QCD thermodynamics from shifted boundary conditions

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Lattice QCD at finite temperature and density, KEK, Ibaraki, Japan, 20-22 January 2014
Contents of this talk

- Introduction
  - finite T with Wilson quarks

- Fixed scale approach
  - quenched results
  - Nf=2+1 QCD results

- Shifted boundary conditions
  - EOS
  - Tc
  - Beta-functions (entropy density)

- Summary
Quark Gluon Plasma in Lattice QCD

Observables in Lattice QCD

- Phase diagram in \((T, \mu, m_{ud}, m_s)\)
- Critical temperature
- Equation of state \((\epsilon/T^4, p/T^4, \ldots)\)
- Hadronic excitations
- Transport coefficients
- Finite chemical potential
- etc...

http://www.gsi.de/fair/experiments/

KEK on finite T & mu QCD

T. Umeda (Hiroshima)
QCD Thermodynamics with Wilson quarks

Most \((T, \mu \neq 0)\) studies at \(m_{\text{phys}}\) are done with Staggered-type quarks

4th-root trick to remove unphysical "tastes"
\[ \rightarrow \text{non-locality "Validity is not guaranteed"} \]

It is important to cross-check with
theoretically sound lattice quarks like Wilson-type quarks

WHOT-QCD collaboration is investigating
QCD at finite \(T\) & \(\mu\) using Wilson-type quarks

Review on WHOT-QCD studies:
S. Ejiri, K. Kanaya, T. Umeda for WHOT-QCD Collaboration,
Recent studies on QCD Thermodynamics

Non-Staggered quark studies at T>0

- Domain-Wall quarks

- Overlap quarks

- twisted mass quarks

- Wilson quarks
  S. Borsanyi et al. (Wuppertal), JHEP08 (2012) 126.

Fixed scale approach is adopted to study T>0
Fixed scale approach to study QCD thermodynamics

Conventional fixed Nt approach
Temperature \( T = \frac{1}{(N_t a)} \) is varied by \( a \) at fixed \( N_t \)

- Coupling constants are different at each \( T \)
  - To study Equation of States
    - \( T=0 \) subtractions at each \( T \)
    - beta-functions at each \( T \)
    - Line of Constant Physics (for full QCD)

\[
\frac{a_{\text{max}}}{a_{\text{min}}} = \frac{T_{\text{max}}}{T_{\text{min}}} > 3
\]

These are done in \( T=0 \) simulations
- larger space-time volume
- smaller eigenvalue in Dirac op.
\( \rightarrow \) larger part of the simulation cost

\( a \) : lattice spacing
\( N_t \) : lattice size in t-direction
Fixed scale approach to study QCD thermodynamics

- Fixed scale approach
  - Temperature $T=1/(N_t a)$ is varied by $N_t$ at fixed $a$

  - Coupling constants are common at each $T$
    - To study Equation of States
      - $T=0$ subtractions are common
      - beta-functions are common
      - Line of Constant Physics is automatically satisfied

  - Cost for $T=0$ simulations can be largely reduced

  - However possible temperatures are restricted by integer $N_t$
    - critical temperature $T_c$
    - EOS

a : lattice spacing
$N_t$ : lattice size in t-direction
We propose the T-integration method to calculate the EOS at fixed scales

T. Umeda et al. (WHOT-QCD), Phys. Rev. D79 (2009) 051501

Our method is based on the trace anomaly (interaction measure),

$$\frac{\epsilon - 3p}{T^4} = \left(\frac{N_t^3}{N_s^3}\right) a \frac{d\beta}{da} \left\langle \frac{dS}{d\beta} \right\rangle_{sub}$$

and the thermodynamic relation.

$$\frac{\epsilon - 3p}{T^4} = T \frac{\partial (p/T^4)}{\partial T}$$

$$\Rightarrow \frac{p}{T^4} = \int_0^T dT' \frac{\epsilon - 3p}{T'^5}$$
Test in quenched QCD

- Our results are roughly consistent with previous results.

