Numerical simulations to understand QCD phase transition at high temperature and density

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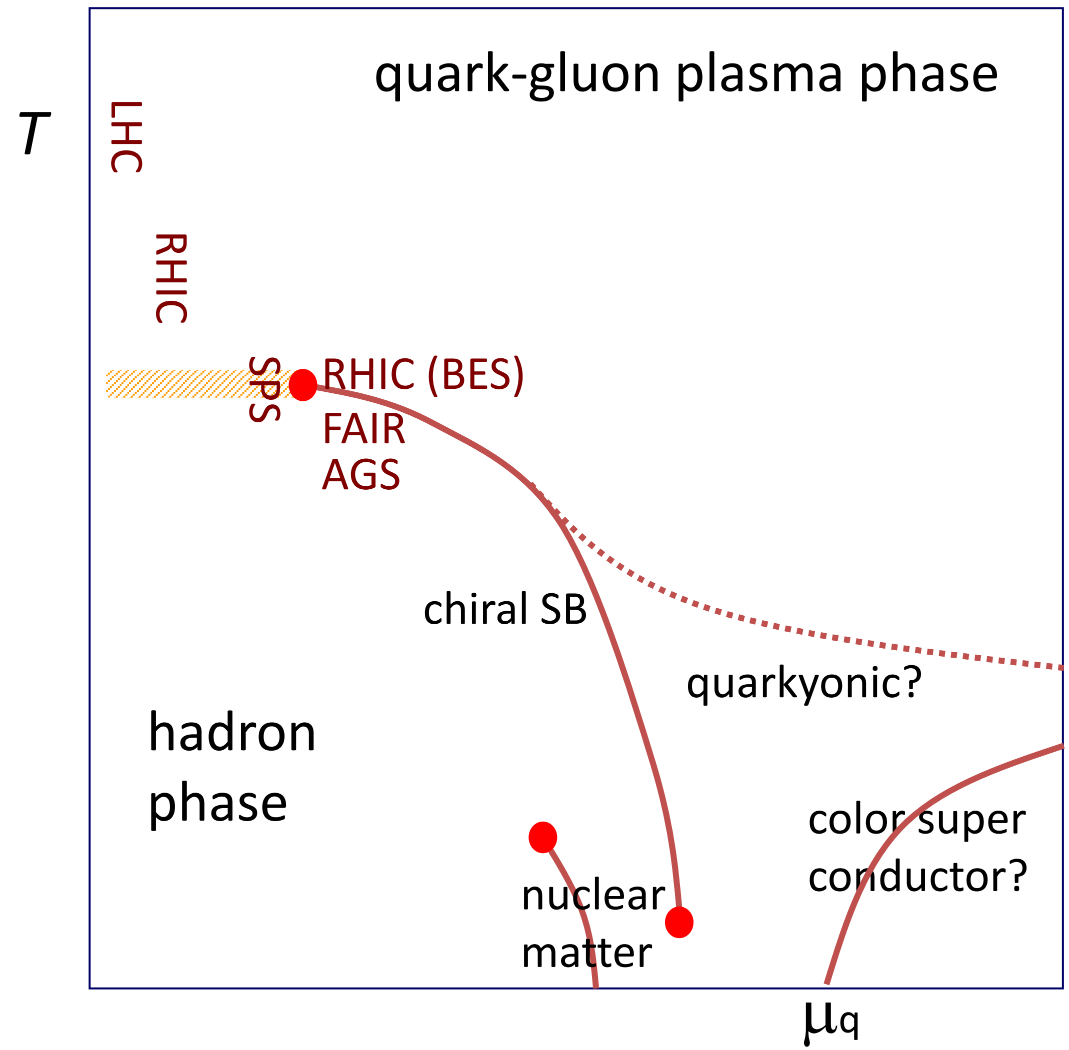
Workshop on Lattice QCD at finite temperature and density
KEK, Tsukuba, Japan, Jan. 20-22, 2014

Phase structure of QCD at high temperature and density

- Phase transition lines
- Critical point
- Equation of state
- Thermodynamic quantities



Lattice QCD Simulations

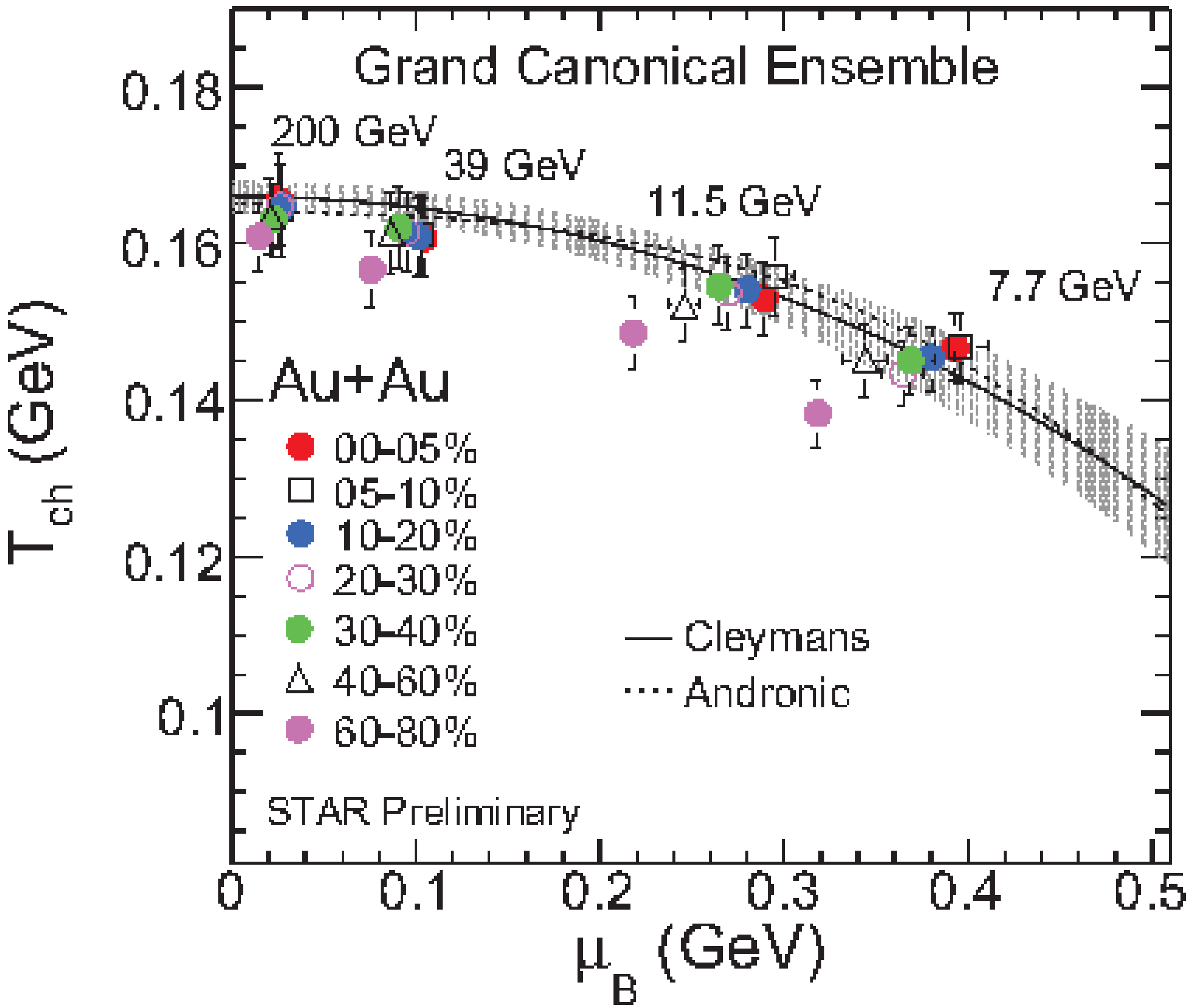
- Direct simulation:
  Impossible at $\mu \neq 0$.

Heavy-ion collisions at RHIC and LHC

- Beam energy scan at RHIC
  - Fluctuations
  - Higher order cumulant — M. Kitazawa
  - Bryon number distribution
  - Canonical partition function — A. Nakamura

RHIC Beam Energy Scan

- Investigate the density dependence changing the beam energy

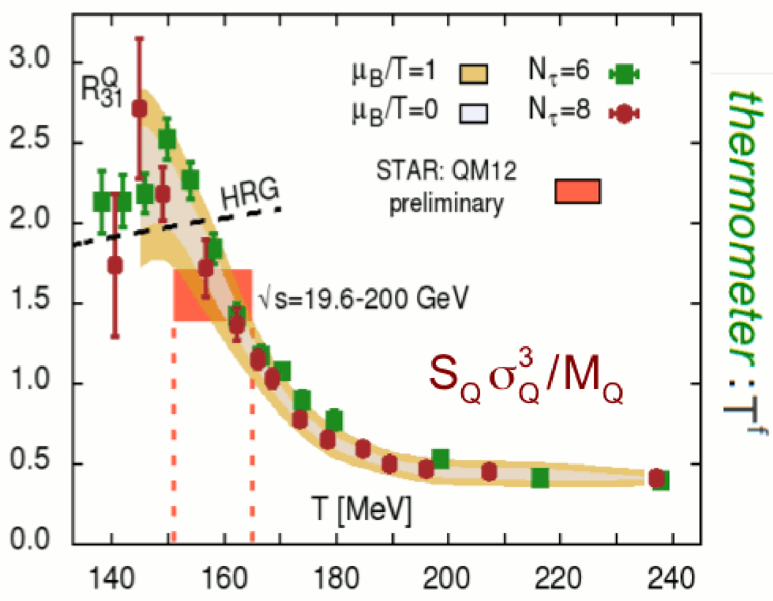


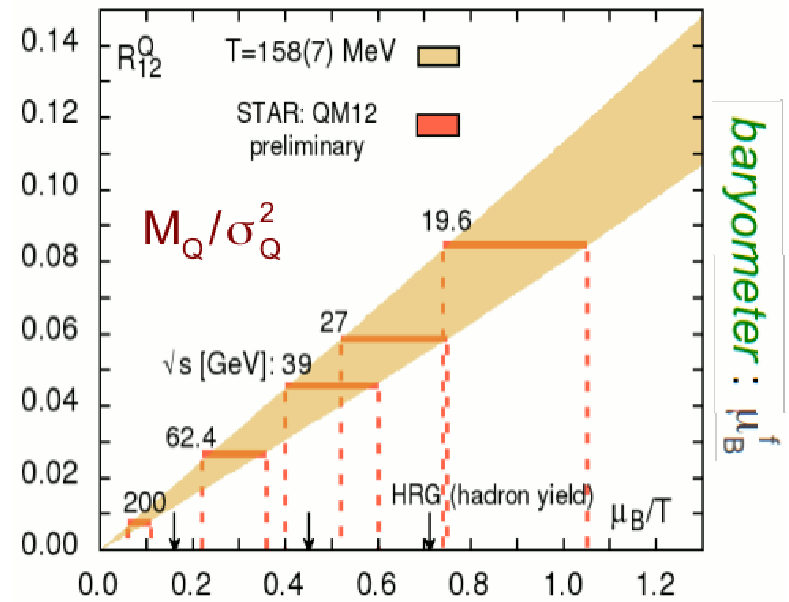
STAR, 2012
BES-I

BES-II: planning

Cumulants of conserved charges

BNL-Bielefeld, Phys. Rev. Lett. 109, 192302 (2012)



