Quantum field theory on the Lefschetz thimble

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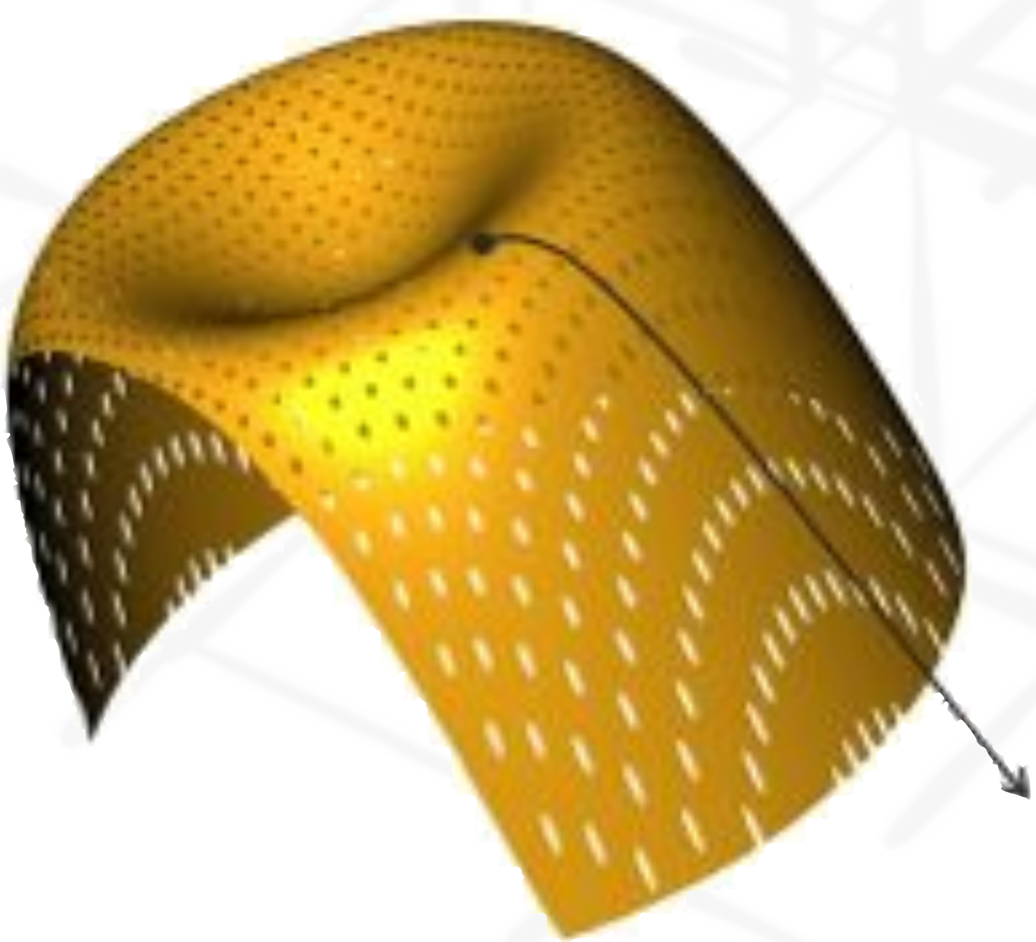


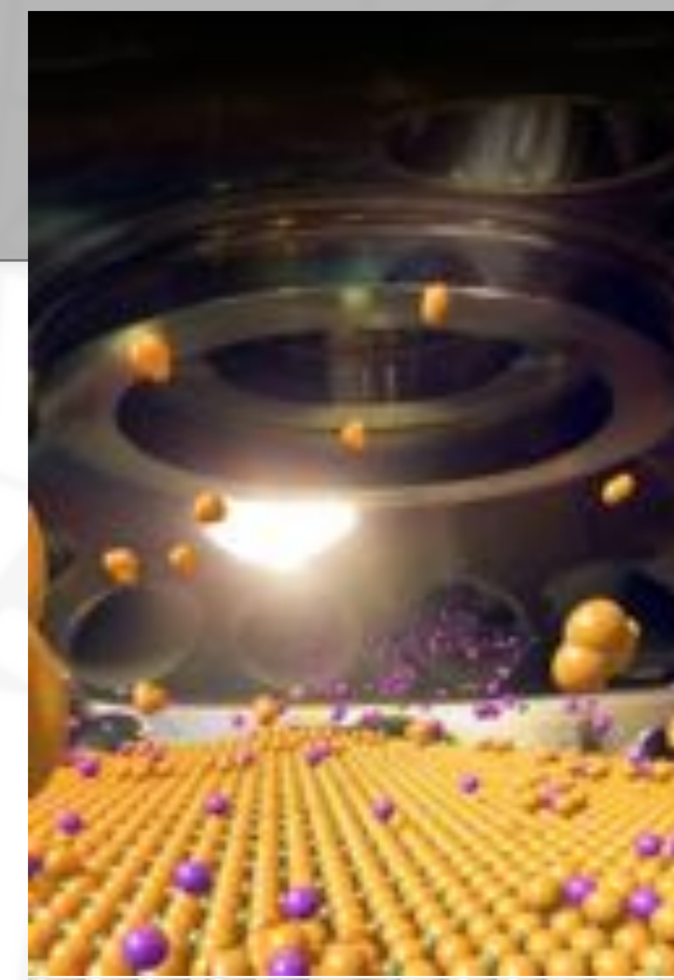
PRD86, 074506 (2012)
PRD Rapid 88, 051501 (2013)
PRD Rapid 88, 051502 (2013)

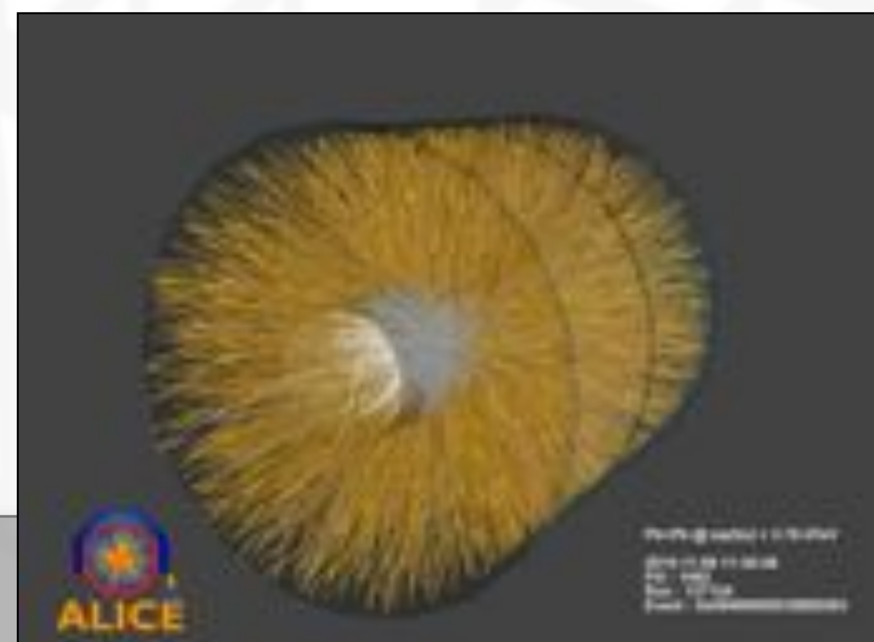
LATTICE QCD AT FINITE TEMPERATURE AND DENSITY
KEK - JANUARY 21 - 2014

Introduction: dense systems

Nuclear physics

Condensed matter





High energy physics

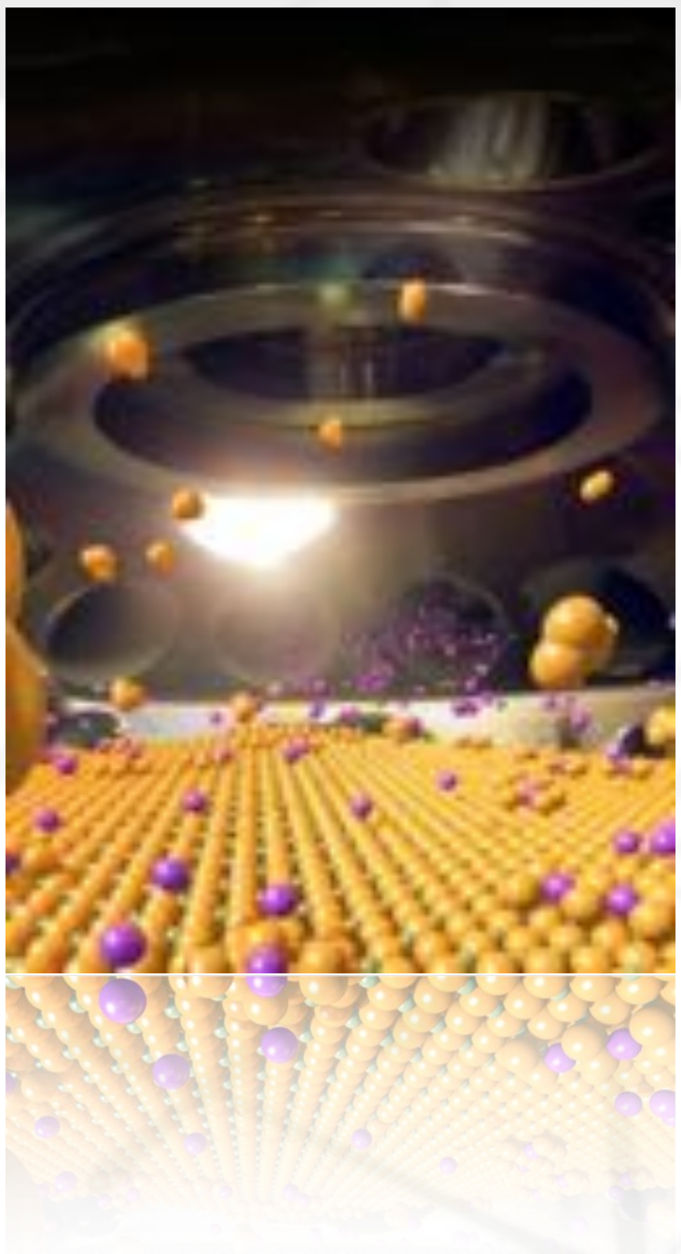
Introduction: dense systems

Studied using Monte Carlo simulations

Introduction: dense systems

Condensed matter

High T_c superconductivity



Hubbard model

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta(\mathcal{H}_K+\mathcal{H}_V+\mathcal{H}_\mu)}\right]$$

$$\mathcal{H}_K = -t \sum_{\langle i,j\rangle,\sigma} (c^\dagger_{i\sigma}c_{j\sigma} + c^\dagger_{j\sigma}c_{i\sigma})$$

$$\mathcal{H}_\mu = -\mu \sum_i (n_{i\uparrow} + n_{i\downarrow})$$

$$\mathcal{H}_V = U \sum_i \left(n_{i\uparrow} - \frac{1}{2}\right)\left(n_{i\downarrow} - \frac{1}{2}\right).$$

Introduction: dense systems

Nuclear physics

stellar nucleosynthesis

Shell model

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta(\mathcal{H}_{MF}+\mathcal{H}_{V_{res}})}\right]$$

$$H_{MF} = \sum_{\alpha} \epsilon_\alpha a^\dagger_\alpha a_\alpha$$

$$H_{V_{res}} = \frac{1}{2}\sum_{\alpha\beta\gamma\delta} \langle\alpha\beta|V|\gamma\delta\rangle a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma$$

Introduction: dense systems

High energy physics

Heavy ion collisions

QCD on the lattice

$$\mathcal{Z} = \int \mathcal{D}U \, \det[M(U)]^{N_f} \, e^{-S_G[U]}$$

$$S_G[U] = \frac{\beta}{3} \sum_{n \in \Lambda} \sum_{\mu < \nu} \mathrm{Re}\,\mathrm{tr}[\mathbf{1} - U_{\mu\nu}(n)]$$

Introduction: sign problem

- ➘Hubbard Model
- ➘Shell Model
- ➘QCD

What do they have in common?