- at higher $T$
  lattice cutoff effects
  ($aT \sim 0.3$ or higher)

- at lower $T$
  finite volume effects
  $V > (2\text{fm})^3$ is necessary $T < T_c$

Anisotropic lattice is a reasonable choice

[+] G. Boyd et al., NPB469, 419 (1996)
EOS for $N_f=2+1$ improved Wilson quarks

\[ S = S_g + S_q \]

\[ S_g = -\beta \left\{ \sum_{x,\mu > \nu} c_0 W_{\mu \nu}^{1 \times 1}(x) + \sum_{x,\mu,\nu} c_1 W_{\mu \nu}^{1 \times 2}(x) \right\} \]

\[ S_q = \sum_{f=u,d,s} \sum_{x,y} \bar{q}_f^x D_{x,y} q_f^y \]

\[ D_{x,y} = \delta_{x,y} - \kappa_f \sum_{\mu} \left\{ (1 - \gamma_\mu) U_{x,\mu} \delta_{x+\tilde{\mu},y} + (1 + \gamma_\mu) U_{x-\tilde{\mu},\mu}^\dagger \delta_{x-\tilde{\mu},y} \right\} - \delta_{x,y} c_{SW} \kappa_f \sum_{\mu > \nu} \sigma_{\mu \nu} F_{\mu \nu} \]

\[ \frac{\epsilon - 3p}{T^4} = \frac{N_t^3}{N_s^3} \left( a \frac{\partial \beta}{\partial a} \left\langle \frac{\partial S}{\partial \beta} \right\rangle_{sub} + a \frac{\partial \kappa_{ud}}{\partial a} \left\langle \frac{\partial S}{\partial \kappa_{ud}} \right\rangle_{sub} + a \frac{\partial \kappa_s}{\partial a} \left\langle \frac{\partial S}{\partial \kappa_s} \right\rangle_{sub} \right) \]

\[ \left\langle \frac{\partial S}{\partial \beta} \right\rangle = N_s^3 N_t \left( - \sum_{x,\mu > \nu} c_0 W_{\mu \nu}^{1 \times 1}(x) + \sum_{x,\mu,\nu} c_1 W_{\mu \nu}^{1 \times 2}(x) \right) + N_f \frac{\partial c_{SW}}{\partial \beta} \kappa_f \left\langle \sum_{x,\mu > \nu} \text{Tr}^{(c,s)} \sigma_{\mu \nu} F_{\mu \nu}(D^{-1})_{x,x} \right\rangle \]

\[ \left\langle \frac{\partial S}{\partial \kappa_f} \right\rangle = N_f N_s^3 N_t \left( \sum_{x,\mu} \text{Tr}^{(c,s)} \left\{ (1 - \gamma_\mu) U_{x,\mu} (D^{-1})_{x+\tilde{\mu},x} + (1 + \gamma_\mu) U_{x-\tilde{\mu},\mu}^\dagger (D^{-1})_{x-\tilde{\mu},x} \right\} \right) + c_{SW} \left\langle \sum_{x,\mu > \nu} \text{Tr}^{(c,s)} \sigma_{\mu \nu} F_{\mu \nu}(D^{-1})_{x,x} \right\rangle \]

**Noise method** ( \#noise = 1 for each color & spin indices )
T=0 & T>0 configurations for $N_f=2+1$ QCD

- **T=0 simulation:** on $28^3 \times 56$
  - RG-improved glue + NP-improved Wilson quarks
  - $V \sim (2 \text{ fm})^3$, $a \approx 0.07 \text{ fm}$, ($m_\pi \sim 634 \text{ MeV}$, $\frac{m_\pi}{m_\rho} = 0.63$, $\frac{m_{\eta_s}}{m_\phi} = 0.74$)
  - configurations available on the ILDG/JLDG


- **T>0 simulations:** on $32^3 \times N_t$ ($N_t=4, 6, \ldots, 14, 16$) lattices
  - RHMC algorithm, same coupling parameters as T=0 simulation

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KEK on finite T & mu QCD

T. Umeda (Hiroshima)
Equation of State in $N_f=2+1$ QCD

T. Umeda et al. (WHOT-QCD)

- T-integration

$$\frac{p}{T^4} = \int_0^T dT' \frac{\epsilon - 3p}{T'^5}$$

is performed by Akima Spline interpolation.

- A systematic error for beta-functions

- Numerical error propagates until higher temperatures
Summary on Fixed scale approach

Fixed scale approach for EOS

- EOS (p, e, s, ...) by T-integral method
- Cost for T=0 simulations can be largely reduced
- possible temperatures are restricted by integer $N_t$
- beta-functions are still a burden

- Some groups adopted the approach
  - S. Borsanyi et al. (Wuppertal), JHEP08 (2012) 126.