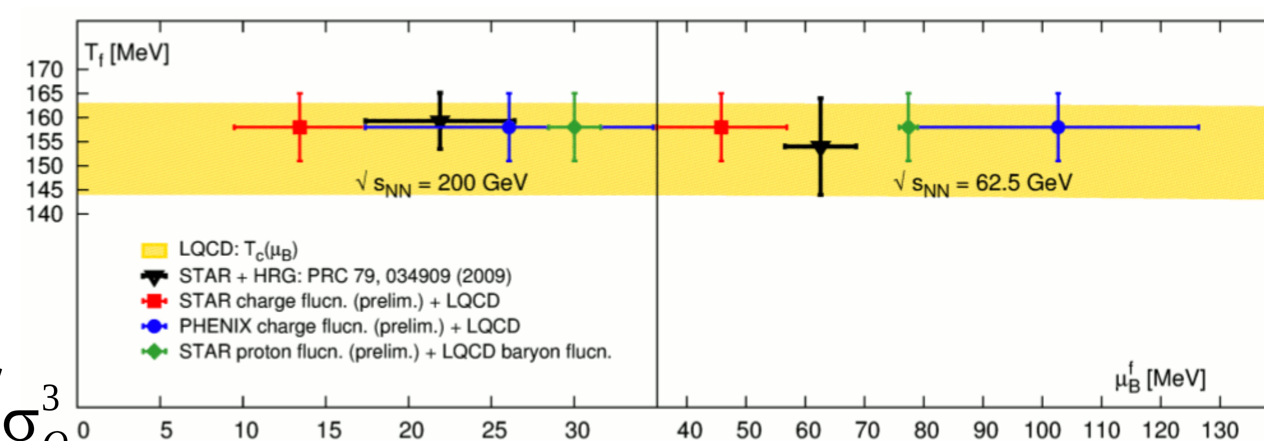


- Taylor expansion method

mean value $M_Q = \langle n_Q \rangle$

deviation $\sigma_Q^2 = \langle (\delta n_Q)^2 \rangle$

skewnes $S_Q = \langle (\delta n_Q)^3 \rangle / \sigma_Q^3$



Mukherjee, PoS CPOD2013, 039 (2013)

Heavy-ion collisions at RHIC and LHC

- Hydrodynamics in HIC
  – Equation of State: Energy density, Pressure

    T. Umeda

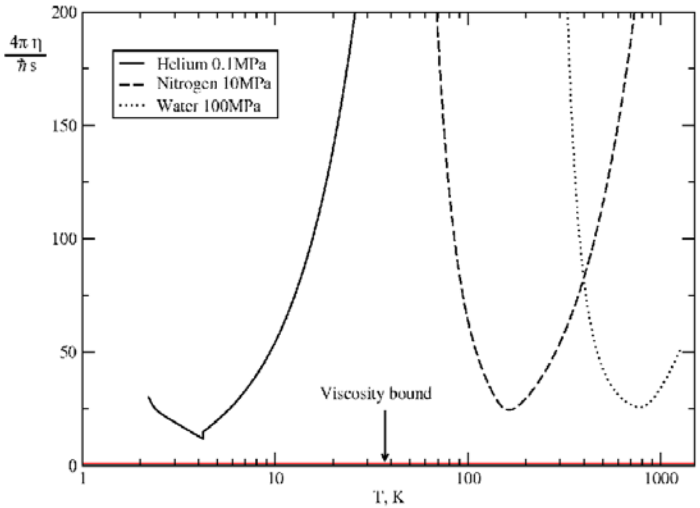
  – Energy-Momentum tensor from Wilson flow: useful?

    T. Hatsuda, H. Suzuki

  – Transport coefficients: Viscosity etc.

    ???

Viscosity in QGP

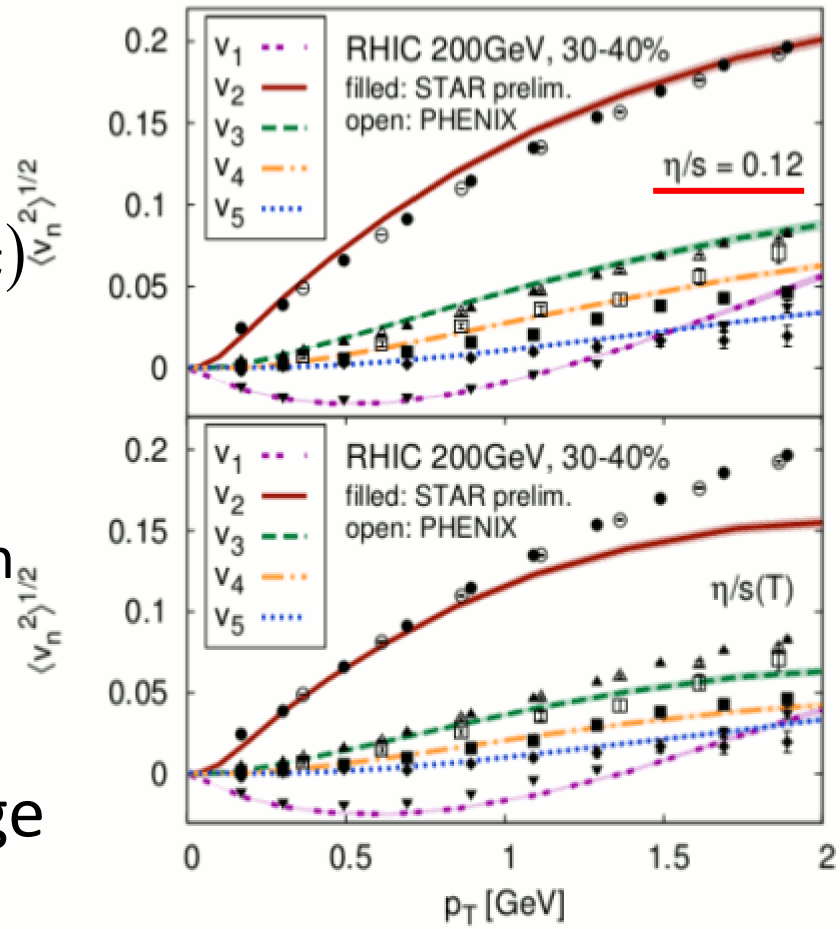


Lower limit

$$\eta/s = 1/(4\pi) \approx 0.08$$

Kovtun, Son, Starinets, PRL 94, 111601 (2005)

Gale et al, PRL 110, 012302 (2012)



- Viscose Hydrodynamics calculation

  e.g. $\eta/s \approx 0.12$

- Lattice calculation: in the stage of quenched approximation

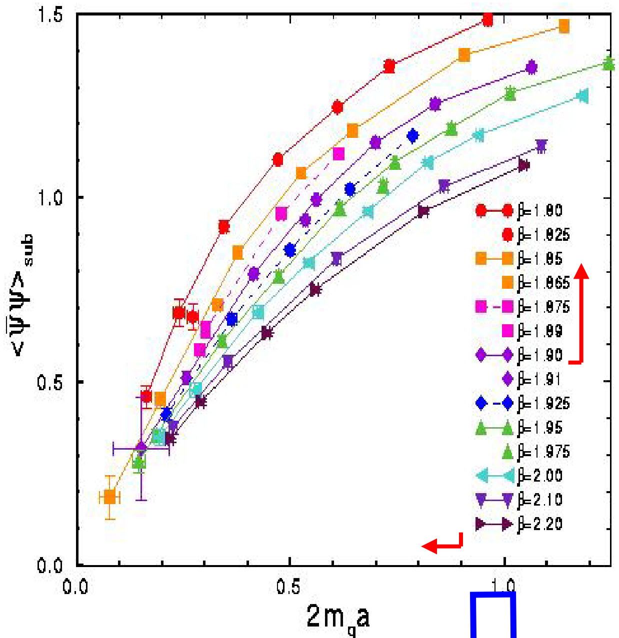
Nature of the phase transitions in terms of quark masses

- 2-flavor QCD: the chiral limit
  - O(4) universality crass or First order transition
  - $U_A(1)$ symmetry? S. Aoki, Y. Taniguchi
- 3-flavor QCD: critical quark mass
  - Improved staggered: very small, m_{PS}<60MeV?
  - Improved Wilson? Large? Y. Nakamura
  - At the physical quark mass?
- Finite chemical potential?