SIGN PROBLEM

Introduction: dense systems

$$\mathcal{Z} = \int \mathcal{D}A \det[\not{D} + m - \mu\gamma_0/2]e^{S_{YM}}$$

The sign problem $\longrightarrow$ **[det M(μ)]* = det M(-μ*)**

at finite chemical potential
the fermionic determinant is complex:
standard Monte Carlo methods fails

det M(μ) = |det M(μ)| $\cancel{e^{i\theta}}$

Unfortunately we cannot simply neglect the phase of the determinant. Phase quenched theory can be very different from the real world

An example of that difference that we will treat later is
the Silver Blaze phenomenon

Introduction: Lefschetz thimble

We want to overcome sign problem for Lattice QCD
We must be extremely careful not destroying physics
[Silver Blaze phenomenon]
whatever machinery we use to solve the theory

Integration on a Lefschetz thimble

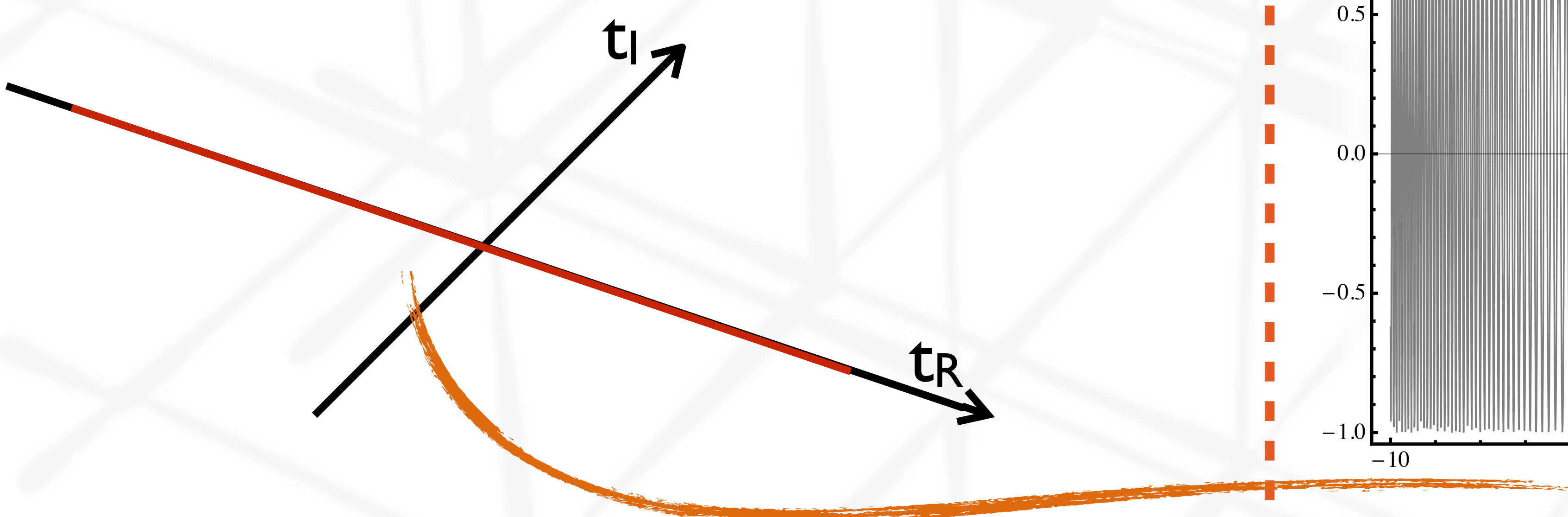
Before applying the idea to full QCD we choose to start from something more manageable: we consider here integration on Lefschetz thimbles for the case of a simple 0-dim field theory and
the 4-dim scalar field with a quartic interaction

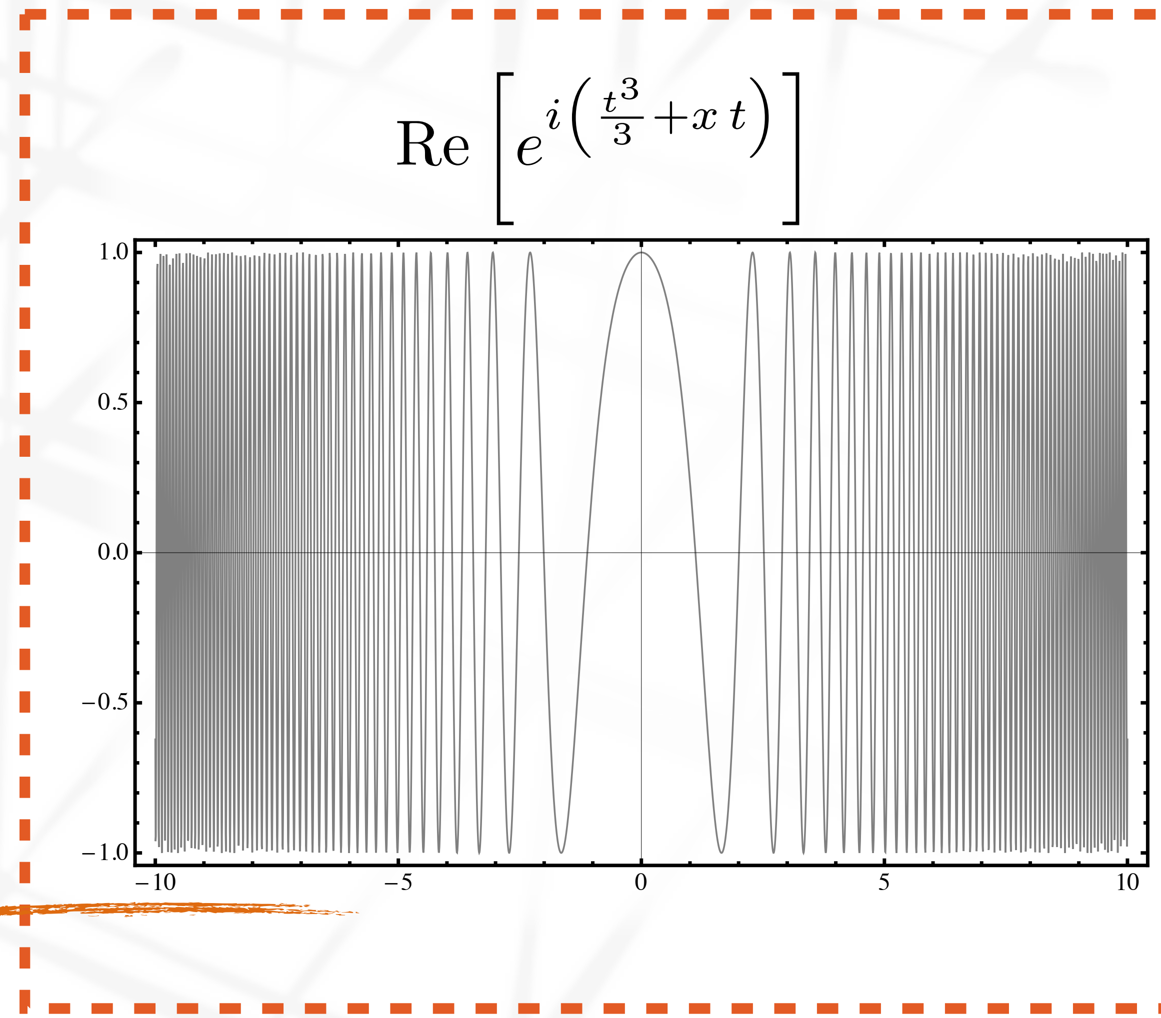
Lefschetz thimble on a lattice

Saddle point integration

the Airy function

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{t^3}{3}+x\,t\right)}\mathrm{d}t$$





Lefschetz thimble on a lattice

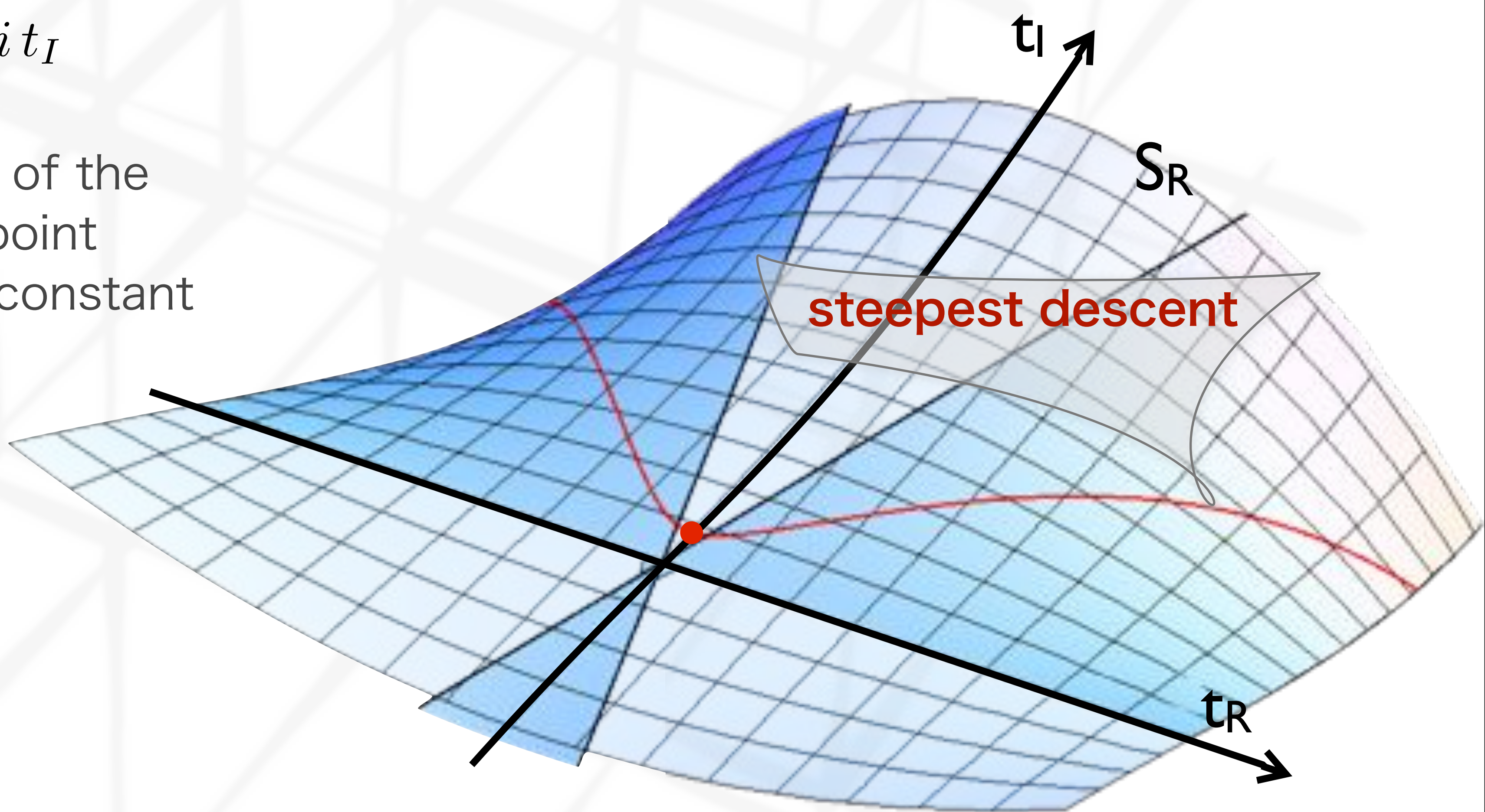
Saddle point integration

the Airy function

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{t^3}{3}+x\,t\right)}\mathrm{d}t$$

$$\xrightarrow{\text{complexify}} \frac{1}{2\pi}\int_\gamma e^{i\left(\frac{z^3}{3}+xz\right)}\mathrm{d}z \xrightarrow{\text{integrate on SD}} \frac{1}{2\pi}e^{i\phi}\int_\gamma e^{\mathbb{R}\left[i\left(\frac{z^3}{3}+xz\right)\right]}\mathrm{d}z$$

- Complexify the variable $t \to t_R + i\,t_I$
- Stationary point
- Steepest descent for the real part of the exponent starting at the stationary point
- Imaginary part of the exponent is constant