- Physical point simulation with Wilson quarks is on going
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- Fixed scale approach
  - quenched results
  - $N_f=2+1$ QCD results

- Shifted boundary conditions
  - EOS
  - $T_c$
  - Beta-functions (entropy density)

- Summary
**Shifted boundary conditions**

Thermal momentum distribution from path integrals with shifted boundary conditions

New method to calculate thermodynamic potentials (entropy density, specific heat, etc.)

The method is based on the partition function

\[
Z(\vec{z}) = Tr \{ e^{-L_0 \hat{H}} e^{i\hat{p}\vec{z}} \}
\]

which can be expressed by Path-integral with shifted boundary condition

\[
\phi(L_0, \vec{x}) = \pm \phi(0, \vec{x} + \vec{z})
\]

- L. Giusti and H. B. Meyer, JHEP 01 (2013) 140
Shifted boundary conditions

Due to the Lorentz invariance of the theory, the free-energy depends on $L_0$ and the boundary shift $\tilde{z}$ only through the combination $\sqrt{L_0^2 + z^2}$

$$f(L_0, \tilde{z}) = f(\sqrt{L_0^2 + z^2}, 0)$$

$$\phi(L_0, \tilde{x}) = \pm \phi(0, \tilde{x} + \tilde{z})$$

$$\tilde{z} = a\tilde{n}$$
Shifted boundary conditions

By using the shifted boundary
various T’s are realized with the same lattice spacing

T resolution is largely improved
while keeping advantages of the fixed scale approach

Figure 3: Inverse temperature values that become accessible with the use of shifted boundary conditions at a fixed lattice spacing \( a \) and for different values of \( L_0/a \). The inverse temperatures accessible with a shift in a single direction, \( \xi = (\xi_1, 0, 0) \), are marked by a double circle.

\[
\beta = \frac{1}{T}, \quad \tilde{z} = L_0 \tilde{\xi}
\]
Test in quenched QCD

Simulation setup
- quenched QCD
- $\beta=6.0$
  - $a \sim 0.1\text{fm}$
- $32^3 \times N_t$ lattices, $N_t = 3, 4, 5, 6, 7, 8, 9$ and $32$ ($T=0$)
  - $T_c(N_f=0) \sim 2 \times T_c(N_f=2+1, m_{\text{phys}})$
- boundary condition
  - spatial: periodic boundary condition
  - temporal: shifted boundary condition
  - $U_\mu(L_0, \vec{x}) = U_\mu(0, \vec{x} + \vec{z})$
- heat-bath algorithm (code for SX-8R)
  - only “even-shift” to keep even-odd structure
  - e.g. $\frac{\vec{z}}{a} = (0,0,0), (1,1,0), (2,0,0), (2,1,1), (2,2,0), (3,1,0), ...$
### Choice of boundary shifts

\[ U_\mu (L_0, \vec{x}) = U_\mu (0, \vec{x} + \vec{z}) \quad \vec{z} = a\vec{n} \]