S. Takeda, SE

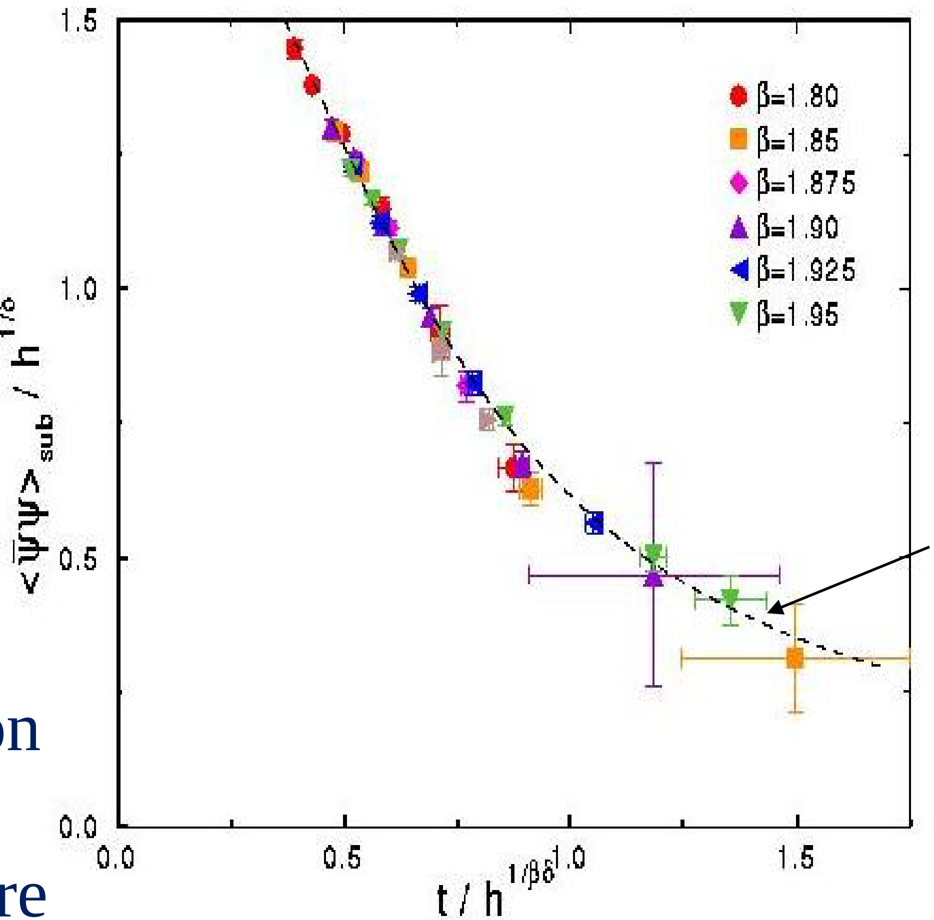
O(4) scaling test by N_f=2 Wilson-type quark

Iwasaki gauge + Clover Wilson (CP-PACS, PRD63.034502(2000))



$$\boxed{M/h^{1/\delta} = f\left(t/h^{1/\beta\delta}\right)}$$

$$\beta_{ct} = 1.469(73)$$
$$\chi^2/N_{df} = 0.816$$



$16^3 \times 4$ lattice

O(4) scaling function

M: Chiral condensation

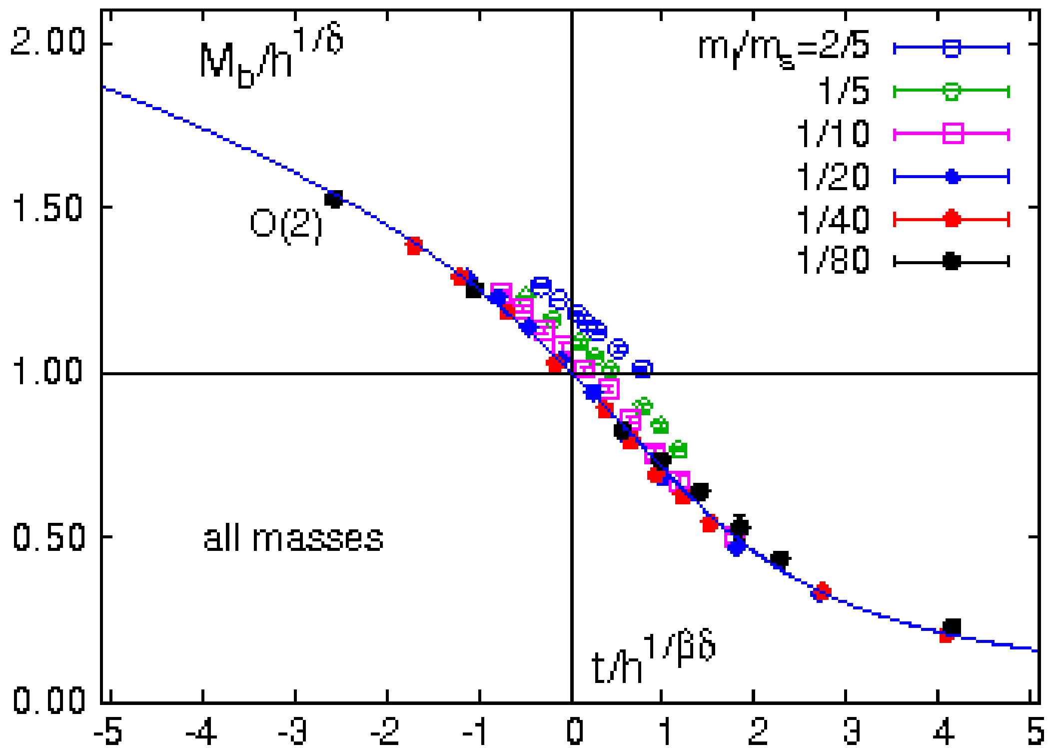
h: quark mass

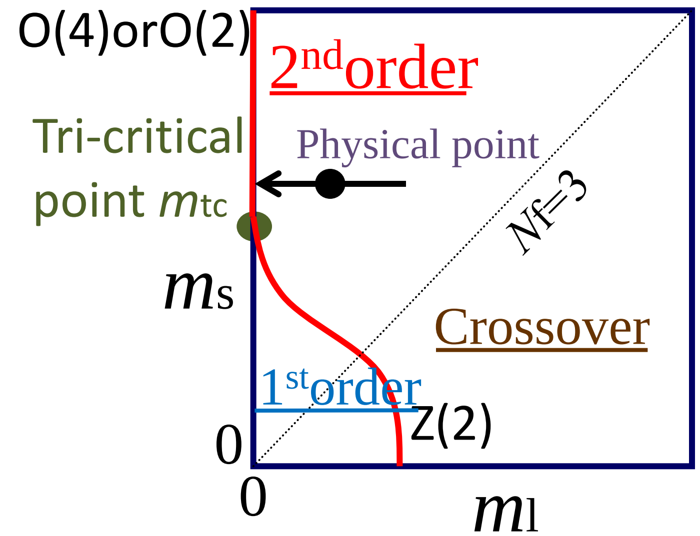
t: reduced temperature

Consistent with O(4) scaling function.

Scaling test by 2+1 flavor staggered quarks

BNL-Bielefeld, PRD80,094505(2009)



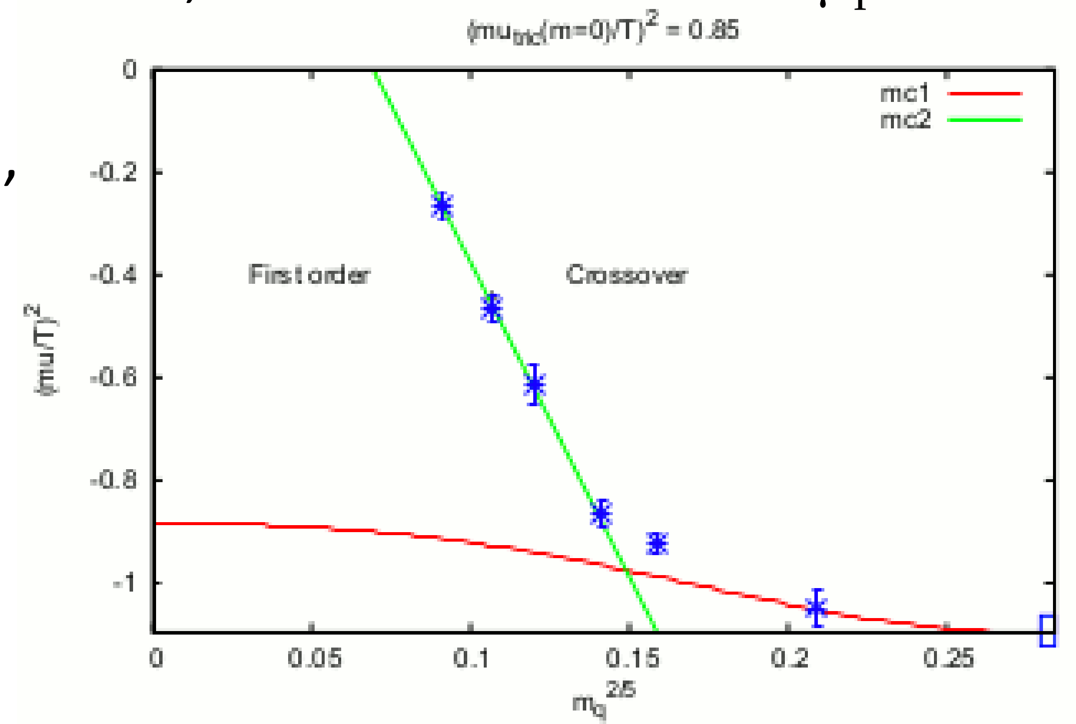


- Fixing the strange quark mass (ms) at the physical point, they investigated the light quark mass (ml) dependence.
- The scaling behavior agrees for ml/ms<1/20 (~ phisical point)
- (Tricritical point) < ms (physical) ➡ the first order region is very small.