Lefschetz thimble on a lattice

Saddle point integration

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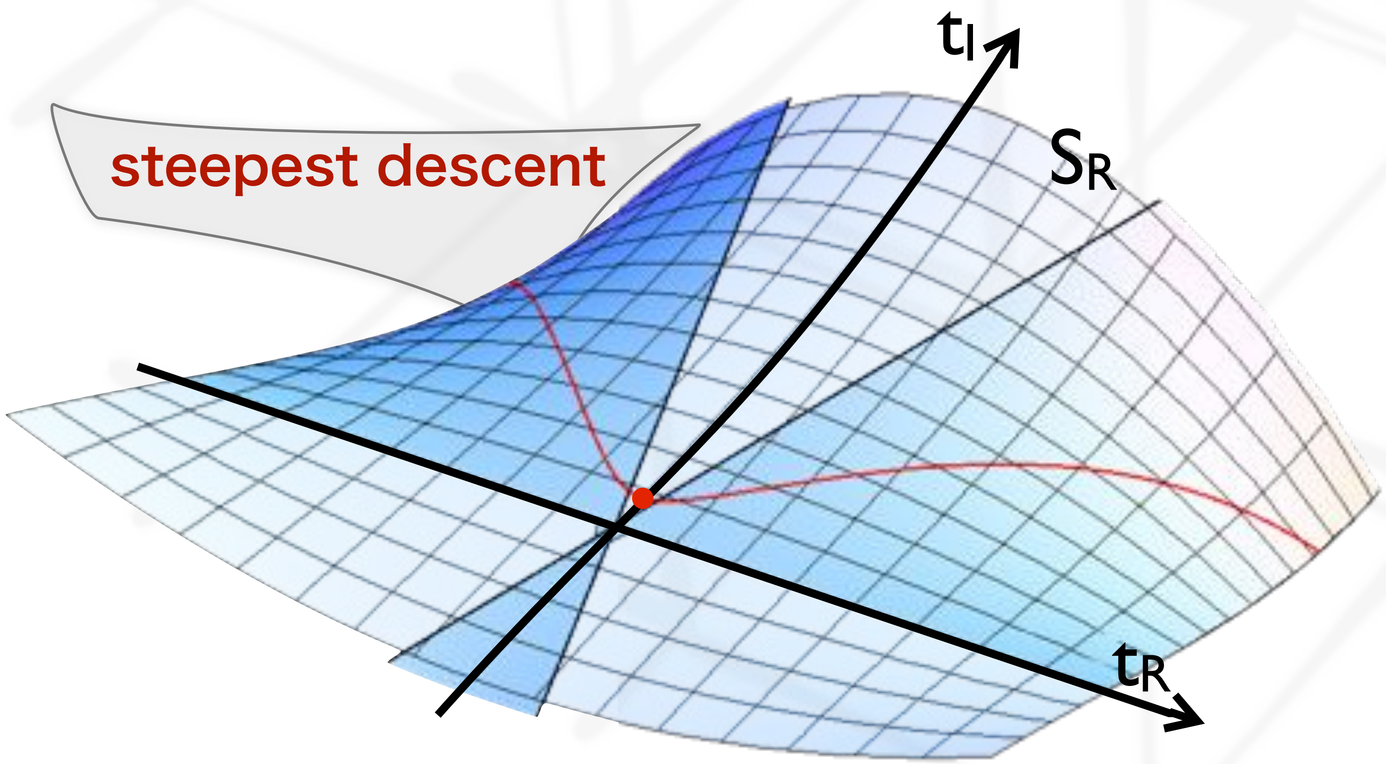
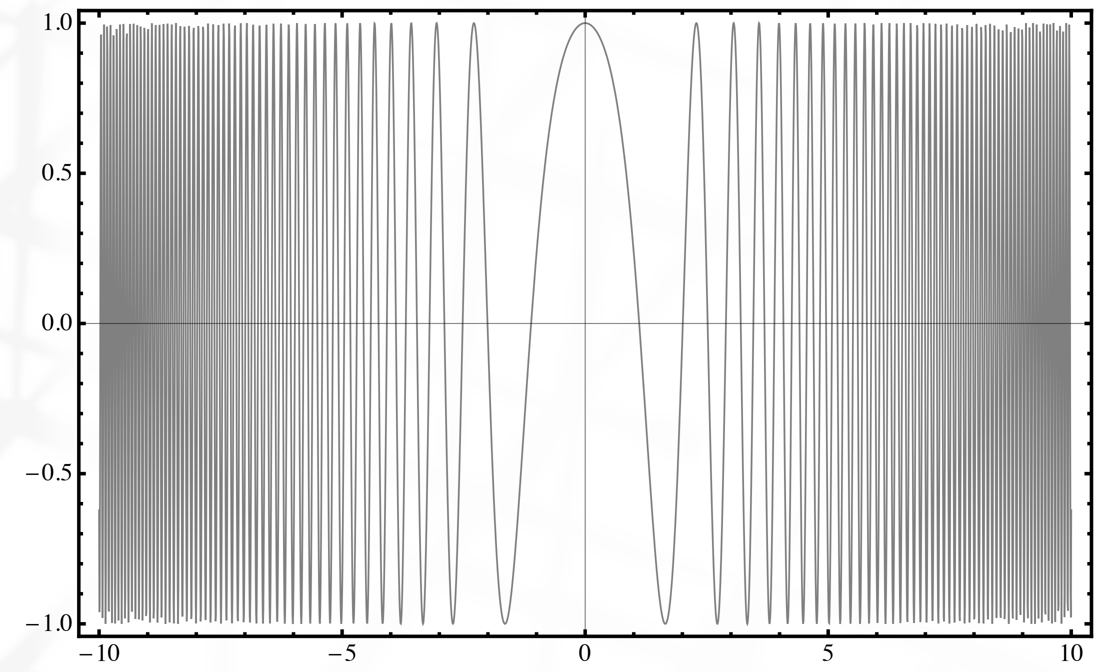
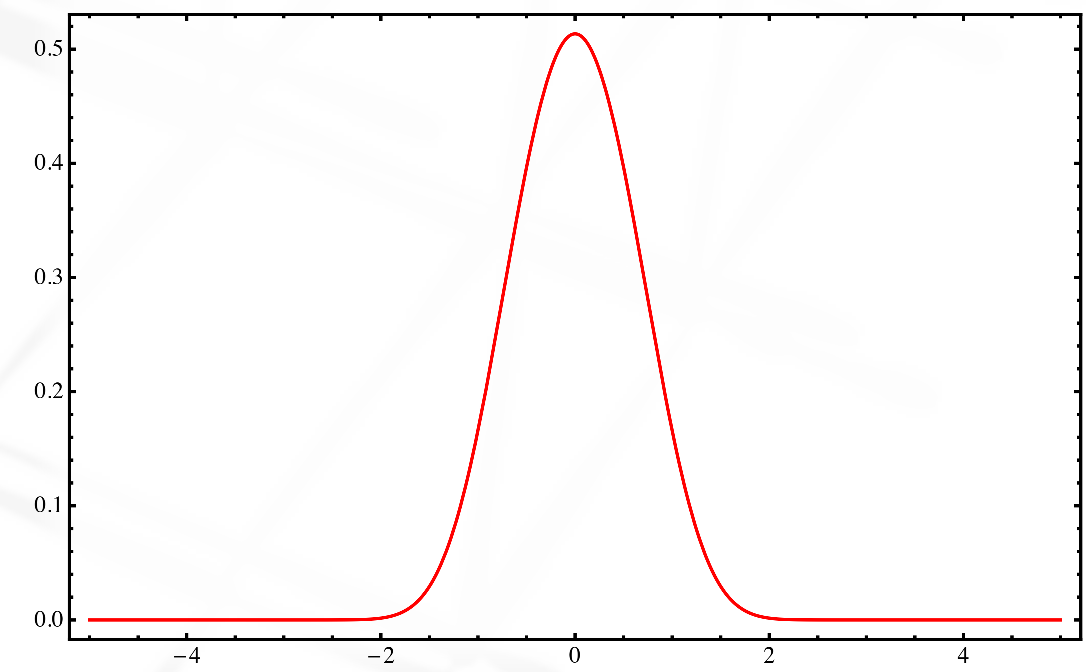


$$\mathrm{Re}\left[e^{i\left(\frac{t^3}{3}+x\,t\right)}\right]$$





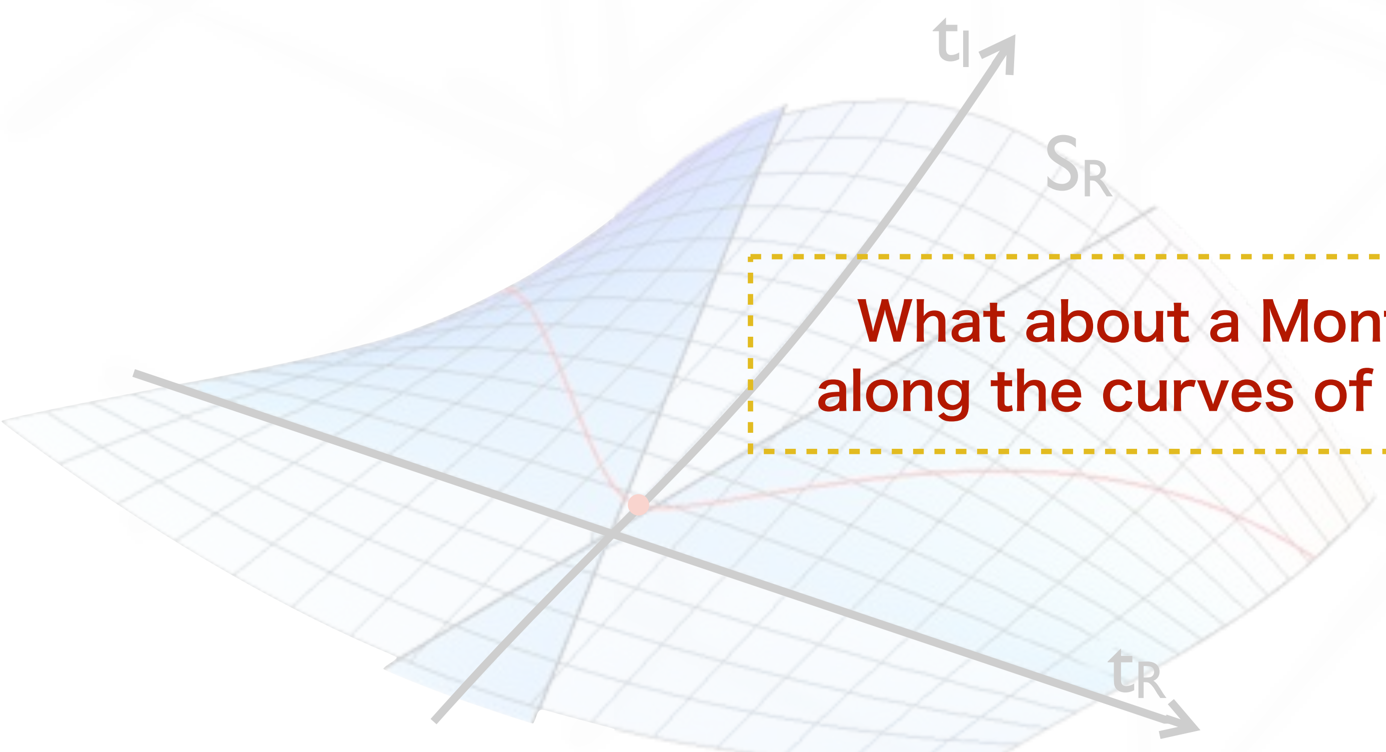
From saddle point to Lefschetz thimble

Saddle point integration

Works extremely well for low dimensional oscillating integrals.

Usually combined with an asymptotic expansion around the stationary point (sort of perturbative expansion).

The phase is stationary +
important contributions localized =
good for sign problem



What about a Monte Carlo integral along the curves of steepest descent

Lefschetz thimble on a lattice

Path integral and Morse theory

E. Witten arXiv:1009.6032 (2010)

- Complexify the degrees of freedom

$$\int_{\mathbb{R}^n} dx^n\, g(x) e^{f(x)} \xrightarrow{z=x+iy} \int_{\mathcal{C}} dz^n\, g(z) e^{f(z)}$$

- Deform appropriately the original integration path (Morse theory)

$$\int_{\mathcal{C}} dz^n\, g(z) e^{f(z)} = \sum_\sigma n_\sigma \int_{\mathcal{L}_\sigma} dz^n g(z) e^{f(z)}$$

$\mathcal{L}_\sigma$ for each stationary point p_σ the L_σ (thimble) is the union of the paths of steepest descent that fall in p_σ at ∞

$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{L}_\sigma$ the thimbles provide a basis of the relevant homology group, with integer coefficients

Generalization of the one dimensional SD to n-dim problems is called Lefschetz thimble

Lefschetz thimble on a lattice

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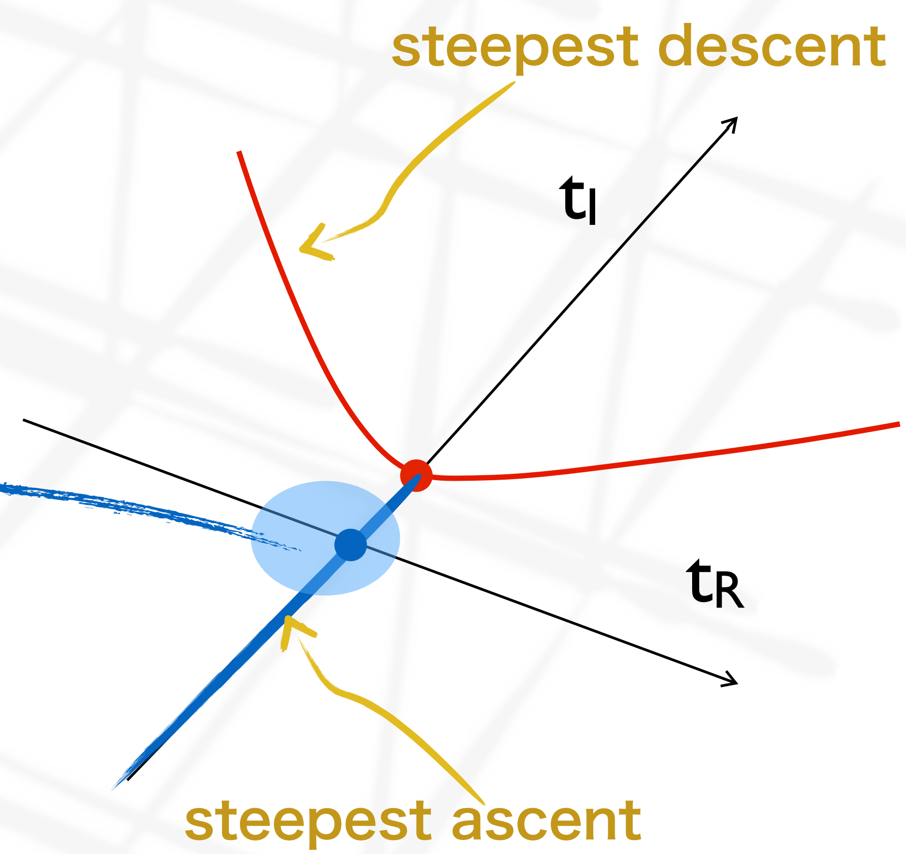
# of intersections between steepest ascent and original integration domain



Lefschetz thimble on a lattice

Lefschetz thimbles and QFT

- Can we use the thimble basis to compute the path integral for a QFT ?

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{C}} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\int_{\mathcal{C}} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}} \longrightarrow \langle \mathcal{O} \rangle = \frac{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}}$$

$\longrightarrow$ In principle yes but we have to discuss “the details”

Lefschetz thimble on a lattice

Lefschetz thimbles and QFT

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{C}} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\int_{\mathcal{C}} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}} \longrightarrow \langle \mathcal{O} \rangle = \frac{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}}$$

- Computing the contribution from all the thimbles is probably not feasible

- **On a Lefschetz thimble the imaginary part of the action is constant** but the measure term does introduce a new **residual phase**, due to the curvature of the thimble

Lefschetz thimble on a lattice

Lefschetz thimbles and QFT

the residual phase

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{J}_0} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\int_{\mathcal{J}_0} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}}$$

Additional phase coming from the Jacobian of the transformation between the canonical complex basis and the tangent space to the thimble

- **Does it lead to a sign problem?**

  No formal proof but ...

  - $d\Phi=1$ at leading order and $<d\Phi> \ll 1$ are strongly suppressed by e^{-S}
  - there is strong correlation between phase and weight (precisely the lack of such correlation is the origin of the sign problem)
  - In fact this residual phase is completely neglected in the saddle point method

Lefschetz thimble on a lattice

Lefschetz thimbles and QFT

the residual phase

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{J}_0} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\int_{\mathcal{J}_0} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}}$$

Additional phase coming from the Jacobian of the transformation between the canonical complex basis and the tangent space to the thimble

Does it lead to a sign problem?

Hopefully not but nevertheless the calculation of that phase cannot be avoid!

$\longrightarrow$ highly demanding in terms of computation power

- H. Fujii et al JHEP 1310 (2013) 147
- Next talk: Y Kikukawa

Lefschetz thimble on a lattice

Lefschetz thimbles and QFT

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{C}} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\int_{\mathcal{C}} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}} \longrightarrow \langle \mathcal{O} \rangle = \frac{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}}$$

- Computing the contribution from all the thimbles is probably not feasible

  $\longrightarrow$ Is it necessary in order to have the correct physics?

  I will try to convince you that in many cases we can consider only one thimble

Lefschetz thimble on a lattice

Lefschetz thimbles and QFT

choosing the stationary point

The study of the stationary points of the complexified theory is mandatory and has to be done on a case-by-case

- ☐ The system has a single global minimum?
- ☐ There are degenerate global minima, that are however connected by symmetries?
- ☐ There are degenerate global minima, with vanishing probability of tunneling?

These cases are good!