| n^2 | n_1 | n_2 | n_3 | e/o | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 |
|-----|-----|-----|-----|-----|----|---|---|---|---|---|---|---|---|
| 0   | 0   | 0   | 0   | 0   | 10.00 | 9.00 | 8.00 | 7.00 | 6.00 | 5.00 | 4.00 | 3.00 |
| 2   | 1   | 1   | 0   | 0   | 10.10 | 9.11 | 8.12 | 7.14 | 6.16 | 5.20 | 4.24 | 3.32 |
| 4   | 2   | 0   | 0   | 0   | 10.20 | 9.22 | 8.25 | 7.28 | 6.32 | 5.39 | 4.47 | 3.61 |
| 6   | 2   | 1   | 1   | 0   | 10.30 | 9.33 | 8.37 | 7.42 | 6.48 | 5.57 | 4.69 | 3.87 |
| 8   | 2   | 2   | 0   | 0   | 10.39 | 9.43 | 8.49 | 7.55 | 6.63 | 5.74 | 4.90 | 4.12 |
| 10  | 3   | 1   | 0   | 0   | 10.49 | 9.54 | 8.60 | 7.68 | 6.78 | 5.92 | 5.10 | 4.36 |
| 12  | 2   | 2   | 2   | 0   | 10.58 | 9.64 | 8.72 | 7.81 | 6.93 | 6.08 | 5.29 | 4.58 |
| 14  | 3   | 2   | 1   | 0   | 10.68 | 9.75 | 8.83 | 7.94 | 7.07 | 6.24 | 5.48 | 4.80 |
| 16  | 4   | 0   | 0   | 0   | 10.77 | 9.85 | 8.94 | 8.06 | 7.21 | 6.40 | 5.66 | 5.00 |
| 18  | 3   | 3   | 0   | 0   | 10.86 | 9.95 | 9.06 | 8.19 | 7.35 | 6.56 | 5.83 | 5.20 |
| 18  | 4   | 1   | 1   | 0   | 10.86 | 9.95 | 9.06 | 8.19 | 7.35 | 6.56 | 5.83 | 5.20 |
| 20  | 4   | 2   | 0   | 0   | 10.95 | 10.05 | 9.17 | 8.31 | 7.48 | 6.71 | 6.00 | 5.39 |
| 22  | 3   | 3   | 2   | 0   | 11.05 | 10.15 | 9.27 | 8.43 | 7.62 | 6.86 | 6.16 | 5.57 |
| 24  | 4   | 2   | 2   | 0   | 11.14 | 10.25 | 9.38 | 8.54 | 7.75 | 7.00 | 6.32 | 5.74 |
| 26  | 4   | 3   | 1   | 0   | 11.22 | 10.34 | 9.49 | 8.66 | 7.87 | 7.14 | 6.48 | 5.92 |
| 26  | 5   | 1   | 0   | 0   | 11.22 | 10.34 | 9.49 | 8.66 | 7.87 | 7.14 | 6.48 | 5.92 |
| 30  | 5   | 2   | 1   | 0   | 11.40 | 10.54 | 9.70 | 8.89 | 8.12 | 7.42 | 6.78 | 6.24 |
| 32  | 4   | 4   | 0   | 0   | 11.49 | 10.63 | 9.80 | 9.00 | 8.25 | 7.55 | 6.93 | 6.40 |
| 34  | 4   | 3   | 3   | 0   | 11.58 | 10.72 | 9.90 | 9.11 | 8.37 | 7.68 | 7.07 | 6.56 |
Trace anomaly \((e-3p)/T^4\)

\[
\frac{\epsilon - 3p}{T^4} = \left( \frac{1}{VT^3} \right) \frac{d}{da} \left( \frac{dS}{d\beta} \right)_{sub}
\]

Reference data
S. Borsanyi et al., JHEP 07 (2012) 056
Precision SU(3) lattice thermodynamics for a large temperature range

- \(N_s/N_t = 8\) near \(T_c\)
- small \(N_t\) dependence at \(T>1.3T_c\)
- peak height at \(N_t=6\) is about 7% higher than continuum value
- assuming \(T_c=293\)MeV

The continuum values referred as “continuum”
Trace anomaly \( \frac{\varepsilon - 3p}{T^4} \)

\[
\frac{\varepsilon - 3p}{T^4} = \left( \frac{1}{VT^3} \right) a \frac{d\beta}{da} \left\langle \frac{dS}{d\beta} \right\rangle_{sub}
\]

no shifted boundary

\( \beta \)-function: Boyd et al. (1998)
Trace anomaly \( (\varepsilon - 3p)/T^4 \)

\[
\frac{\varepsilon - 3p}{T^4} = \left( \frac{1}{VT^3} \right) a \frac{d\beta}{da} \left\{ \frac{dS}{d\beta} \right\}_{\text{sub}}
\]

\[
T = \frac{1}{a \sqrt{N_t^2 + \vec{n}^2}} \quad V = \prod_{i=1}^{3} \frac{aN_{s_i}}{\sqrt{1 + \left( \frac{n_{s_i}}{N_t} \right)^2}}
\]

\( w/o \) shifted boundary

\( w/ \) shifted boundary

beta-function: Boyd et al. (1998)

KEK on finite T & mu QCD

T. Umeda (Hiroshima)
Lattice artifacts from shifted boundaries

- Lattice artifacts are suppressed at larger shifts
- Non-interacting limit with fermions should be checked

Figure 2: Pressure at finite lattice spacing for the SU(N) Yang–Mills theory in the non-interacting limit. The discretization used is the Wilson action and the 'clover' form of the lattice field strength tensor. The inverse temperature is given by $\beta = L_0 \sqrt{1 + \xi^2}$, and $a$ is the lattice spacing.
Critical temperature $T_c$

Polyakov loop is difficult to be defined because of misalignment of time and compact directions.