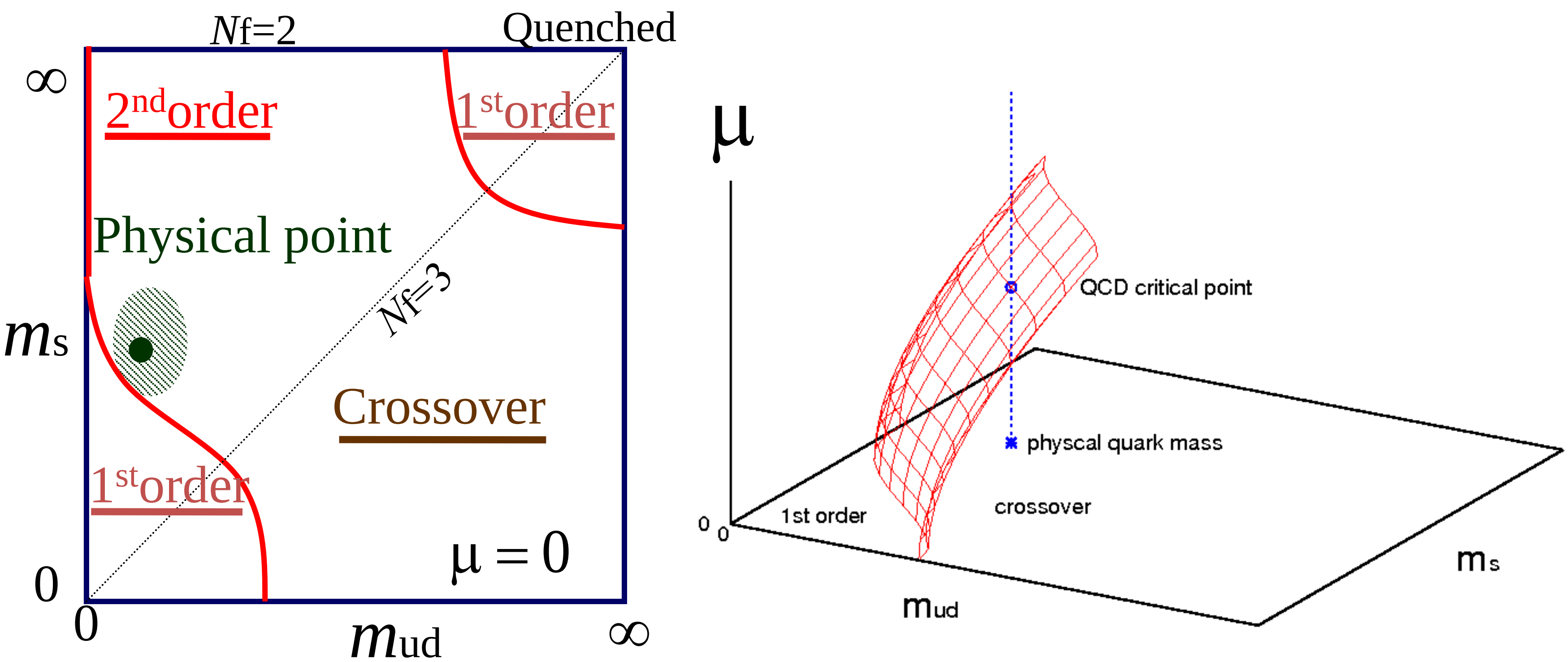
First order transition in 2-flavor QCD by staggered quarks

- The critical line separating first order and crossover is determined in QCD with an imaginary chemical potential μ_I.
- Extrapolate to light mass direction, the critical line crosses $\mu_I = 0$.
- But, Nt=4

  (Bonati, D' Elia, de Forcrand, Philipsen, Sanfilippo, arXiv:1311.0473, Lattice 2013 proc.)



Quark Mass dependence of QCD phase trantion



- The determination of the boundary of 1st order region: important.
- On the line of physical mass, the crossover at low density ⇒ 1st order transition at high density.
- However, the 1st order region is very small, and simulations with very small quark mass are required. ⇒ Difficult to study.

New and Old Approaches

- Complex Langevin I-O. Stamatescu
- Simulations on Lefschets Thimbles

  M. Cristoforetti, Y. Kikukawa
- Strong Coupling effective theory A. Ohnishi
- Reduction formula, Lee-Yang Zero K. Nagata
- Reweighting method SE
- TBA Handa??

Histogram method (Reweighting method)

- Monte-Carlo method

(Sg: gauge action, M: qaurk matrix)

  - Generate configurations with the probability of the Boltzmann weight.

$$\langle O \rangle_{(m,T,\mu)} = \frac{1}{Z}\int DU\, O\left(\det M(m,\mu)\right)^{N_{\rm f}} e^{-S_g} \approx \frac{1}{N_{\rm conf.}} \sum_{\{\rm conf.\}} O$$

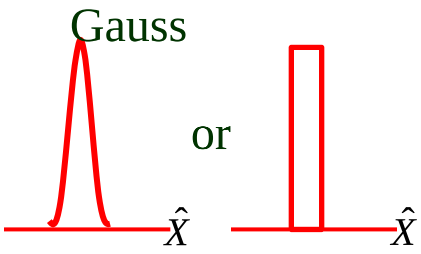
- Distribution function in Density of state method (Histogram method)

X: order parameters, total quark number, average plaquette etc.

$$W(X;m,T,\mu) \equiv \int DU\, \delta(X\text{-}\hat{X})\left(\det M(m,\mu)\right)^{N_{\rm f}}\ e^{-S_g}$$

$$\frac{W(X)}{Z} \approx \frac{1}{N_{\rm conf.}} \sum_{\{\rm conf.\}} \delta(X\text{-}\hat{X})$$

$$\delta(\hat{X}) \approx$$



  - Expectation values

$$\langle O[X] \rangle_{(m,T,\mu)} = \frac{1}{Z}\int dX\, O[X]\, W(X,m,T,\mu), \quad Z(m,T,\mu) = \int dX\, W(X,m,T,\mu)$$

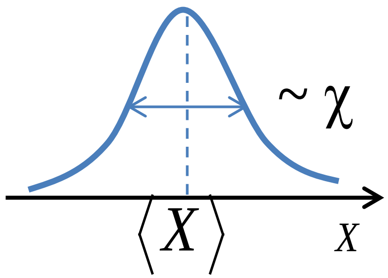
Effective potential $V_{\mathrm{eff}}(X)$

Probability distribution function (histogram)

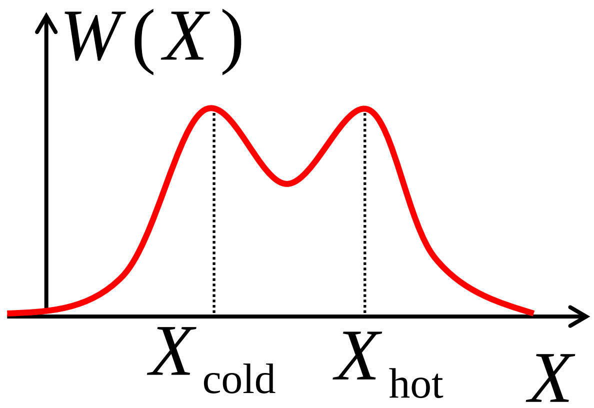
histogram

- If W(X) is a Gaussian distribution,
  - The peak position of $W(X)$ ⟶ ($\langle X \rangle$)
  - The width of $W(X)$ ⟶ susceptibilities

$$\chi = V \left\langle \left(X - \langle X \rangle \right)^2 \right\rangle$$



$$W(X) \approx \sqrt{\frac{A}{\pi}}\, e^{-A\left(X - \langle X \rangle\right)^2} \qquad A \propto V/\chi$$



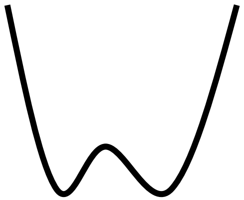
- First order phase transition

  Two phases coexists at T_c

- If $W(X)$ have two peaks,

  ⟶ first order transition

- Effective potential: $V_{\mathrm{eff}}(X) \equiv -\ln\left(W(X)\right)$

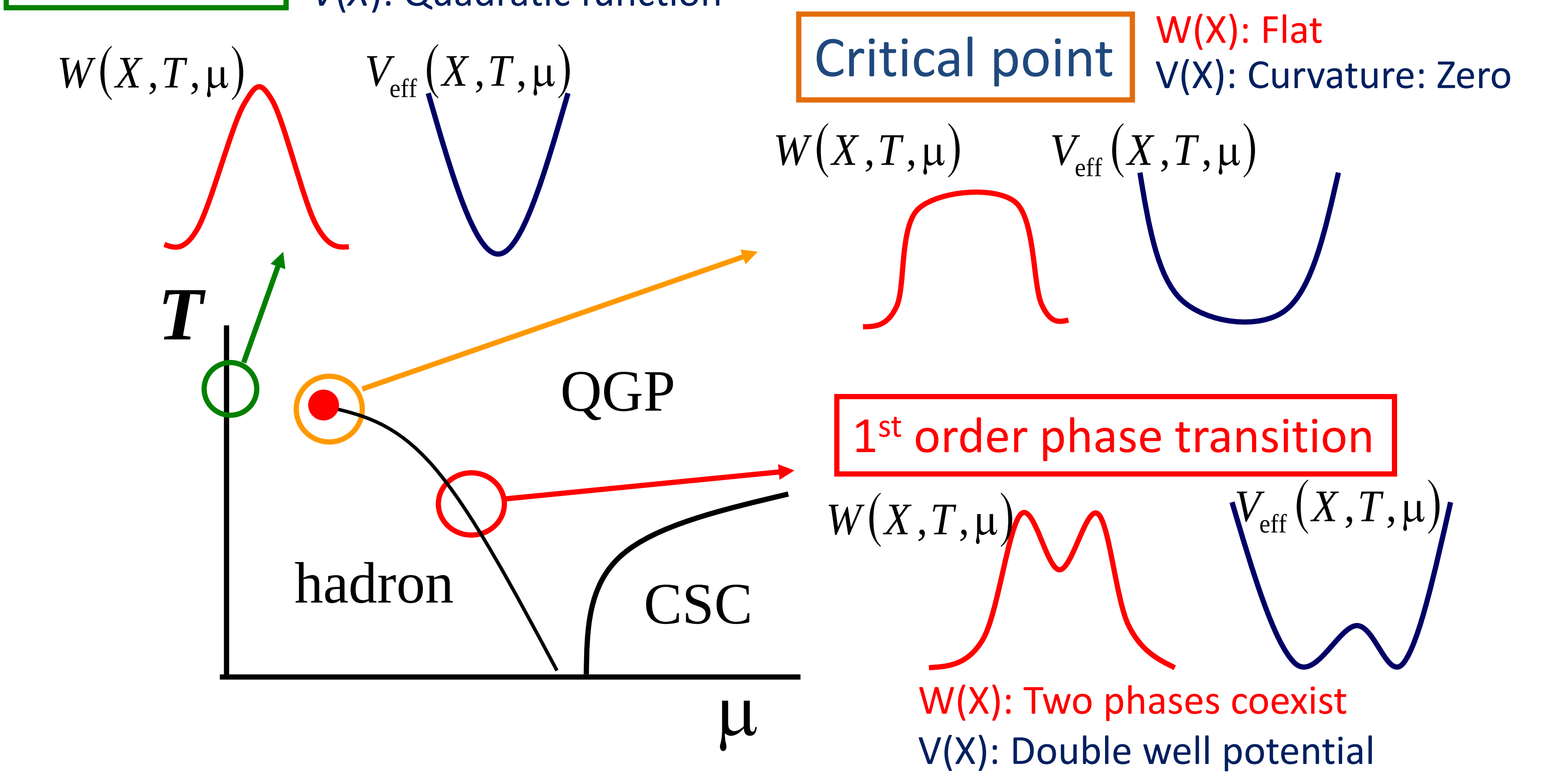


Distribution function & the effective potential

$$W\left(X;m,T,\mu\right)\equiv\int DU\delta\left(X-\hat{X}\right)\left(\det M\left(m,\mu\right)\right)^{N_\mathrm{f}}e^{-S_g}\qquad \text{(Histogram)}$$