- ☐ There is a large number of stationary points that accumulate near the global minimum giving a finite contribution?

This case can be bad!

Lefschetz thimble on a lattice

Lefschetz thimbles and QFT

choosing the stationary point

$$\langle \mathcal{O} \rangle = \frac{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}}$$

This is an exact reformulation of the original path integral.
For a QFT reproducing the original integral is both impractical and unnecessary

- **Consider the stationary point with the lower value of the real part of the action and with $n_\sigma \neq 0$**
  Define a QFT on the thimble attached to this point. If
  - → The degrees of freedom are the same
  - → The symmetries are the same
  - → The same perturbative expansion
  - → The same continuum limit

By universality this is a legitimate regularisation of the original QFT

Lefschetz thimble on a lattice

Lefschetz thimbles and QFT

choosing the stationary point

$$\langle \mathcal{O} \rangle = \frac{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}} \longrightarrow$$

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Universality is not a theorem BUT it is an assumed property studying QFTs on a lattice

By universality this is a legitimate regularisation of the original QFT

Lefschetz thimble on a lattice

Lefschetz thimbles and QFT

choosing the stationary point

$$\langle \mathcal{O} \rangle = \frac{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}} \longrightarrow \langle \mathcal{O} \rangle = \frac{\int_{\mathcal{J}_0} \prod_x \mathrm{d}\phi_x e^{-S[\phi]} \mathcal{O}[\phi]}{\int_{\mathcal{J}_0} \prod_x \mathrm{d}\phi_x e^{-S[\phi]}}$$

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Universality is not a theorem BUT it is an assumed property studying QFTs on a lattice

By universality this is a legitimate regularisation of the original QFT

Lefschetz thimble: algorithm

Integration on a Lefschetz thimble

M. C., F. Di Renzo and L. Scorzato
PRD86, 074506 (2012)

Is it numerically applicable to QFT's on a Lattice?

Before applying the idea to full QCD we choose to start from something more manageable: we consider here integration on the Lefschetz thimbles for the case of

→ **a 0 dimensional field theory with U(1) symmetry**

→ **the four dimensional scalar field with a quartic interaction**

Lefschetz thimble: algorithm

Langevin

PRD Rapid 88, 051501 (2013)

$$\frac{\mathrm{d}\phi_i^R(\tau)}{\mathrm{d}\tau} = -\frac{\delta S^R(\phi(\tau))}{\delta\phi_i^R(\tau)} + \eta_i^R(\tau)$$

$$\frac{\mathrm{d}\phi_i^I(\tau)}{\mathrm{d}\tau} = -\frac{\delta S^R(\phi(\tau))}{\delta\phi_i^I(\tau)} + \eta_i^I(\tau)$$

$$\frac{\mathrm{d}\eta_i(\tau)}{\mathrm{d}\tau} = \sum_k \eta(\tau)_k \partial_k \partial_j S_R$$

projection of the noise on the tangent space

Metropolis

PRD Rapid 88, 051502 (2013)

$$\frac{\mathrm{d}\phi_i(r)}{\mathrm{d}r} = \frac{1}{r}\overline{\frac{\delta S}{\delta\phi_i(r)}} \dashrightarrow \phi_i(n+1) = \phi_i(n) + \delta r \overline{\frac{\delta S}{\delta\phi_i}}$$

Other methods (example HMC)

- H. Fujii et al JHEP 1310 (2013) 147
- Next talk: Y Kikukawa

Lefschetz thimble: algorithm

Langevin

PRD Rapid 88, 051501 (2013)

We want to compute this:

boundend from below on J_0

$$\langle \mathcal{O} \rangle = \frac{1}{Z_0} e^{-iS_I} \int_{\mathcal{J}_0} \prod_x \mathrm{d}\phi_x e^{-S_R[\phi]} \mathcal{O}[\phi]$$

constant on J_0

preserve J_0 by construction

We can use a Langevin algorithm but **how can we stay on the thimble?**

$$\frac{\mathrm{d}\phi_i^R(\tau)}{\mathrm{d}\tau} = -\frac{\delta S^R(\phi(\tau))}{\delta \phi_i^R(\tau)} + \eta_i^R(\tau)$$

$$\frac{\mathrm{d}\phi_i^I(\tau)}{\mathrm{d}\tau} = -\frac{\delta S^R(\phi(\tau))}{\delta \phi_i^I(\tau)} + \eta_i^I(\tau)$$

Need to be projected on the tangent space to J_0

Lefschetz thimble: algorithm

Langevin
PRD Rapid 88, 051501 (2013)

We can use a Langevin algorithm but **how can we stay on the thimble?**

preserve J_0 by construction

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Need to be projected on the tangent space to J_0

The tangent space at the stationary point is easy to compute

We can get tangent vectors at any point if we can transport the noise along the gradient flow so that it remains tangent to the thimble

$$\rightarrow \quad \mathcal{L}_{\partial S_R}(\eta) = 0 \quad \leftrightarrow \quad [\partial S_R, \eta] = 0 \quad \leftrightarrow \quad \frac{\mathrm{d}\eta_i(\tau)}{\mathrm{d}\tau} = \sum_k \eta(\tau)_k \partial_k \partial_j S_R$$

projection of the noise on the tangent space

Lefschetz thimble: algorithm

Langevin

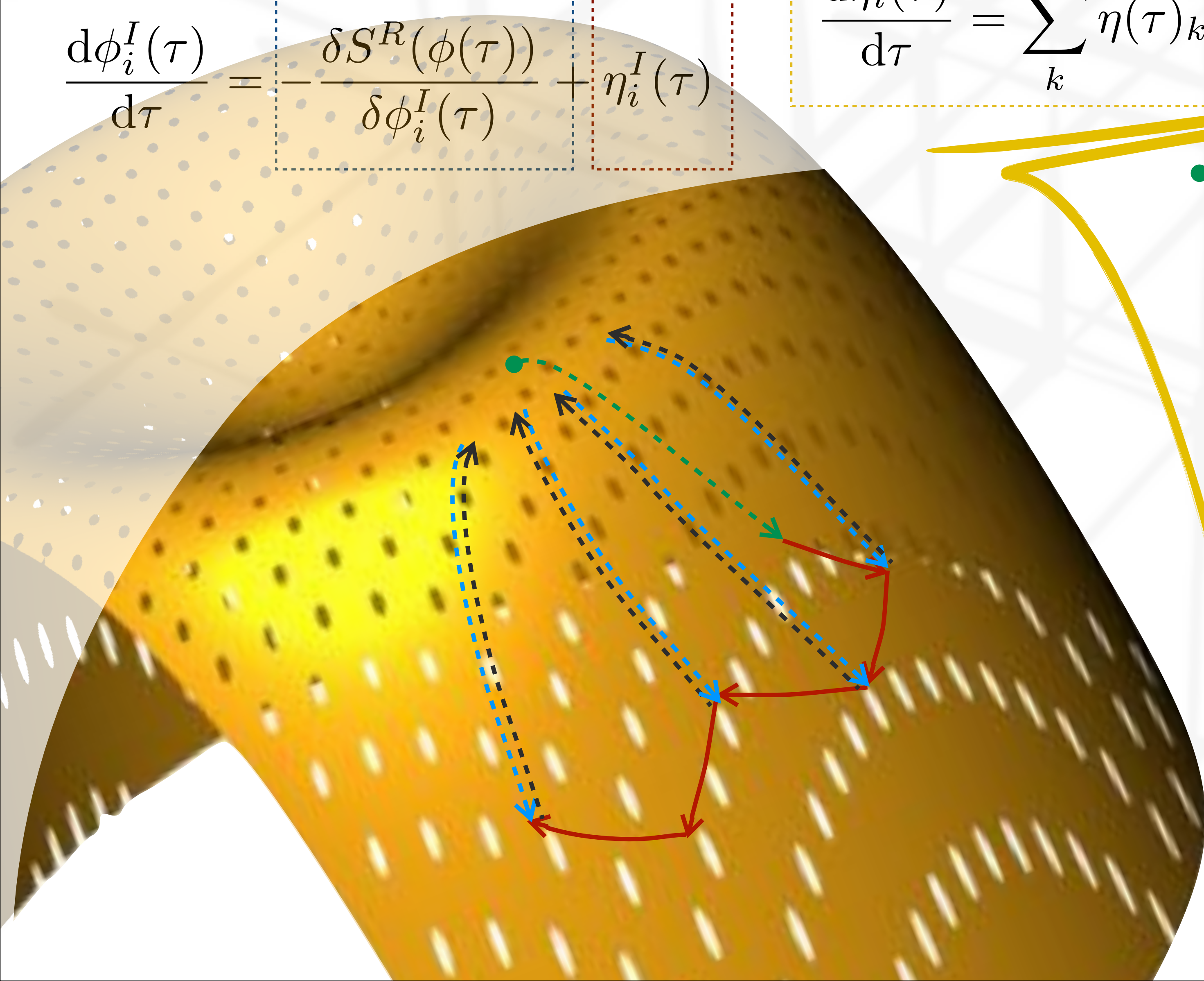
PRD Rapid 88, 051501 (2013)

$$\langle \mathcal{O} \rangle = \frac{1}{Z_0} e^{-iS_I} \int_{\mathcal{J}_0} \prod_x \mathrm{d}\phi_x e^{-S_R[\phi]} \mathcal{O}[\phi]$$

$$\frac{\mathrm{d}\phi_i^R(\tau)}{\mathrm{d}\tau} = \boxed{-\frac{\delta S^R(\phi(\tau))}{\delta \phi_i^R(\tau)}} + \boxed{\eta_i^R(\tau)}$$

$$\frac{\mathrm{d}\phi_i^I(\tau)}{\mathrm{d}\tau} = \boxed{-\frac{\delta S^R(\phi(\tau))}{\delta \phi_i^I(\tau)}} + \boxed{\eta_i^I(\tau)}$$

$$\boxed{\frac{\mathrm{d}\eta_i(\tau)}{\mathrm{d}\tau} = \sum_k \eta(\tau)_k \partial_k \partial_j S_R}$$



Start from the global minimum of the real part of the action, generate a noise vector projected on the thimble and follow the steepest ascent

Perform a Langevin step using the noise evolved along the steepest descent and compute the observables

Go back along the steepest descent until you are in a region where quadratic approx. is valid and then project the configuration on the thimble

Generate a new noise and go back along the steepest ascent

* in the plot you have $-S_R$ that is the exponent in the integrand of the partition function : where I wrote ascent (descent) you see the opposite in the figure

Lefschetz thimble: algorithm

Langevin on the Lefschetz thimble vs Complex Langevin

Lefschetz Langevin

$$\frac{\mathrm{d}\phi_i^R(\tau)}{\mathrm{d}\tau} = -\frac{\delta S^R(\phi(\tau))}{\delta \phi_i^R(\tau)} + \eta_i^R(\tau)$$

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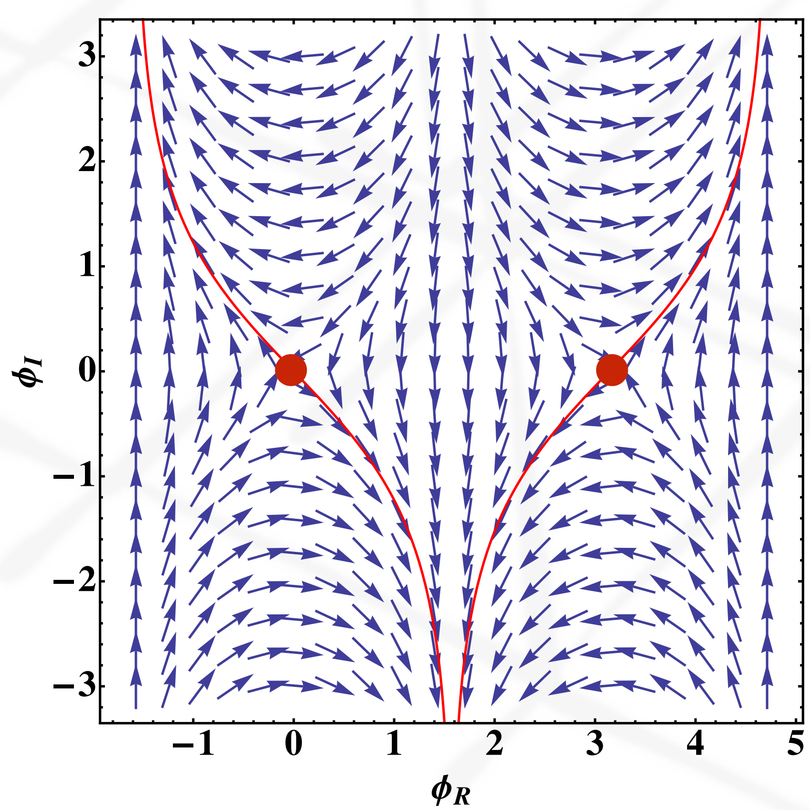
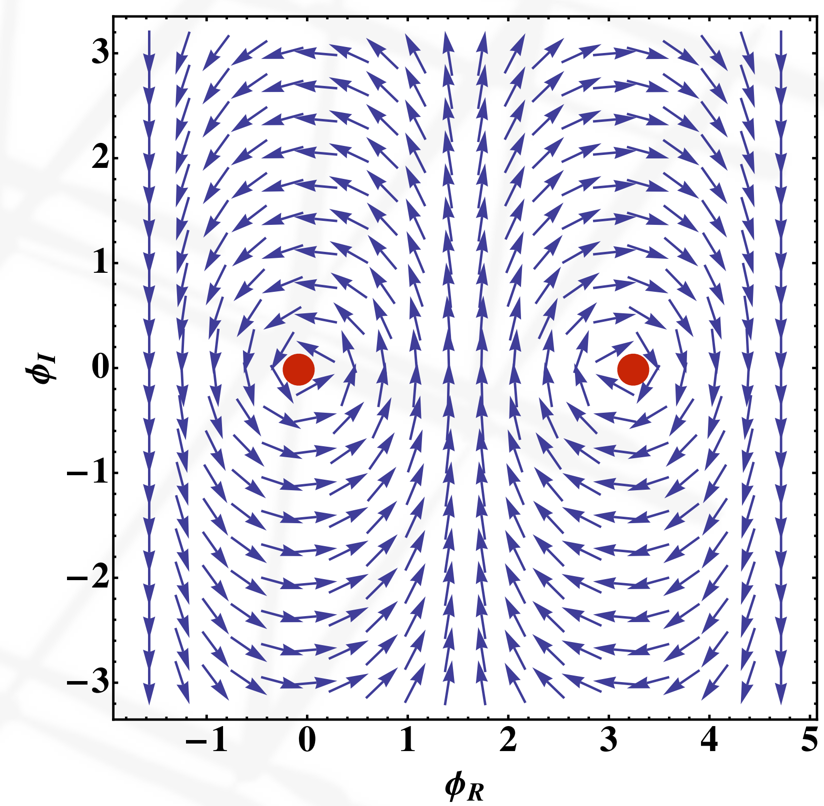
Complex Langevin

$$\frac{\mathrm{d}\phi_i^R(\tau)}{\mathrm{d}\tau} = -\frac{\delta S^R(\phi(\tau))}{\delta \phi_i^R(\tau)} + \eta_i^R(\tau)$$

$$\frac{\mathrm{d}\phi_i^I(\tau)}{\mathrm{d}\tau} = +\frac{\delta S^R(\phi(\tau))}{\delta \phi_i^I(\tau)} + \eta_i^I(\tau)$$



The relation between the two approaches should be studied carefully!