Dressed Polyakov loop

E. Bilgici et al.,

Polyakov loop defined with light quarks

$$\Sigma_n(m, V) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i\phi n} \frac{1}{V} \langle Tr[(m + D_\phi)^{-1}] \rangle_G$$

KEK on finite T & mu QCD

FIG. 2 (color online). The dressed Polyakov loop at $m = 100$ MeV in units of GeV$^3$ as a function of the temperature $T$ in MeV.
Critical temperature $T_c$

Plaquette value

$$\langle P \rangle = \frac{1}{6N_s^3N_t} \sum_P \langle 1 - \frac{1}{3}ReTrU_P \rangle$$

Plaquette susceptibility

$$\chi_P = N_s^3N_t \left( \langle P^2 \rangle - \langle P \rangle^2 \right)$$

Plaq. suscep. has a peak around $T = 293$ MeV
Beta-functions (in case of quenched QCD)

$$\frac{\epsilon - 3p}{T^4} = \left( \frac{1}{VT^3} \right) a \frac{d\beta}{da} \left\langle \frac{dS}{d\beta} \right\rangle_{sub}$$

In the fixed scale approach, beta-func at the simulation point is required.

However, T=0 simulations near the point are necessary to calculate the beta-function.

We are looking for new methods to calculate beta-function:
- Reweighting method
- Shifted boundary condition
Entropy density from shifted boundaries

Entropy density $s/T^3$
from the cumulant of the momentum distribution

$$\frac{s(T)}{T^3} = \lim_{a \to 0} \frac{2K(T, \vec{z}, a)}{|\vec{z}|^2T^5V}$$

$$K(T, \vec{z}, a) = -\ln \frac{Z(T, \vec{z}, a)}{Z(T, \vec{0}, a)}$$

$Z(T, \vec{z}, a)$ : partition function with shifted boundary

where $\vec{z} = (0, 0, n_z a)$, $n_z$ being kept fixed when $a \to 0$

FIG. 1 (color online). Scaling behavior of $s/T^3$; see Eq. (15). The Stefan-Boltzmann value reached in the high-$T$ limit is also displayed.
Entropy density from shifted boundaries

- Entropy density at a temperature \(T_0\) by the new method with shifted b.c.
  \[ s(T_0) \]

- Entropy density w/o beta-function by the T-integral method
  \[ s(T)/a \frac{d\beta}{da} \]

\[
\frac{\epsilon - 3p}{T^4} = \left( \frac{N_t^3}{N_s^3} \right) \frac{d\beta}{da} \left\langle \frac{dS}{d\beta} \right\rangle_{sub} \\
\frac{p}{T^4} = \int_0^T dT' \frac{\epsilon - 3p}{T'^5} \]

\[ Ts = \epsilon + p \]

Beta-func is determined by matching of entropy densities at \(T_0\)

**FIG. 1** (color online). Scaling behavior of \(s/T^3\); see Eq. (15). The Stefan-Boltzmann value reached in the high-\(T\) limit is also displayed.
momentum distribution

\[
\frac{R(\vec{p})}{V} = \frac{Tr\{e^{-L_0\hat{H}}\hat{P}(\vec{p})\}}{Tr\{e^{-L_0\hat{H}}\}}
\]

\(L_0\) : Temporal extent

\(\hat{H}\) : Hamiltonian

\(\hat{P}(\vec{p})\) : projector onto states with total momentum \(p\)

The generating function \(K(z)\) of the cumulants of the mom. dist. is defined

\[
e^{-K(\vec{z})} = \frac{1}{V} \sum_{\vec{p}} e^{i\vec{p} \cdot \vec{z}} R(\vec{p})
\]

the cumulants are given by

\[
k\{2n_1,2n_2,2n_3\} = (-1)^{n_1+n_2+n_3+1} \frac{\partial^{2n_1}}{\partial z_1^{2n_1}} \frac{\partial^{2n_2}}{\partial z_2^{2n_2}} \frac{\partial^{2n_3}}{\partial z_3^{2n_3}} \frac{K(\vec{p})}{V} \bigg|_{\vec{z}=0}
\]
The generating func. $K(p)$ can be written with the partition function

$$e^{-K(p)} = \frac{Z(\vec{\tau})}{Z}$$

$$Z(\vec{\tau}) = Tr\{e^{-L_0 \hat{H} e^{i\vec{\tau} \vec{\theta}}}\}$$

$Z(z)$ can be expressed as a path integral with the field satisfying the shifted b.c.