X: order parameters, total quark number, average plaquette, etc.

$$V_\mathrm{eff}(X)=-\ln W(X)$$



Reweighting method for plaquette distribution function

$$W(P,\beta,m,\mu) \equiv \int DU\,\delta(\hat{P}-P) \prod_{f=1}^{N_f} \det M(m_f,\mu_f)\, e^{6N_{\rm site}\beta\hat{P}}$$

$$S_g = -6N_{\rm site}\beta\hat{P}$$

$$(\beta = 6/g^2)$$

plaquette P (1x1 Wilson loop for the standard action)

$$R(P,\beta,\beta_0 m,m_0,\mu) \equiv W(P,\beta,m,\mu)/W(P,\beta_0,m_0,0)$$

(Reweight factor)

$$R(P) = \frac{\left\langle \delta(\hat{P}-P)\, e^{6N_{\rm site}(\beta-\beta_0)\hat{P}} \prod_f \frac{\det M(m_f,\mu_f)}{\det M(m_0,0)} \right\rangle_{(\beta_0,\mu=0)}}{\left\langle \delta(\hat{P}-P) \right\rangle_{(\beta_0,\mu=0)}} \equiv \left\langle e^{6N_{\rm site}(\beta-\beta_0)\hat{P}} \prod_f \frac{\det M(m_f,\mu_f)}{\det M(m_0,0)} \right\rangle_{P:\text{fixed}}$$

Effective potential:

$$V_{\rm eff}(P,\beta,m,\mu) = -\ln[W(P,\beta,m,\mu)] = V_{\rm eff}(P,\beta_0,m_0,0) - \ln R(P,\beta,\beta_0 m,m_0,\mu)$$

$$\ln R(P) = \underline{6N_{\rm site}(\beta-\beta_0)P} + \ln \left\langle \underline{\prod_f \frac{\det M(m_f,\mu_f)}{\det M(m_0,0)}} \right\rangle_{P:\text{fixed}}$$

Sign problem

$$\left\langle \left(\frac{\det M(m,\mu)}{\det M(m_0,0)} \right)^{N_f} \right\rangle_{X \text{ fixed}} = \left\langle e^{i\theta} \left| \frac{\det M(m,\mu)}{\det M(m_0,0)} \right|^{N_f} \right\rangle_{X \text{ fixed}}$$

θ: complex phase of $(\det M)^{N_f}$

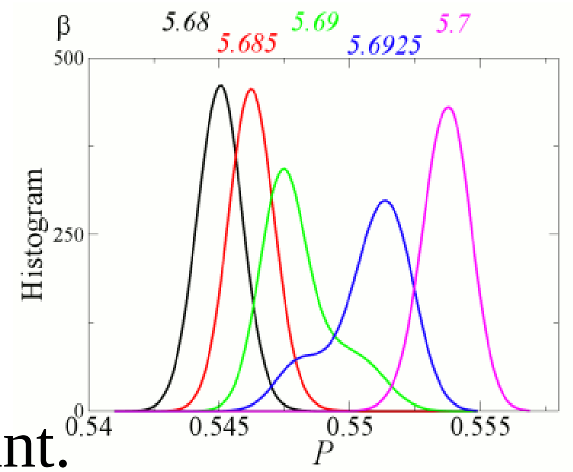
- Sign problem: if $e^{i\theta}$ changes the sign frequently,

$$W(X) \sim \left\langle e^{i\theta} \left| \frac{\det M(m,\mu)}{\det M(m_0,0)} \right|^{N_f} \right\rangle_{X \text{ fixed}} \ll \text{(statistical error)}$$

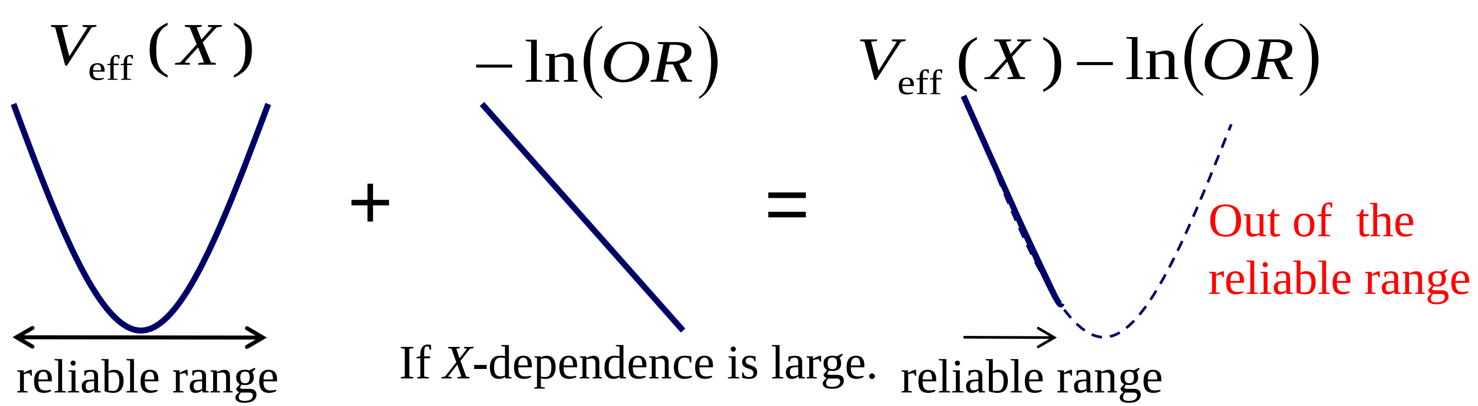
Overlap problem

$$\langle OR \rangle = \frac{1}{Z}\int ORW(X)\,dX \;= \frac{1}{Z}\int \exp\left(-V_{\rm eff}(X) + \ln(OR)\right) dX$$

$$V_{\rm eff}(X) = -\ln W(X)$$

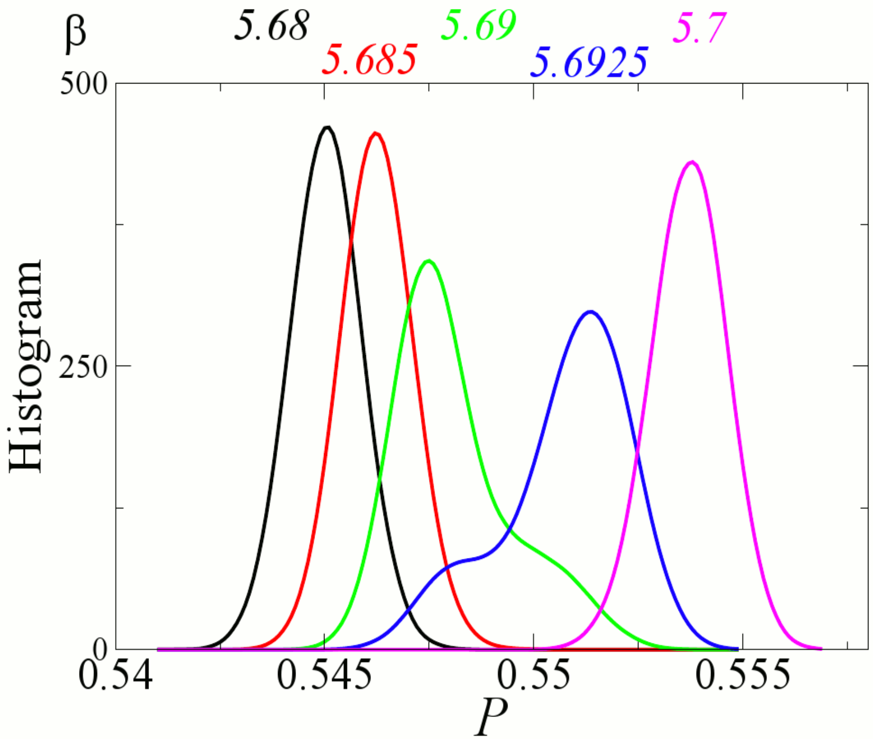


- W is computed from the histogram.
- Distribution function around X where $V_{\rm eff}(X) - \ln(OR)$ is minimized: important.
- $V_{\rm eff}$ must be computed in a wide range.

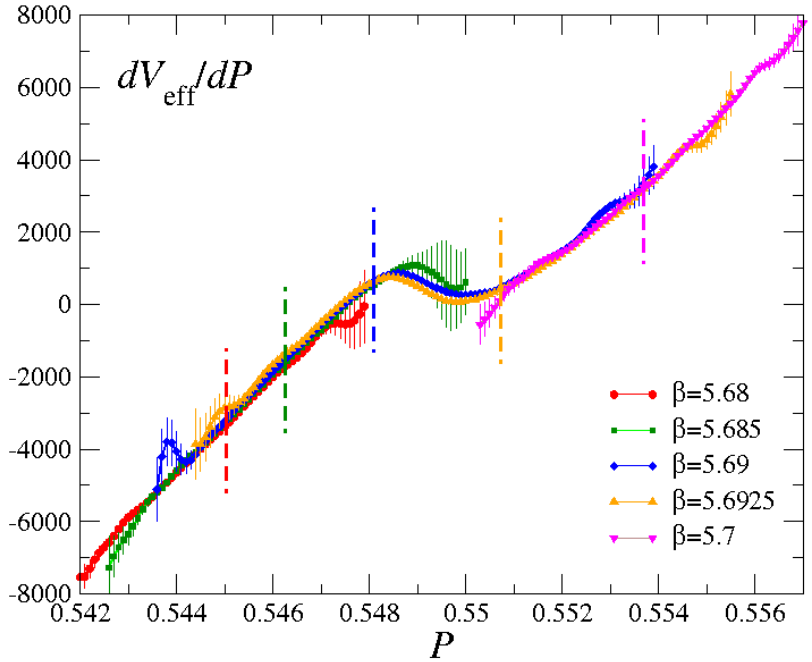


Distribution function in quenched simulations

Plaquette histogram at $K=1/m_q=0$.