Lefschetz thimble: algorithm

Metropolis

PRD Rapid 88, 051502 (2013)

In the neighbourhood of a critical point →

$$S[\phi] = S[\phi_0] + S_G[\eta] + \mathcal{O}(|\eta|^3)$$

$$S_G = \frac{1}{2}\sum_k \lambda_k \eta_k^2$$

$$\phi_i = \phi_i^0 + \sum_k w_{ki}\eta_k$$

G_0 is the flat thimble associated to the gaussian action S_G

The $\boldsymbol{\lambda}$ and **w** are solutions of $Hw_k = \lambda_k \bar{w}_k$ where **H** is the Hessian

$\boldsymbol{\eta}$ real are the direction of steepest descent of S_R and the equations of steepest descent of η for the Gaussian action can be explicitly solved in term of a new parameter $r=e^{-\tau}$

$$\rightarrow \quad \frac{\mathrm{d}\eta_k}{\mathrm{d}r} = \frac{1}{r}\frac{\partial \bar{S}_G}{\partial \eta_k} = \frac{1}{r}\lambda_k \eta_k \quad \rightarrow \quad \eta_k \propto r^{\lambda_k}$$

but for $r=\varepsilon$ infinitesimal the Lefschetz and Gaussian thimbles coincide

$$\phi_i(\epsilon) = \phi_i^0 + \sum_k \epsilon^{\lambda_k} w_{ki}\eta_k$$

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}r} = \frac{1}{r}\frac{\partial \bar{S}}{\partial \phi_i} \qquad r \in [\epsilon, 1]$$

Start with a random real η vector, compute $\Phi(\varepsilon)$ and evolve using steepest descent

Lefschetz thimble: algorithm

Metropolis

PRD Rapid 88, 051502 (2013)

In the neighbourhood of a critical point →

$$S[\phi] = S[\phi_0] + S_G[\eta] + \mathcal{O}(|\eta|^3)$$

$$S_G = \frac{1}{2}\sum_k \lambda_k \eta_k^2$$

$$\phi_i = \phi_i^0 + \sum_k w_{ki}\eta_k$$

G_0 is the flat thimble associated to the gaussian action S_G

The $\boldsymbol{\lambda}$ and **w** are solutions of

$$H w_k = \lambda_k \bar{w}_k$$

where **H** is the Hessian

$\boldsymbol{\eta}$ → n-dim random vector living on the manifold defined by the eigenvectors of the Hessian computed at the critical point with positive eigenvalues

$|\boldsymbol{\eta}|$ → distance along the thimble

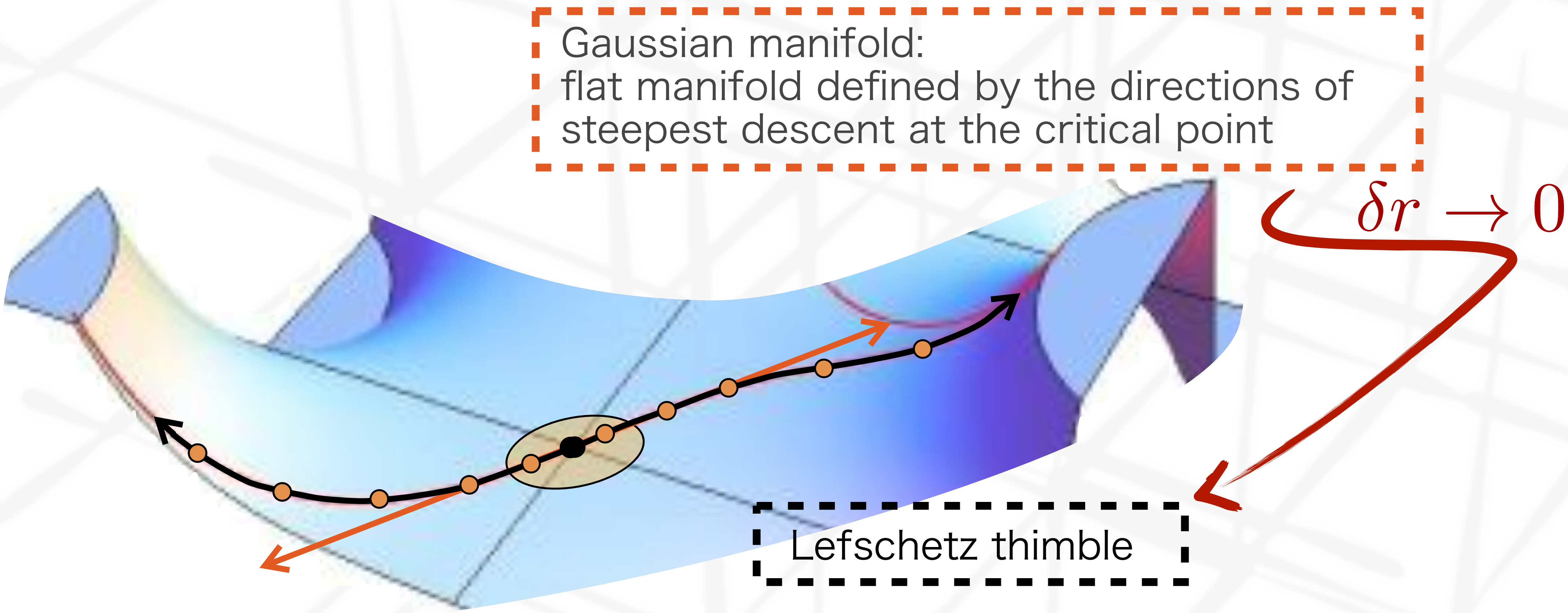
$|\boldsymbol{\eta}|/\boldsymbol{\delta}$**r** → number of steps along the steepest descent

$$\frac{\mathrm{d}\phi_i(r)}{\mathrm{d}r} = \frac{1}{r}\overline{\frac{\delta S}{\delta\phi_i(r)}}$$

$$\dashrightarrow \quad \phi_i(n+1) = \phi_i(n) + \delta r\,\overline{\frac{\delta S}{\delta\phi_i}}$$

Lefschetz thimble: algorithm

Gaussian thimble



$|\eta|/\delta r = N$ → number of steps along the steepest descent

- Decreasing δr your manifold get closer and closer to the Lefschetz thimble

- If the action decreases fast away from the stationary point integrating on the Gaussian thimble can be sufficient

U(1) one plaquette model

PRD Rapid 88, 051502 (2013)

- Can be seen as the limiting case of the more interesting three-dimensional XY model
- One dimensional problem: the integration on the Lefschetz thimble can be plotted

ACTION $$S = -i\frac{\beta}{2}\left(U + U^{-}1\right) = -i\beta\cos\phi$$

OBSERVABLE $$\langle e^{i\phi}\rangle = i\frac{J_1(\beta)}{J_0(\beta)}$$

On the thimble

$$\langle \mathcal{O}(\phi)\rangle = \frac{\sum_\sigma m_\sigma \int_{\mathcal{J}_\sigma} d\phi \mathcal{O}(\phi) e^{-S(\phi)}}{\sum_\sigma m_\sigma \int_{\mathcal{J}_\sigma} d\phi (\phi) e^{-S(\phi)}}$$

$$S_R = -\beta\sin\phi_R\sinh\phi_I$$

$$S_I = -\beta\cos\phi_R\cosh\phi_I$$

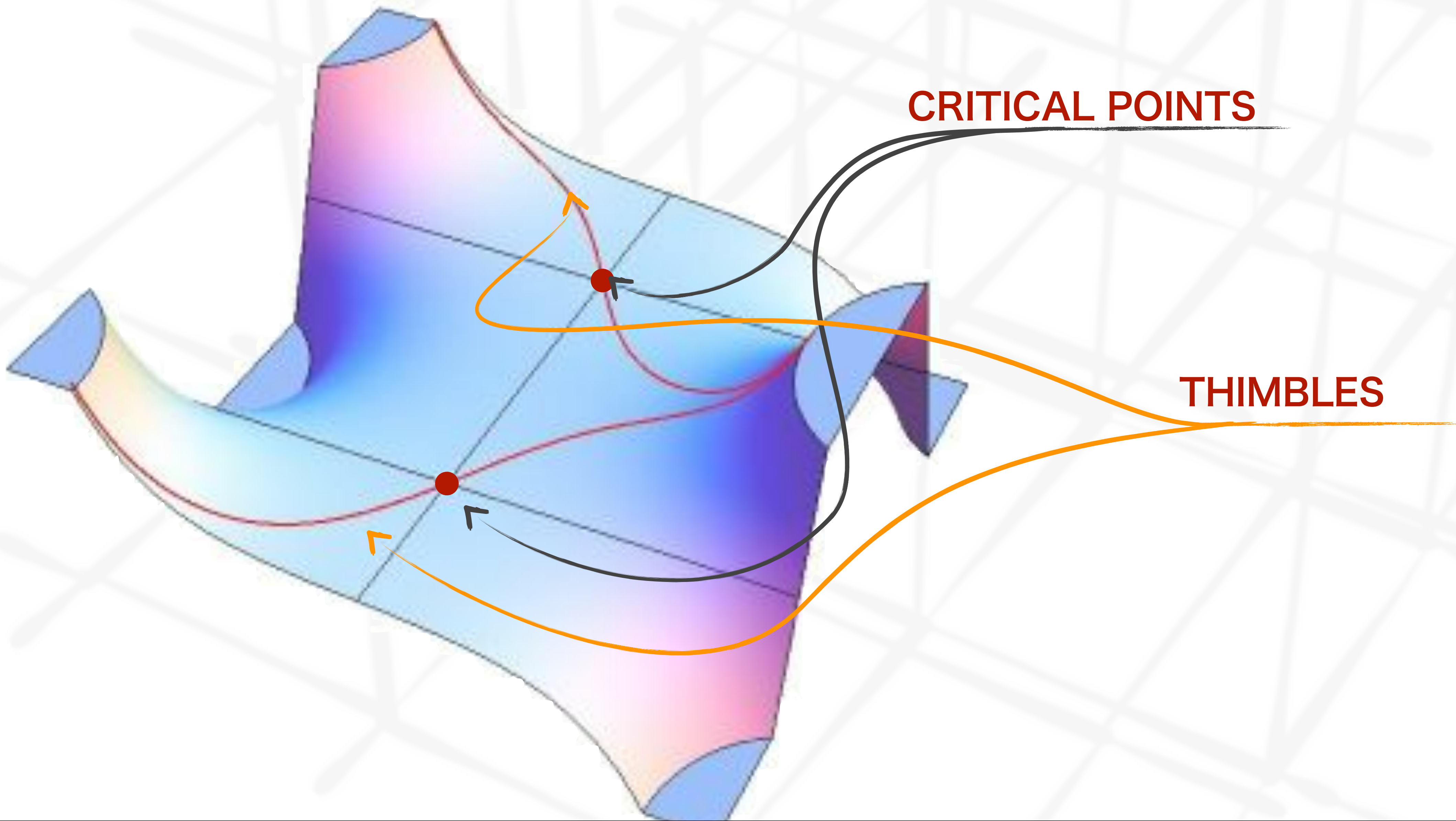
constant on the thimble

U(1) one plaquette model

PRD Rapid 88, 051502 (2013)