By the Ward Identities, the cumulant is related to the entropy density “$s$”

$$k_{\{0,0,2\}} = T(\epsilon + p) = T^2 s$$

$$s = -\frac{1}{T^2} \lim_{V \to \infty} \frac{1}{V} \frac{d^2}{d\tau^2} \ln Z(\{0,0,\tau\})|_{\tau=0}$$

The specific heat and speed of sound can be also obtained in the method.
How to calculate $k_{\{0,0,2\}}$

Evaluation of $Z(z)/Z$ with reweighting method

$$\frac{Z(\bar{z})}{Z} = \prod_{i=1}^{n-1} \frac{Z(r_i)}{Z(r_{i+1})}$$

$$\bar{S}(U, r_i) = r_i S(U) + (1 - r_i) S(U^z)$$

$$\frac{Z(r_i)}{Z(r_{i+1})} = \langle e^{\bar{S}(U, r_{i+1}) - \bar{S}(U, r_i)} \rangle_{r_{i+1}}$$

$$K(\bar{z}) = -\ln \frac{Z(\bar{z})}{Z} = -\sum_{i=0}^{n-1} \ln \frac{Z(r_i)}{Z(r_{i+1})}$$

$$k_{\{2n_1, 2n_2, 2n_3\}} = (-1)^{n_1+n_2+n_3+1} \frac{\partial^{2n_1}}{\partial \bar{z}_1^{2n_1}} \frac{\partial^{2n_2}}{\partial \bar{z}_2^{2n_2}} \frac{\partial^{2n_3}}{\partial \bar{z}_3^{2n_3}} \frac{K(\bar{z})}{V} \bigg|_{\bar{z}=0}$$
Summary & outlook

We presented our study of the QCD Thermodynamics by using **Fixed scale approach** and **Shifted boundary conditions**

- **Fixed scale approach**
  - Cost for $T=0$ simulations can be largely reduced
  - First result in $N_f=2+1$ QCD with Wilson-type quarks

- **Shifted boundary conditions** are promising tool to improve the fixed scale approach
  - Fine resolution in Temperature
  - Suppression of lattice artifacts at larger shifts
  - $T_c$ determination could be possible
  - New method to estimate beta-functions

- **Test in full QCD** $\rightarrow N_f=2+1$ QCD at the physical point
Thank you for your attention!
\[\epsilon_{\nu} \langle \partial_{\mu} T_{\mu \nu}(x) O_1 \cdots O_n \rangle = - \sum_{i=1}^{n} \langle O_1 \cdots \delta_{\epsilon} O_i \cdots O_n \rangle \]

\[O(y) = T_{0k}(y)\]

\[\partial_{0}^{x} \left\{ \langle \tilde{T}_{0k}(x_0) T_{0k}(y) \rangle - \delta(x_0 - y_0) \langle T_{kk} + \mathcal{L} \rangle \right\} = 0\]

\[\partial_{k}^{w} \left\{ \langle \tilde{T}_{0k}(w_0) T_{0k}(z) \rangle - \delta(w_k - z_k) \langle T_{00} + \mathcal{L} \rangle \right\} = 0\]

\[L_0 \langle \tilde{T}_{0k}(x_0) T_{0k}(y) \rangle - L_k \langle \tilde{T}_{0k}(w_k) T_{0k}(z) \rangle = \langle T_{00} \rangle - \langle T_{kk} \rangle\]

\[V \to \infty \quad L_0 \langle \tilde{T}_{0k}(x_0) T_{0k}(y) \rangle = \langle T_{00} \rangle - \langle T_{kk} \rangle = -(e + p) = -Ts\]

\[\langle \tilde{T}_{03}(x_0) T_{03}(y) \rangle = -k_{\{0,0,2\}}\]