Derivative of $V_{\rm eff}$ at β=5.69



$$V_{\rm eff}(\beta_2) = V_{\rm eff}(\beta_1) - 6N_{\rm site}(\beta_2-\beta_1)P$$

$$\frac{dV_{\rm eff}}{dP}(\beta_2) = \frac{dV_{\rm eff}}{dP}(\beta_1) - 6N_{\rm site}(\beta_2-\beta_1)$$

First order phase transition

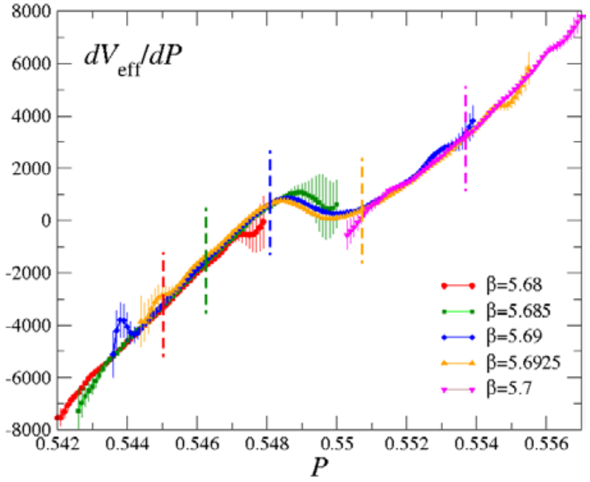
$dV_{\rm eff}/dP=0$ at the peak position of $V_{\rm eff}(P)$.

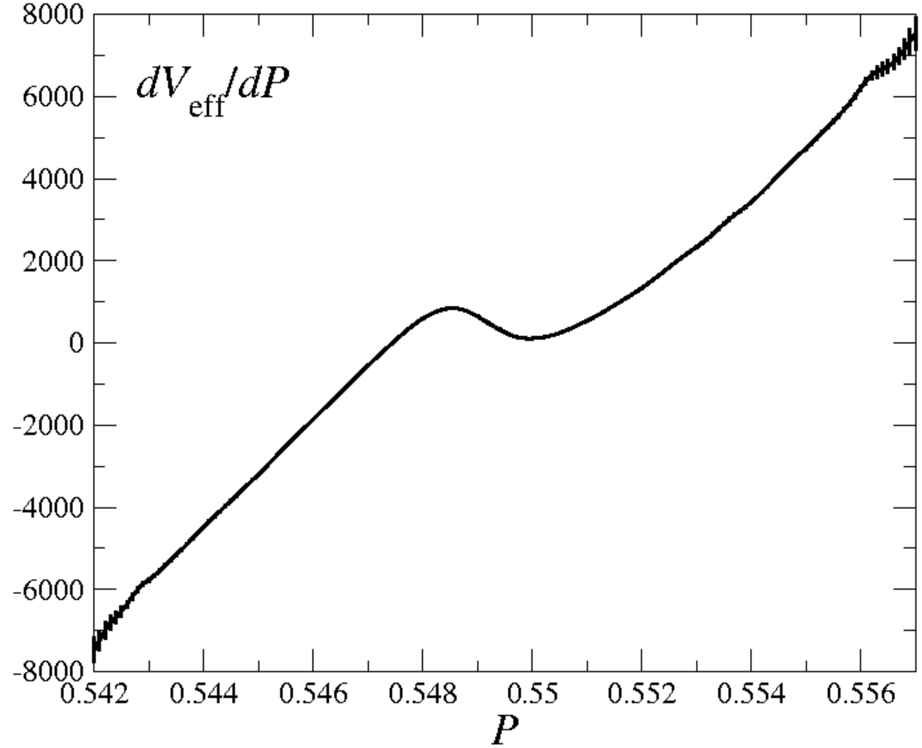
In this case, the curvature of $V_{\rm eff}$ is independent of β.

$$N_{\rm site} = 24^3 \times 4$$

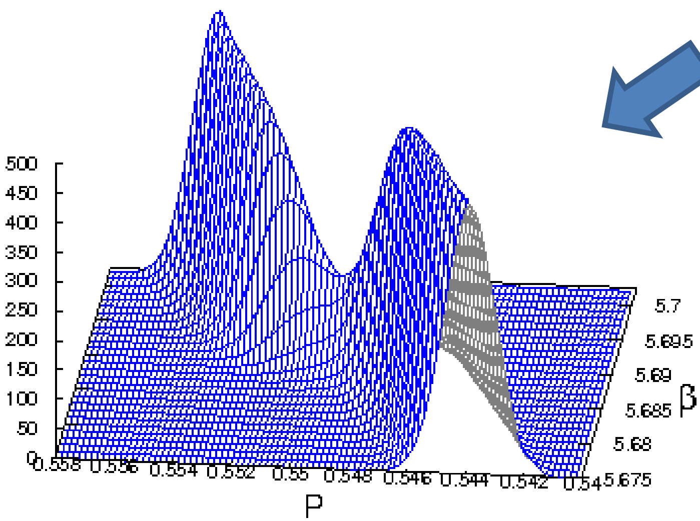
Distribution function in a quenched simulation

Derivative of the plaquette effective potential





Plaquette distribution function

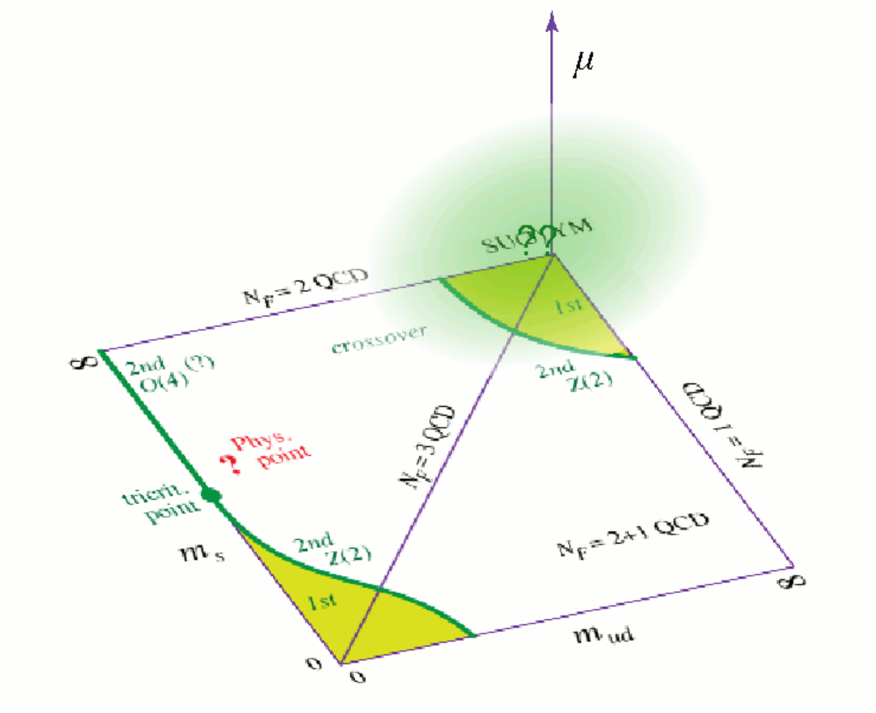


multi-point reweighting method

- Adopting β, average with the weight of N_{conf}
- Ferrenberg-Swendsen, Phys.Rev.Lett. 63, 1195 (1989); S.E., Phys. Rev. D78, 074507 (2008); WHOT-QCD, arXiv:1309.2445.

Distribution function in the heavy quark region

(WHOT-QCD Collab., Phys.Rev.D84, 054502(2011); arXiv:1309.2445)



- We study the properties of $W(X)$ in the heavy quark region.
- Performing quenched simulations + Reweighting.
- We find the critical surface.
- Standard Wilson quark action + plaquette gauge action, $S_g = -6 N_{\rm site} \beta P$
- 24^3x4 lattice

Hopping parameter expansion

$$\boxed{\kappa \sim 1/(\text{quark mass})}$$

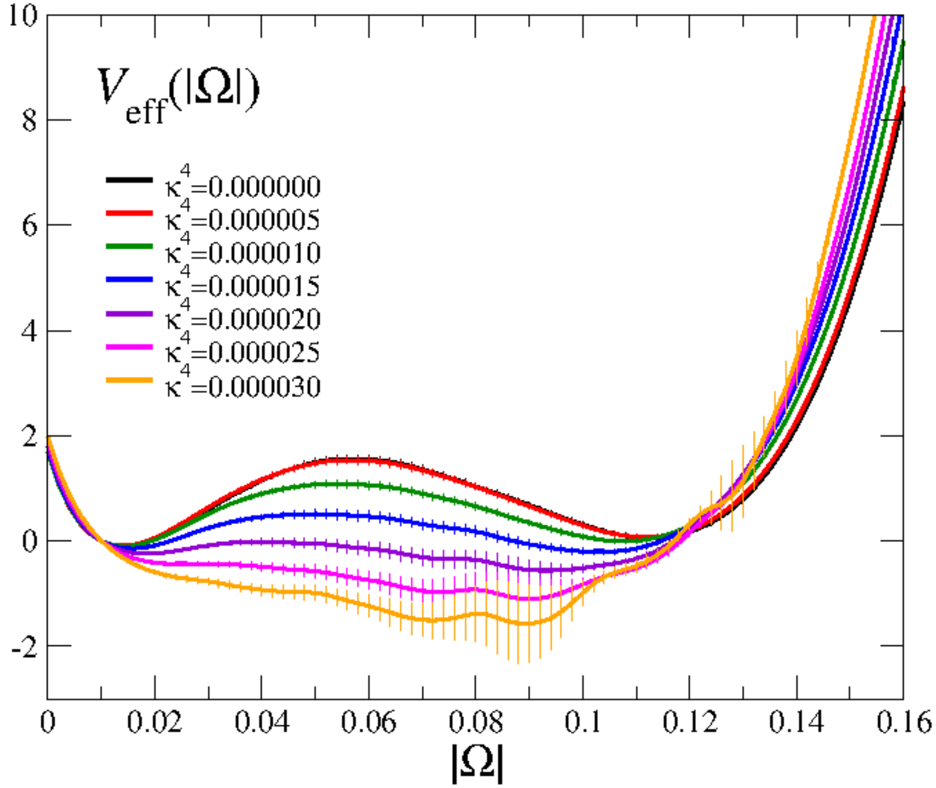
$$N_{\rm f} \ln\left(\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right) = N_{\rm f}\left(288 N_{\rm site}\kappa^4 P + 12\cdot 2^{N_t} N_s^3 \kappa^{N_t}\left(\cosh(\mu/T)\Omega_R + \underline{i\sinh(\mu/T)\Omega_I}\right) + \cdots\right)$$

phase

P: plaquette, $\Omega = \Omega_R + i\Omega_I$: Polyakov loop $\qquad \det M(0,0) = 1$

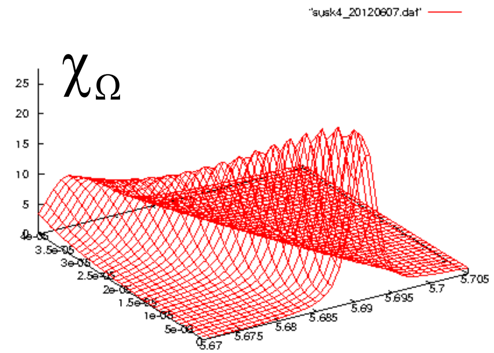
Order of the phase transition
Polyakov loop distribution (2-flavor)

Effective potential of $|\Omega|$
on the pseudo-critical line at $\mu=0$



$$\text{Critical point}: \kappa^4 \approx 2.0\times10^{-5}$$

- The pseudo-critical line is determined by χ_Ω peak.