- The stationary points are in (0,0) and (π,0) and the thimble can be computed also analytically
  - Exact thimbles: have to pass from the critical point and the imaginary part of the action has to be constant

$$S_I(\tau) = -\beta \cos\phi_R(\tau) \cosh\phi_I(\tau) = S_I^{\rm cp}$$



U(1) one plaquette model

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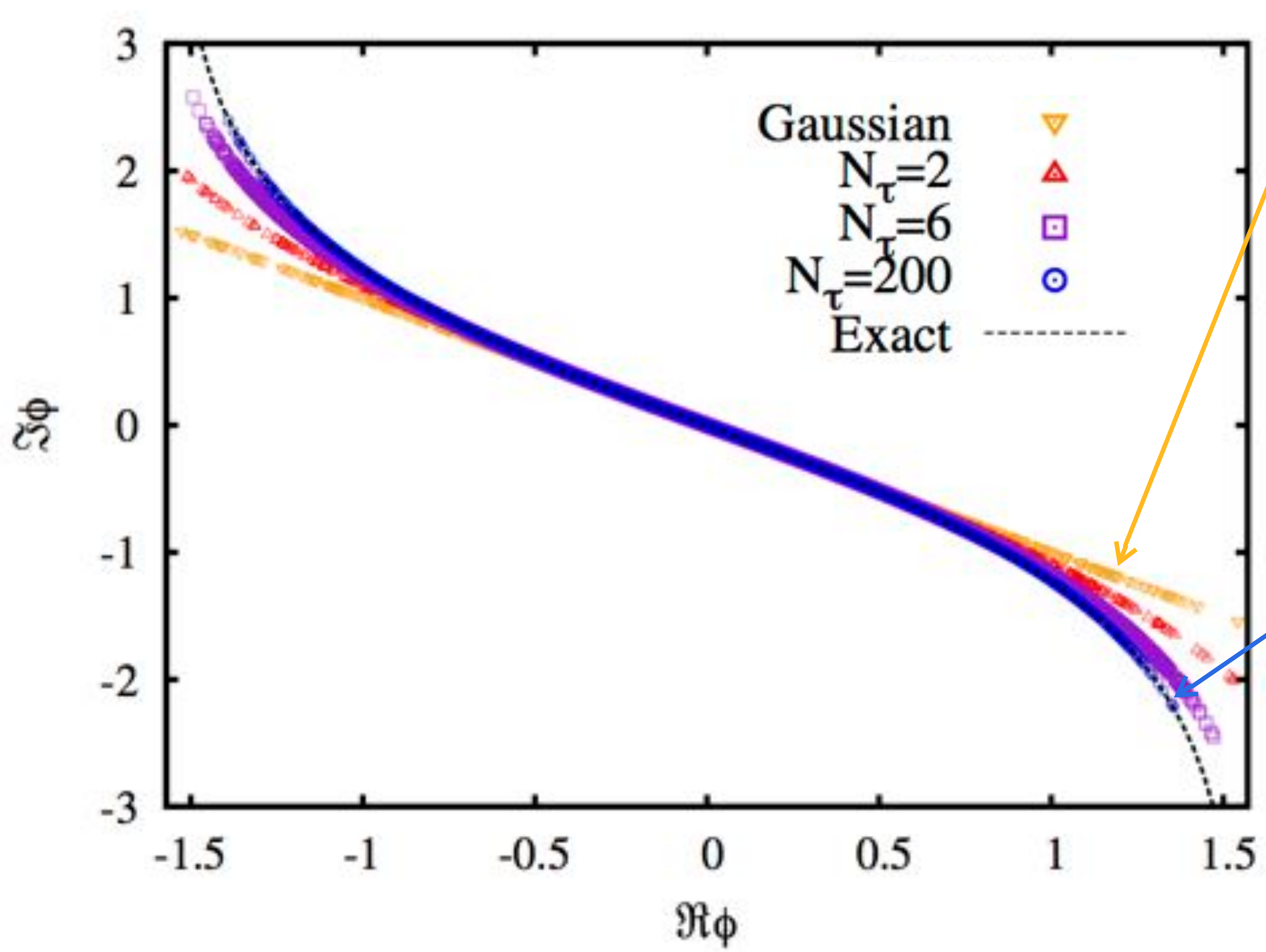
- The stationary points are in (0,0) and (π,0) and the thimble can be computed also analytically
  - In order to perform the integration on the thimble we use a Metropolis algorithm



Gaussian manifold

Increasing the accuracy in the integration of the steepest descent we move closer to the exact thimble

U(1) one plaquette model

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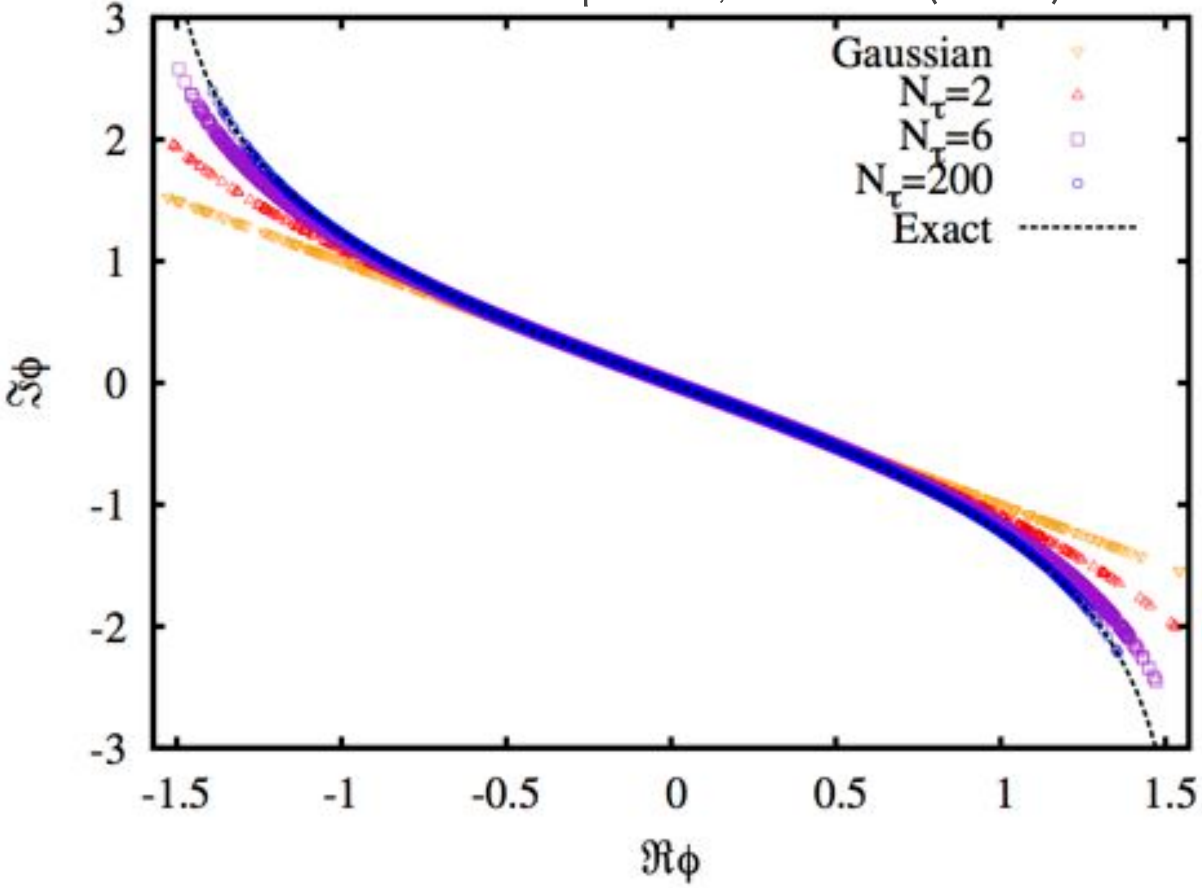
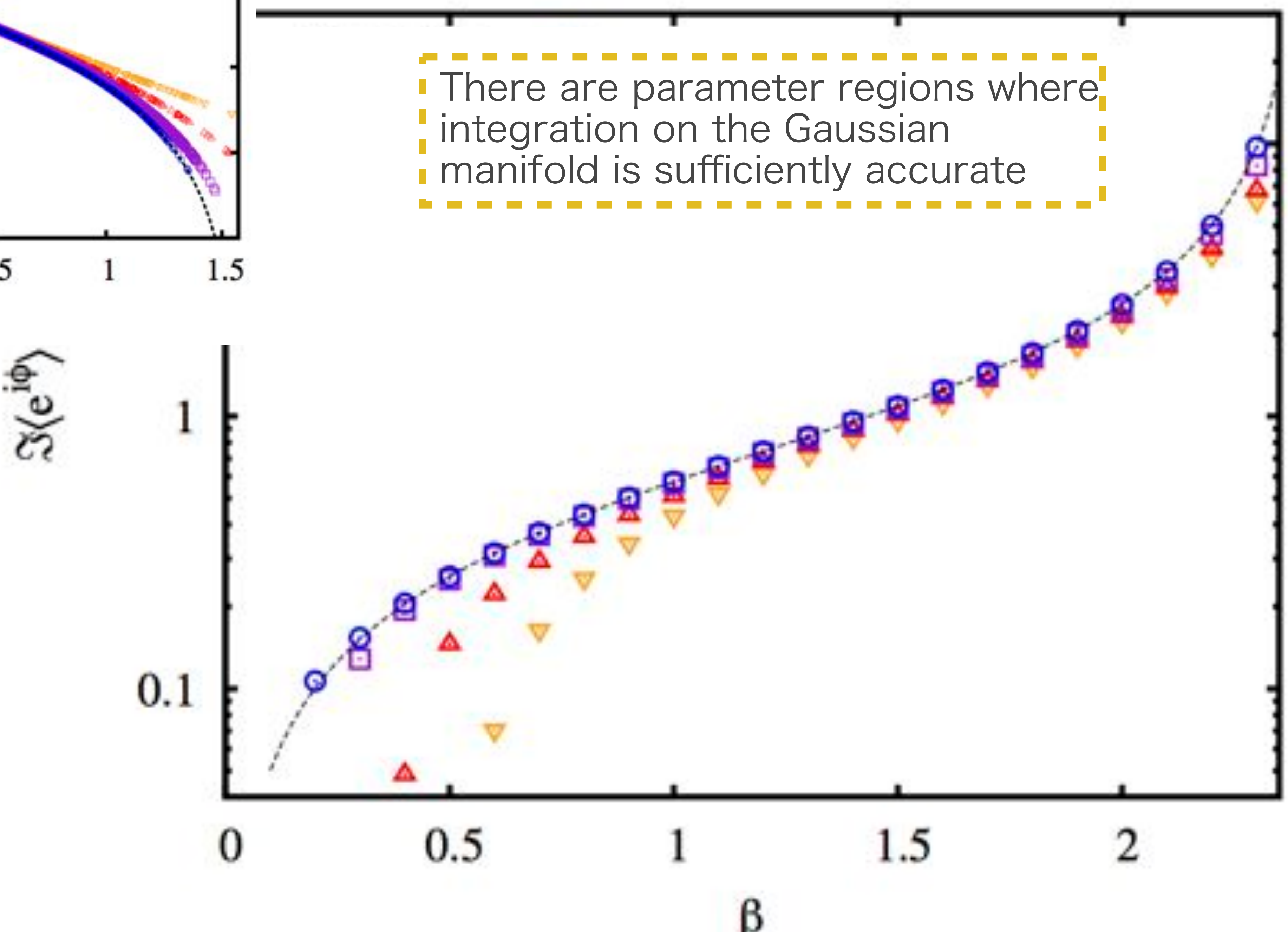


OBSERVABLE

$$\langle e^{i\phi}\rangle = i\frac{J_1(\beta)}{J_0(\beta)}$$

There are parameter regions where integration on the Gaussian manifold is sufficiently accurate



U(1) one plaquette model

PRD Rapid 88, 051502 (2013)

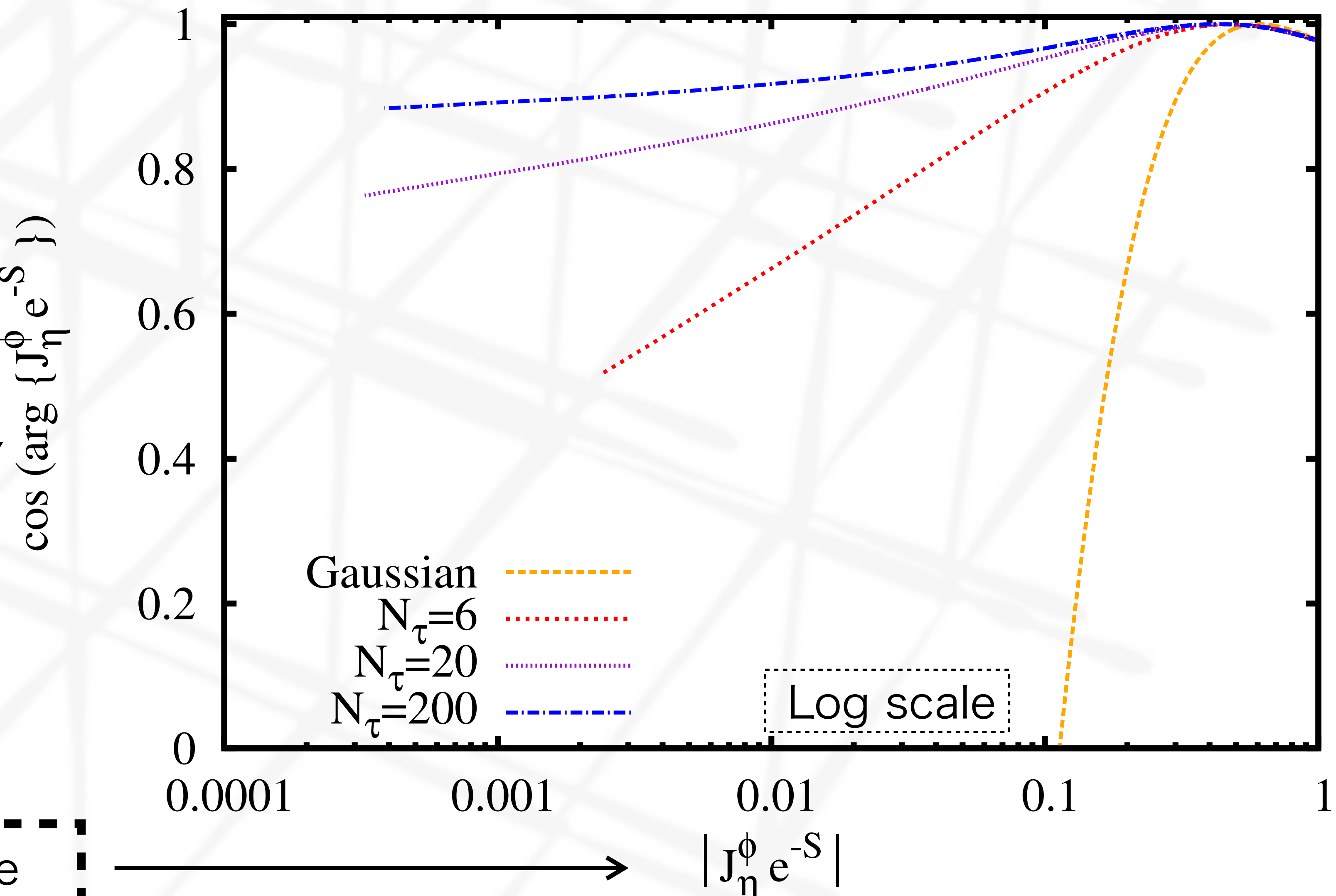
- Residual phase is well under control and is not a source of additional sign problem (at least in this case)

$$\langle \mathcal{O}(\phi) \rangle = \frac{\sum_\sigma m_\sigma \int_{\mathcal{J}_\sigma} d\phi \mathcal{O}(\phi) e^{-S(\phi)}}{\sum_\sigma m_\sigma \int_{\mathcal{J}_\sigma} d\phi (\phi) e^{-S(\phi)}}$$

There is an additional phase coming from the Jacobian of the transformation between the canonical complex basis and the tangent space to the thimble

This phase should be essentially constant over the portion of phase space which dominates the integral.

