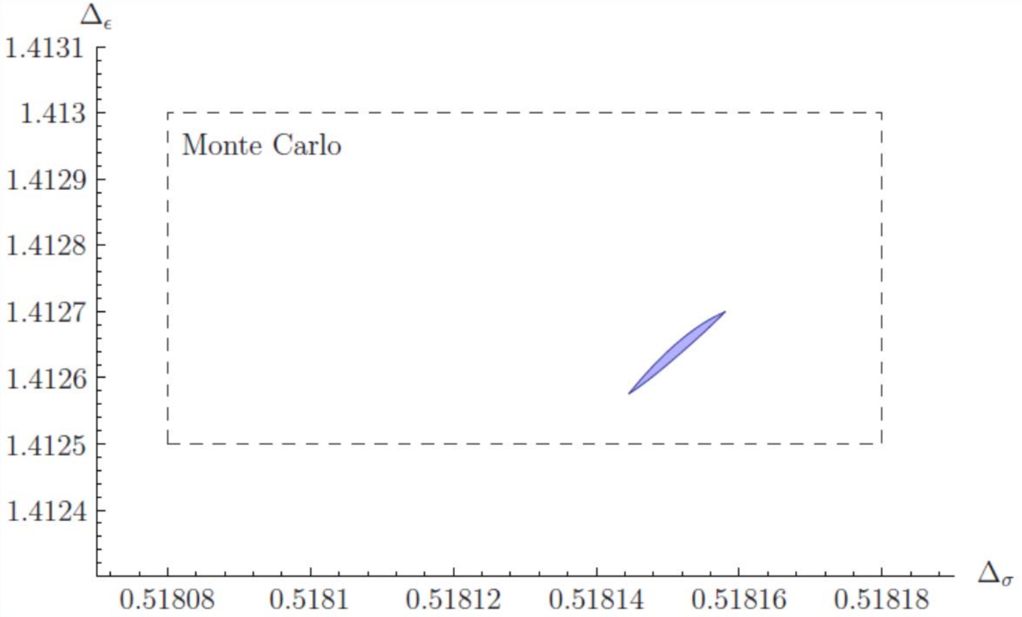
- So far we have been considering only single correlator, $\langle\phi\phi\phi\phi\rangle$.
- Of course there is another opearator $\epsilon(x)$ and we are able to consider the bootstrap equation for
$$\langle\phi\phi\varepsilon\varepsilon\rangle, \langle\phi\phi\phi\phi\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle$$
simultaneously.
- The linear functional argument can be equally applied, but the machinery there is semi-definite programming (a generalization of linear programming).

Universality hypothesis given proof? (Kos *et. al.*,1406.4858)

- Assume that a CFT has only two relevant operator, ϕ, ϵ.
- Then the allowed region for $(\Delta_\phi, \Delta_\epsilon)$ is



Bootstrap state of the art (Simmons-Duffin, 1502.02033)



- Note that the error estimate is rigorous here!!

What's the next?

- Getting more precision?
- The study of correlators with non-scalar operator, like EM-tensor 4pt function
  $$\langle T_{\mu_1\nu_1}(x_1)T_{\mu_2\nu_2}(x_2)T_{\mu_3\nu_3}(x_3)T_{\mu_4\nu_4}(x_4)\rangle$$
  This is now possible, if we have the expression (good approximation algorithm) for the conformal block.
- Now it has become fairly easy to start the bootstrap study, thanks to a user-friendly package, "SDPB" in 1502.02033!
  Anyway all we have to do is to implement the conformal blocks.