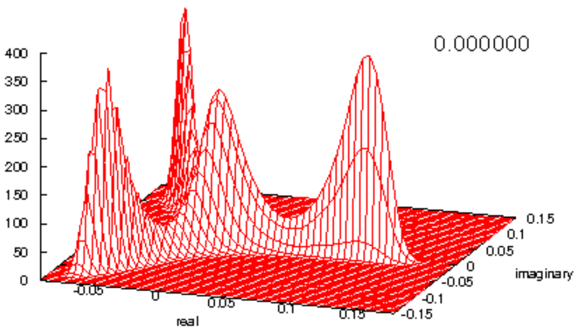
- Double-well at small κ
  - First order transition
- Single-well at large κ
  - Crossover

κ∼1/(quark mass)

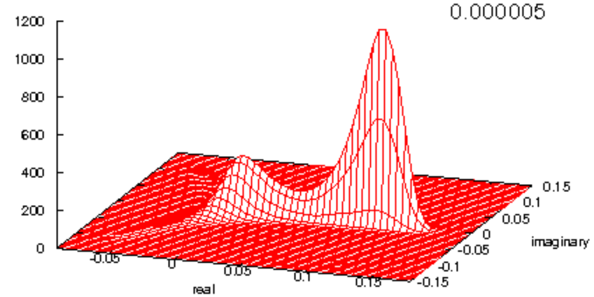
Polyakov loop distribution in the complex plane

(2-flavor, μ=0)

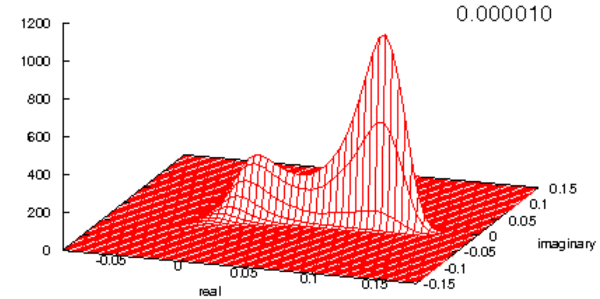
$\kappa^4 = 0.0$ Z(3) symmetric



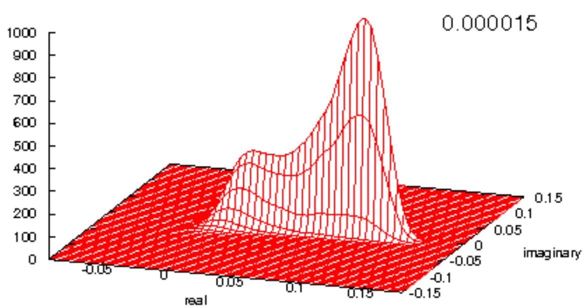
$\kappa^4 = 5.0 \times 10^{-6}$



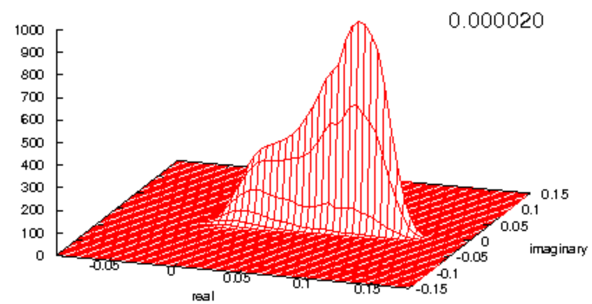
$\kappa^4 = 1.0 \times 10^{-5}$



$\kappa^4 = 1.5 \times 10^{-5}$

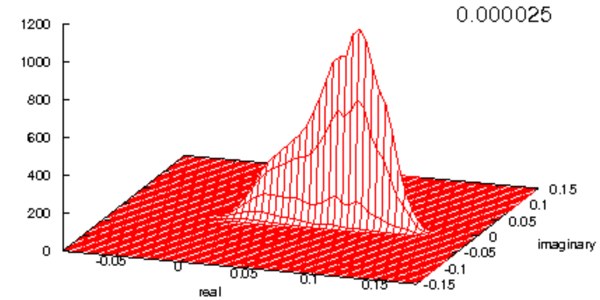


$\kappa^4 = 2.0 \times 10^{-5}$



critical point

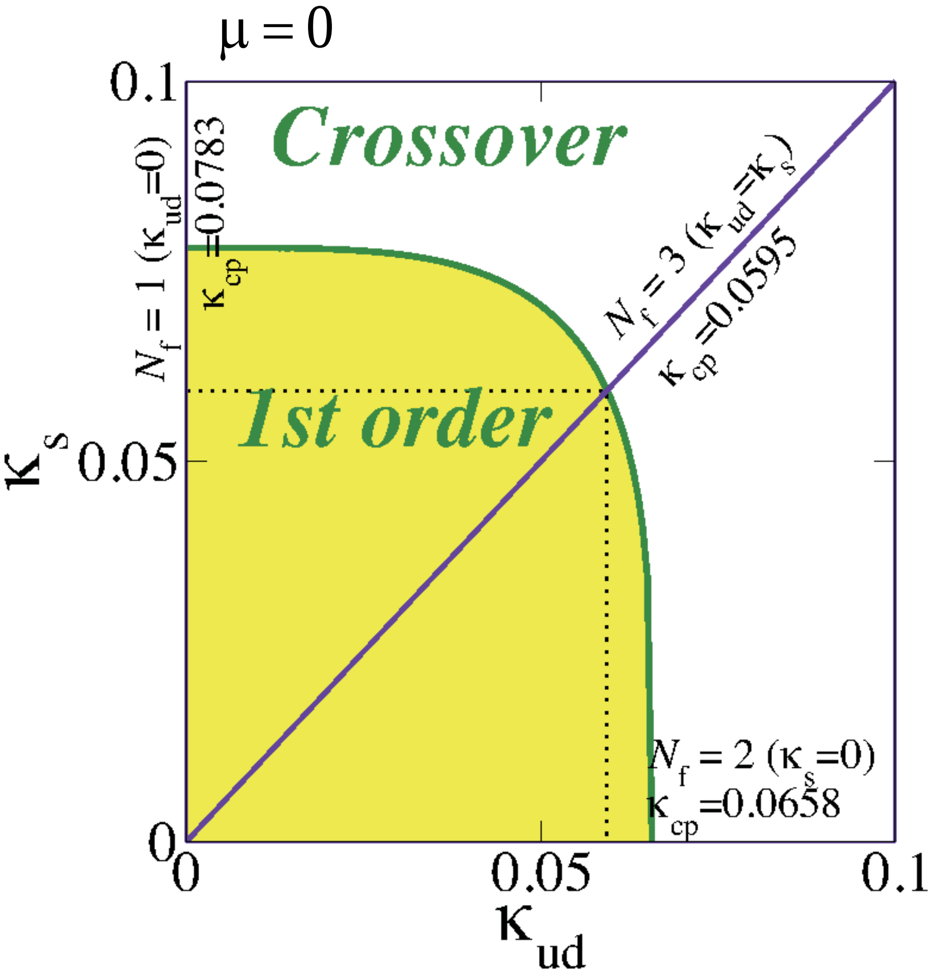
$\kappa^4 = 2.5 \times 10^{-5}$



- on β_{pc} measured by the Polyakov loop susceptibility.

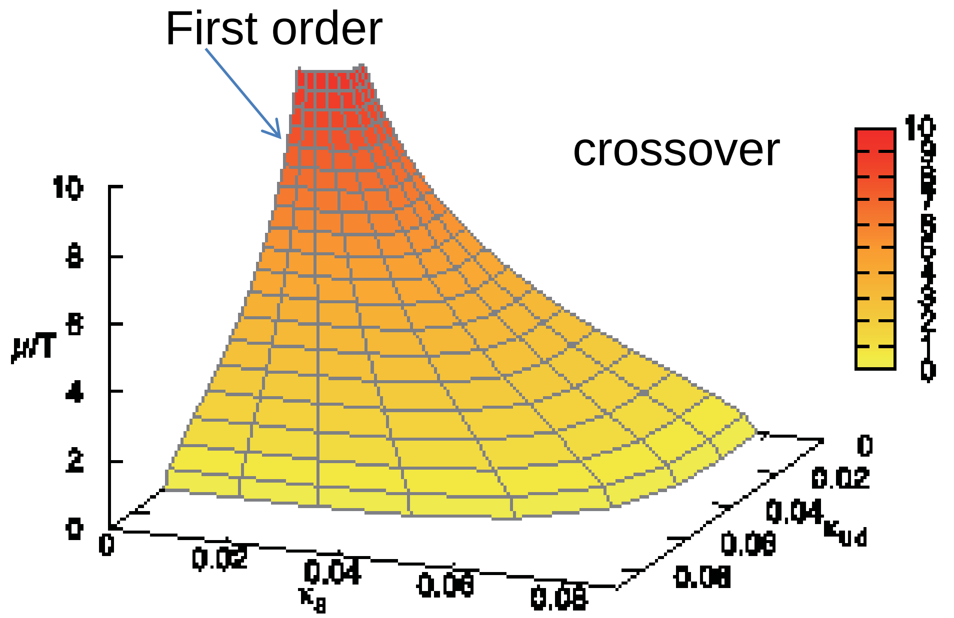
Critical surface in the heavy quark region of (2+1)-flavor QCD