Residual phase



Probability measure

$\lambda\,\Phi^4$ theory on the lattice

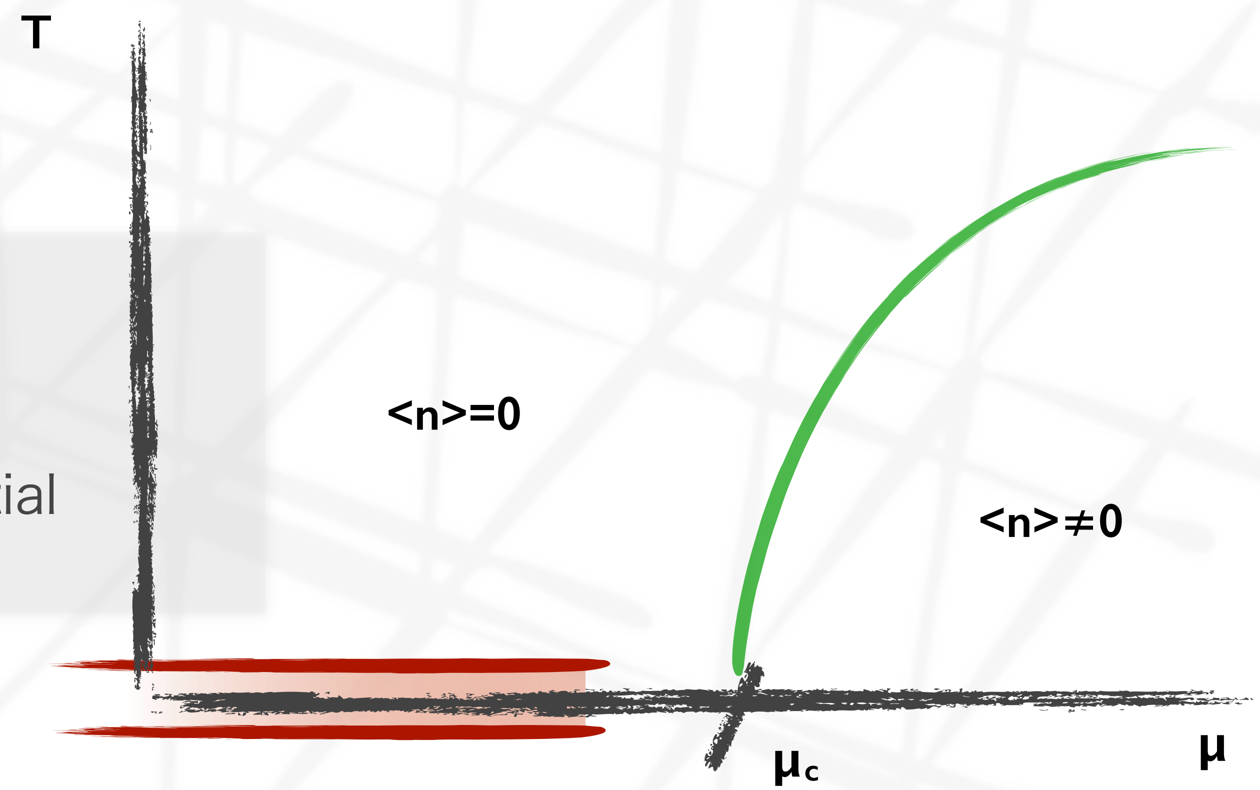
PRD Rapid 88, 051501 (2013)

$$S[\phi,\phi^*] = \int d^4x\left(|\partial_\nu\phi|^2 + (m^2-\mu^2)|\phi|^2 + \lambda|\phi|^4 + \mu(\phi^*\partial_0\phi - \partial_0\phi^*\phi)\right)$$

Continuum action

Silver Blaze problem

when T=0 and $\mu<\mu_c$ physics is independent from the chemical potential



We will study the system at zero temperature

$\lambda\,\Phi^4$ theory on the lattice

PRD Rapid 88, 051501 (2013)

$$S[\phi,\phi^*] = \int d^4x(|\partial_\nu\phi|^2 + (m^2-\mu^2)|\phi|^2 + \lambda|\phi|^4 + \mu(\phi^*\partial_0\phi - \partial_0\phi^*\phi)$$

Continuum action

$$S[\phi,\phi^*] = \sum_x [(2d+m^2)\phi_x^*\phi_x + \lambda(\phi_x^*\phi_x)^2$$
$$-\sum_{\nu=0}^{4}(\phi_x^* e^{-\mu\delta_{\nu,0}}\phi_{x+\hat\nu} + \phi_{x+\hat\nu}^* e^{\mu\delta_{\nu,0}}\phi_x))$$

Lattice action:
chemical potential introduced as an imaginary constant vector potential in the temporal direction

in term of real fields $\phi_a (a=1,2)$ $\quad \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$S[\phi_a] = \sum_x \left[\frac{1}{2}(2d+m^2)\phi_{a,x}^2 + \frac{\lambda}{4}(\phi_{a,x}^2)^2 - \sum_{\nu=1}^{3}\phi_{a,x}\phi_{a,x+\hat{i}} - \cosh\mu\,\phi_{a,x}\phi_{a,x+\hat{0}} + i\sinh\mu\,\varepsilon_{ab}\phi_{a,x}\phi_{b,x+\hat{0}}\right]$$

$\lambda\,\Phi^4$ theory on the lattice

PRD Rapid 88, 051501 (2013)

PHASE QUENCHED

$$S[\phi,\phi^*] = \int d^4x\left(|\partial_\nu\phi|^2 + (m^2-\mu^2)|\phi|^2 + \lambda|\phi|^4 + \mu(\phi^*\partial_0\phi - \partial_0\phi^*\phi)\right)$$

$$\langle\mathcal{O}\rangle_{\rm full} = \frac{\int \mathcal{D}\phi|e^{-S}|e^{i\theta}\mathcal{O}}{\int \mathcal{D}\phi|e^{-S}|e^{i\theta}} = \frac{\langle e^{i\theta}\mathcal{O}\rangle_{\rm pq}}{\langle e^{i\theta}\rangle_{\rm pq}} \dashrightarrow \frac{0}{0}$$

Let us try ignoring the phase

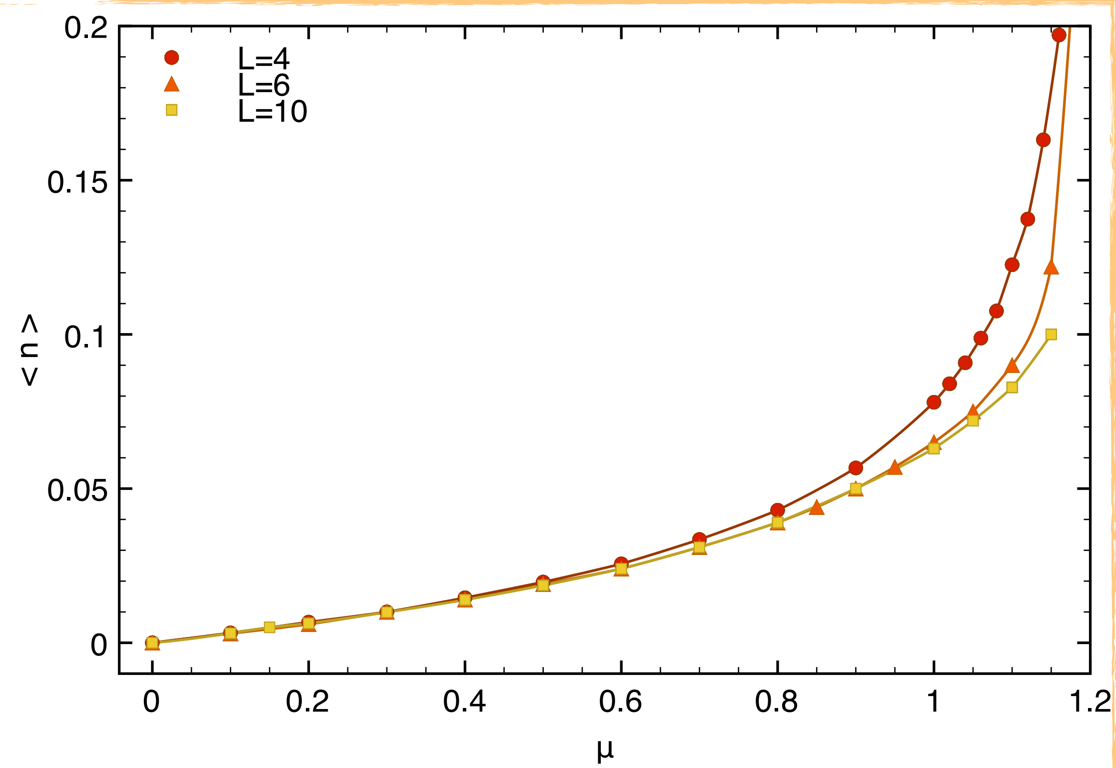
$$\langle\mathcal{O}\rangle_{\rm pq} = \frac{\int \mathcal{D}\phi|e^{-S}|\mathcal{O}}{\int \mathcal{D}\phi|e^{-S}|}$$

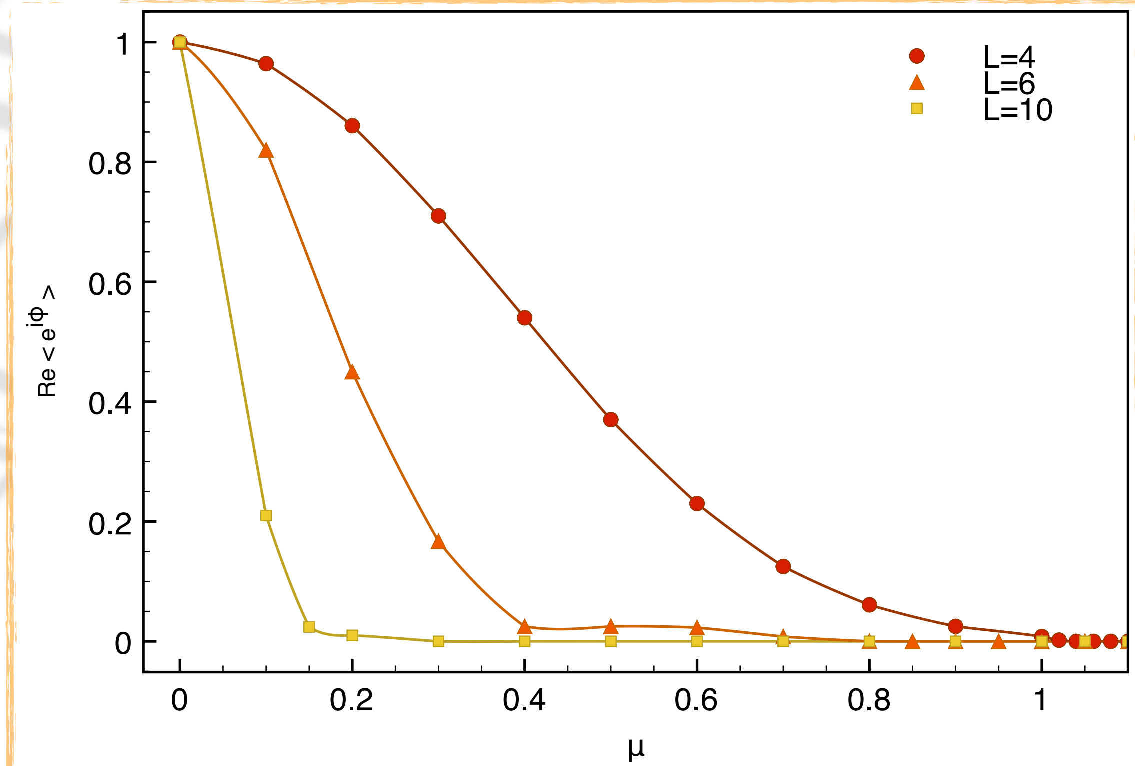
$\lambda\,\Phi^4$ theory on the lattice

PRD Rapid 88, 051501 (2013)

PHASE QUENCHED

$$S[\phi,\phi^*] = \int d^4x\left(|\partial_\nu\phi|^2 + (m^2-\mu^2)|\phi|^2 + \lambda|\phi|^4 + \cancel{\mu(\phi^*\partial_0\phi - \partial_0\phi^*\phi)}\right)$$





$$\langle n\rangle = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu}$$

$\lambda\,\Phi^4$ on a Lefschetz thimble

PRD Rapid 88, 051501 (2013)

On the Lefschetz thimble

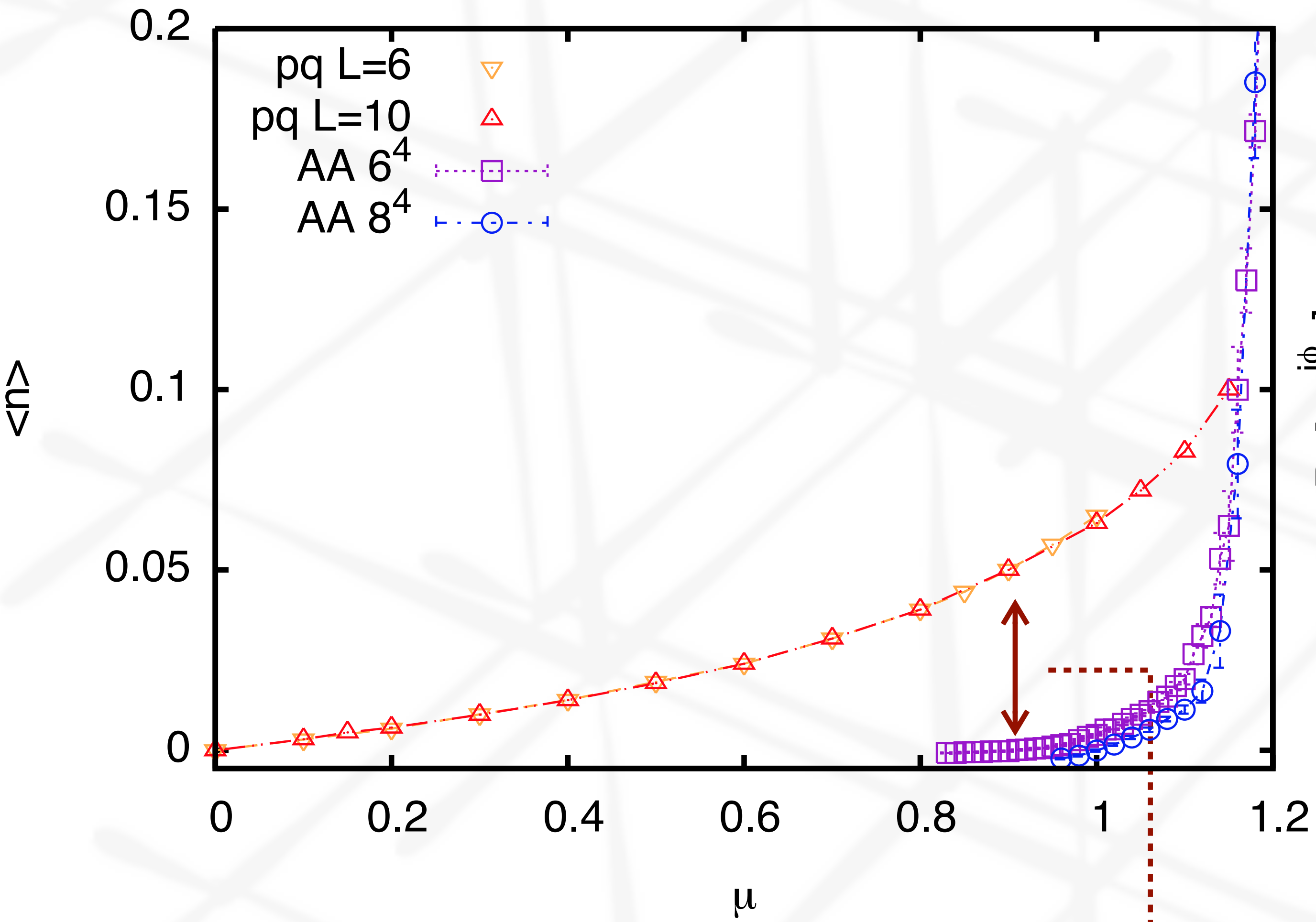
M. C., F. Di Renzo, A. Mukherjee and L. Scorzato
arXiv:1303.7204 (2013)

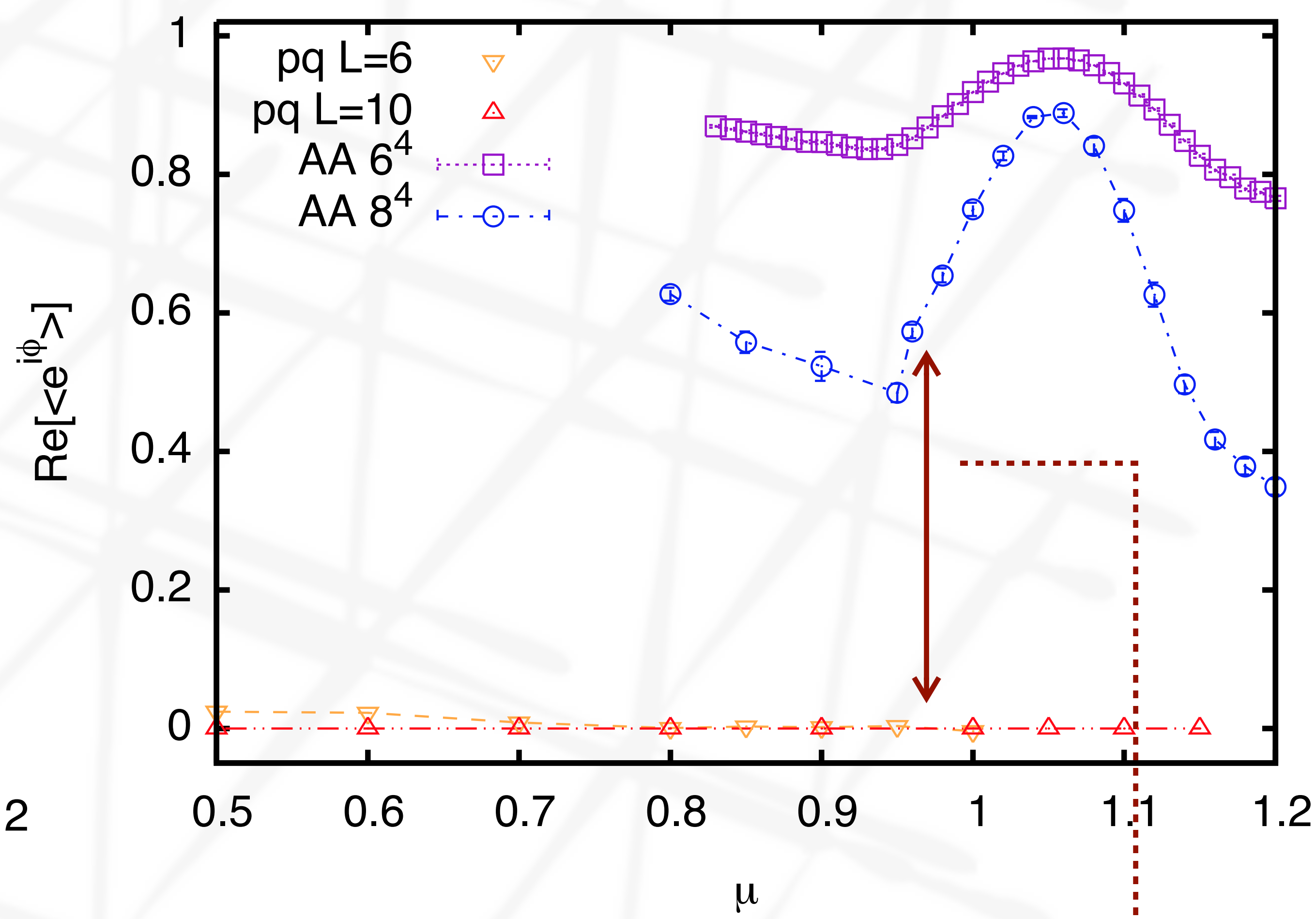
- ☐ Fields are complexified $\phi_a \to \phi_a^R + i\phi_a^I$
- ☐ The integration on the thimble performed with a Langevin algorithm
- ☐ In this case calculations in Gaussian approximation are sufficient to obtain the exact result

$\lambda\,\Phi^4$ on a Lefschetz thimble

PRD Rapid 88, 051501 (2013)

On the Lefschetz thimble



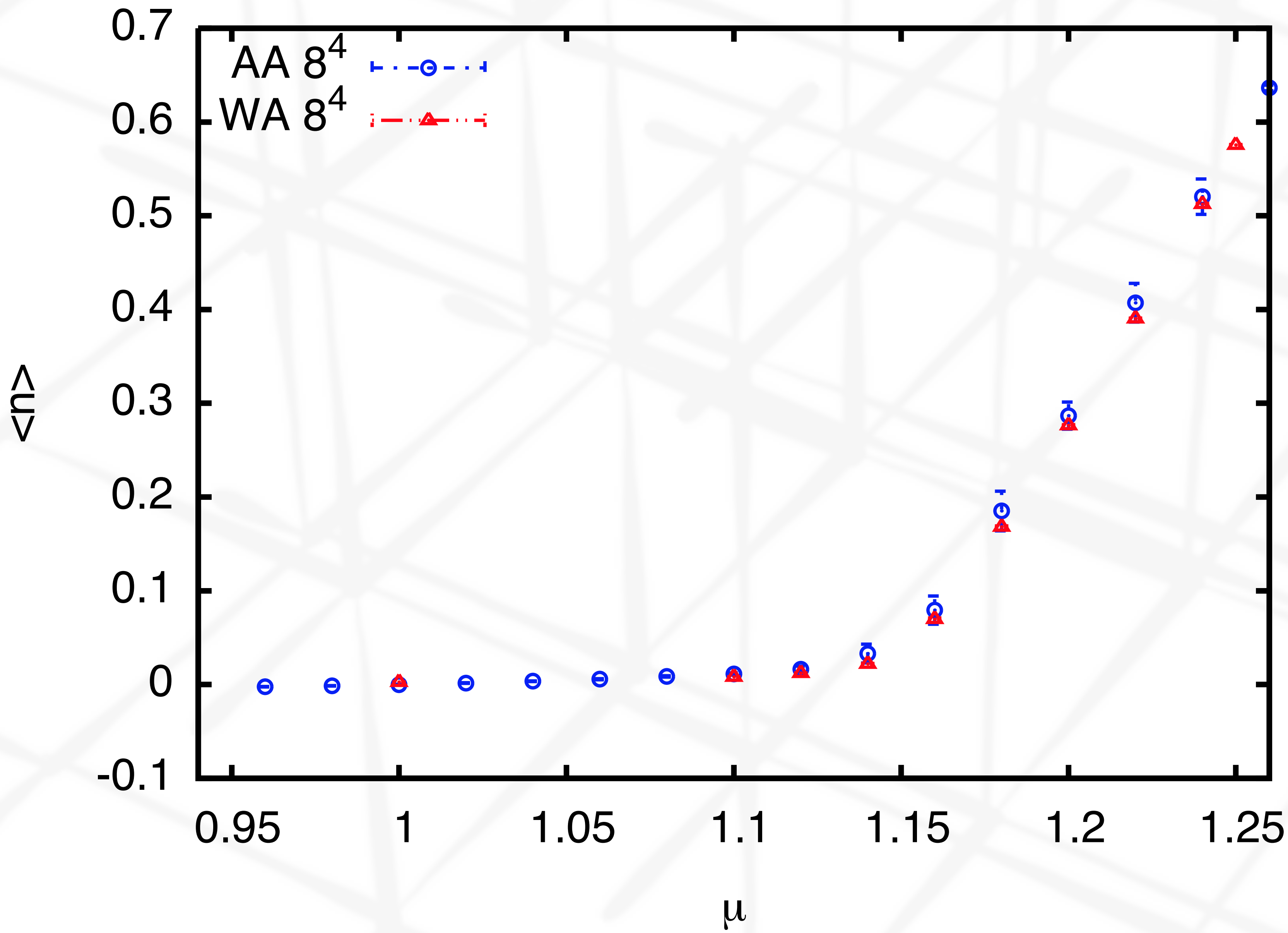


Silver Blaze

solving sign problem we have the correct physics

$\lambda\,\Phi^4$ on a Lefschetz thimble

On the Lefschetz thimble



Comparison with Worm Algorithm
(courtesy of C. Gattringer and T. Kloiber)

$$\langle n \rangle = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu}$$

What about QCD?

- **Consider the stationary point with the lower value of the real part of the action and with $n_\sigma \neq 0$**
  Define a QFT on the thimble attached to this point.

  ↘ In QCD this is the trivial vacuum

↘ Complexification: $A^a_\nu(x) \to A^{a,R}_\nu(x) + iA^{a,R}_\nu(x) \qquad a = 1 \ldots N_c^{-1}$

$$SU(3)^{4V} \to SL(3,\mathbb{C})^{4V}$$

↘ Covariant derivative: $\nabla_{x,\nu,a} F[U] := \frac{\partial}{\partial \alpha} F[e^{i\alpha T_a} U_\nu(x)]_{|\alpha=0}$

$$\nabla_{x,\nu,a} = \nabla^R_{x,\nu,a} - i\nabla^I_{x,\nu,a}$$

$$\overline{\nabla}_{x,\nu,a} = \nabla^R_{x,\nu,a} + i\nabla^I_{x,\nu,a}$$

↘ Equation of steepest descent: $\frac{\mathrm{d}}{\mathrm{d}\tau} U_\nu(x;\tau) = (-iT_a \overline{\nabla}_{x,\nu,a} \overline{S}[U]) U_\nu(x;\tau)$

↘ Defining the thimble for gauge theories is possible: substitute the concept of non-degenerate critical point with that of non-degenerate critical manifold

All the ingredients are there

What about QCD?

→ The symmetries are the same? Yes

↘ It can be proved starting from the invariance of the SD equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}U_\nu(x;\tau) = (-iT_a\overline{\nabla}_{x,\nu,a}\overline{S}[U])U_\nu(x;\tau)$$

Under gauge transformations it changes as:

$$(T_a\overline{\nabla}_{x,\nu,a}\overline{S}[U]) \to (\Lambda(x)^{-1})^\dagger(T_a\overline{\nabla}_{x,\nu,a}\overline{S}[U])\Lambda(x)^\dagger$$

$$U_\nu(x) \to \Lambda(x)U_\nu(x)\Lambda(x+\hat{\nu})^{-1}$$

The full SD equation is **invariant only under the SU(3)** subgroup of SL(3,C)

this is very interesting: the gauge links are not in SU(3) but the gauge invariance is exactly the same!

→ The perturbative expansion is the same? Yes

↘ It is an expansion around the trivial vacuum where the integrand in the partition function has the form of a gaussian times polynomials (let me skip the details)

Something else on a Lefschetz thimble

Next steps

- Theoretical questions (single thimble, residual phase, reflection positivity ...)

- 0-dim Φ^4 (finished)

- Chiral random matrix theory

- Thirring model

- Hubbard model (repulsive almost done)

- SU(3) pure gauge with theta term

- QCD in 0+1 dimension

- . . .

- QCD

thank **you**