$\left(24^3 \times 4 \text{ lattice}\right)$



$$\frac{T_c}{m_\pi} \approx 0.02$$ at κ_{cp} for 2-flavor

Critical surface at finite density



Control Parameters in $W(X)$

- Distribution function

$$W(X;\kappa,\beta,\mu) \equiv \int DU\delta\left(X-\hat{X}\right)\prod_{f=1}^{N_{\mathrm{f}}} \det M\left(\kappa_f,\mu_f\right)e^{-S_g}$$

$$S_g = -6N_{\mathrm{site}}\beta\underline{\hat{P}}$$

- Hopping parameter expansion

$$\ln\left(\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right) = 288N_{\mathrm{site}}\kappa^4\underline{\hat{P}} + 12\cdot 2^{N_t}N_s^3\kappa^{N_t}\left(\cosh(\mu/T)\underline{\hat{\Omega}_R} + i\sinh(\mu/T)\underline{\hat{\Omega}_I}\right)+\cdots$$

- Three quantities in W: P, Ω_R, Ω_I
- Three parameters

$$\left(0 \le \left|\tanh\left(\mu_f/T\right)\right| < 1\right)$$

$$\underline{\beta^* \equiv \beta + \prod_{f=1}^{N_{\mathrm{f}}} 48\kappa_f^4}, \quad \underline{\prod_{f=1}^{N_{\mathrm{f}}}\kappa_f^{N_t}\cosh\left(\mu_f/T\right)}, \quad \prod_{f=1}^{N_{\mathrm{f}}}\kappa_f^{N_t}\sinh\left(\mu_f/T\right)$$

$$= \prod_{f=1}^{N_{\mathrm{f}}}\kappa_f^{N_t}\cosh\left(\mu_f/T\right)\underline{\tanh\left(\mu_f/T\right)}$$

Distribution function of Ω_R at finite density

$$W(\Omega_R,\beta,\kappa,\mu)=\int DU\,\delta\left(\Omega_R-\hat{\Omega}_R\right)\left(\det M(\kappa)\right)^{N_f}\,e^{-6N_{\mathrm{site}}\hat{P}}$$

- Hopping parameter expansion

$$\frac{W(\beta,\kappa,\mu)}{W(\beta_0,0,0)}=\left\langle \exp\left[\left(6(\beta-\beta_0)+288N_f\kappa^4\right)N_{\mathrm{site}}\hat{P}-12\times 2^{N_t}N_f N_s^3\kappa^{N_t}\cosh(\mu/T)\hat{\Omega}_R+i\theta\right]\right\rangle_{\Omega_R;\beta_0,\kappa=\mu=0}$$

$$\left(\theta=12\cdot 2^{N_t}N_s^3 N_f\kappa^{N_t}\sinh(\mu/T)\hat{\Omega}_I\right)$$

- Adopting $\beta_0\equiv\beta+48N_f\kappa^4$,
- Effective potential: $V_{\mathrm{eff}}(\Omega_R;\beta,\kappa,\mu)=-\ln W(\Omega_R;\beta,\kappa,\mu)$

$$V_{\mathrm{eff}}(\beta,\kappa,\mu)=V_{\mathrm{eff}}(\beta_0,0,0)-12\times 2^{N_t}N_f N_s^3\kappa^{N_t}\cosh(\mu/T)\Omega_R-\ln\left\langle e^{i\theta}\right\rangle_{\Omega_R;\beta_0,\kappa=\mu=0}$$

$$\equiv V_0(\beta,\kappa,\mu)\qquad -\ln\left\langle e^{i\theta}\right\rangle_{\Omega_R;\beta_0,\kappa=\mu=0}$$

Phase-quenched part　　Phase average

- V_0 is $V_{\mathrm{eff}}(\mu=0)$ when we replace $\kappa^{N_t}\Rightarrow\kappa^{N_t}\cosh(\mu/T)$

(at $\mu=0$, $V_{\mathrm{eff}}(\beta,\kappa,0)=V_{\mathrm{eff}}(\beta_0,0,0)-12\times 2^{N_t}N_f N_s^3\kappa^{N_t}\Omega_R$)

Avoiding the sign problem

(SE, Phys.Rev.D77,014508(2008), WHOT-QCD, Phys.Rev.D82, 014508(2010))

θ: complex phase $\quad \theta \equiv \mathrm{Im} \ln \det M \quad \approx 12 \cdot 2^{N_t} N_s^3 N_f \kappa^{N_t} \sinh(\mu/T) \Omega_I$

- Sign problem: If $e^{i\theta}$ changes its sign,

$$\left\langle e^{i\theta} \right\rangle_{\Omega_R \text{ fixed}} << \text{(statistical error)}$$

- Cumulant expansion

$$\left\langle e^{i\theta} \right\rangle_{\Omega_R} = \exp\left[\underbrace{\cancel{i\langle\theta\rangle_C}}_{\to 0} - \frac{1}{2}\langle\theta^2\rangle_C \underbrace{\cancel{- \frac{i}{3!}\langle\theta^3\rangle_C}}_{\to 0} + \frac{1}{4!}\langle\theta^4\rangle_C \cancel{+ \cdots} \right]$$

cumulants

$$\langle\theta\rangle_C = \langle\theta\rangle_{\Omega_R}, \quad \langle\theta^2\rangle_C = \langle\theta^2\rangle_{\Omega_R} - \langle\theta\rangle_{\Omega_R}^2, \quad \langle\theta^3\rangle_C = \langle\theta^3\rangle_{\Omega_R} - 3\langle\theta^2\rangle_{\Omega_R}\langle\theta\rangle_{\Omega_R} + 2\langle\theta\rangle_{\Omega_R}^3, \quad \langle\theta^4\rangle_C = \cdots$$

– Odd terms vanish from a symmetry under $\mu \leftrightarrow -\mu$ ($\theta \leftrightarrow -\theta$)
  Source of the complex phase

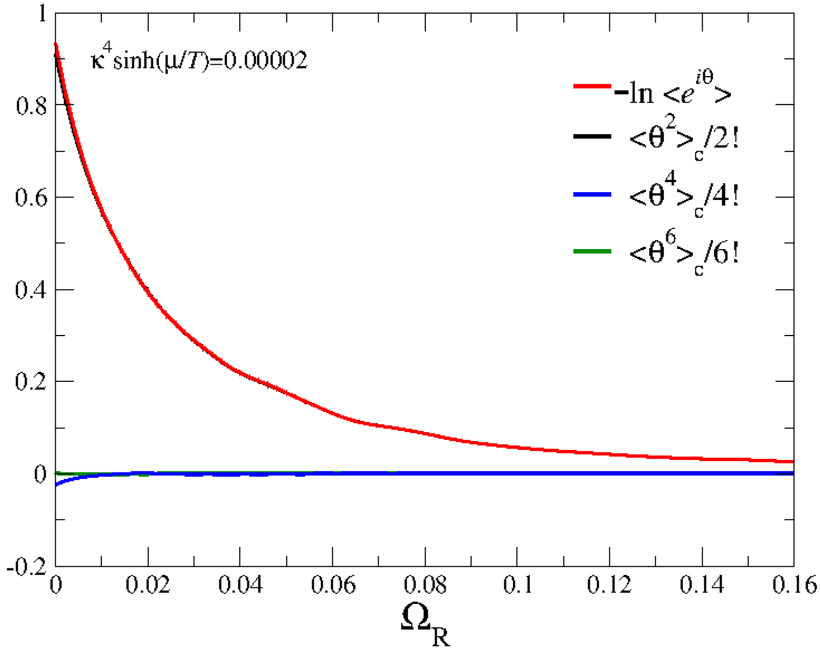
If the cumulant expansion converges, No sign problem.

Cumulant expansion

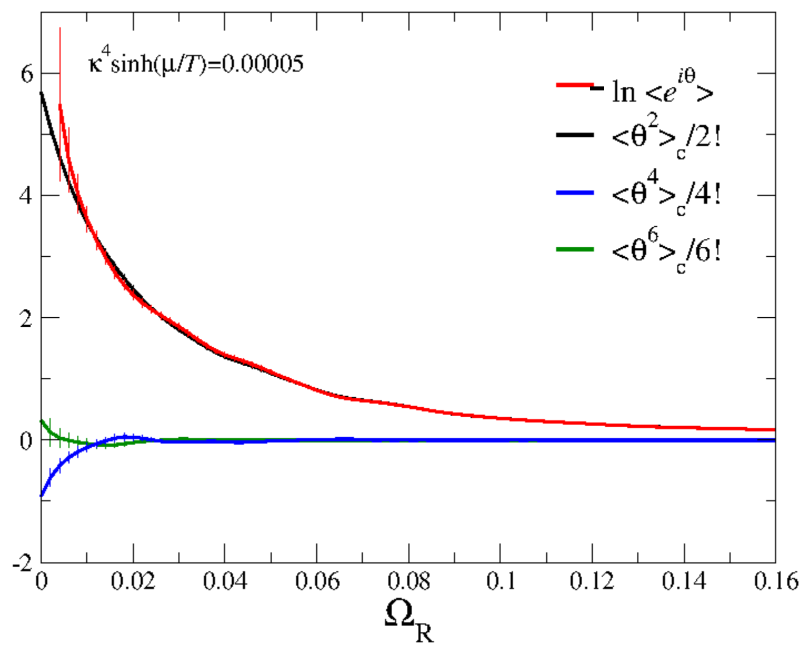
$\beta_0 = 5.69$

$$\ln\left\langle e^{i\theta}\right\rangle_{\Omega_R} = -\frac{1}{2}\left\langle\theta^2\right\rangle_C + \frac{1}{4!}\left\langle\theta^4\right\rangle_C - \frac{1}{6!}\left\langle\theta^6\right\rangle_C + \cdots$$

$K^4(\mu)\sinh(\mu/T) = 0.00002$



$K^4(\mu)\sinh(\mu/T) = 0.00005$



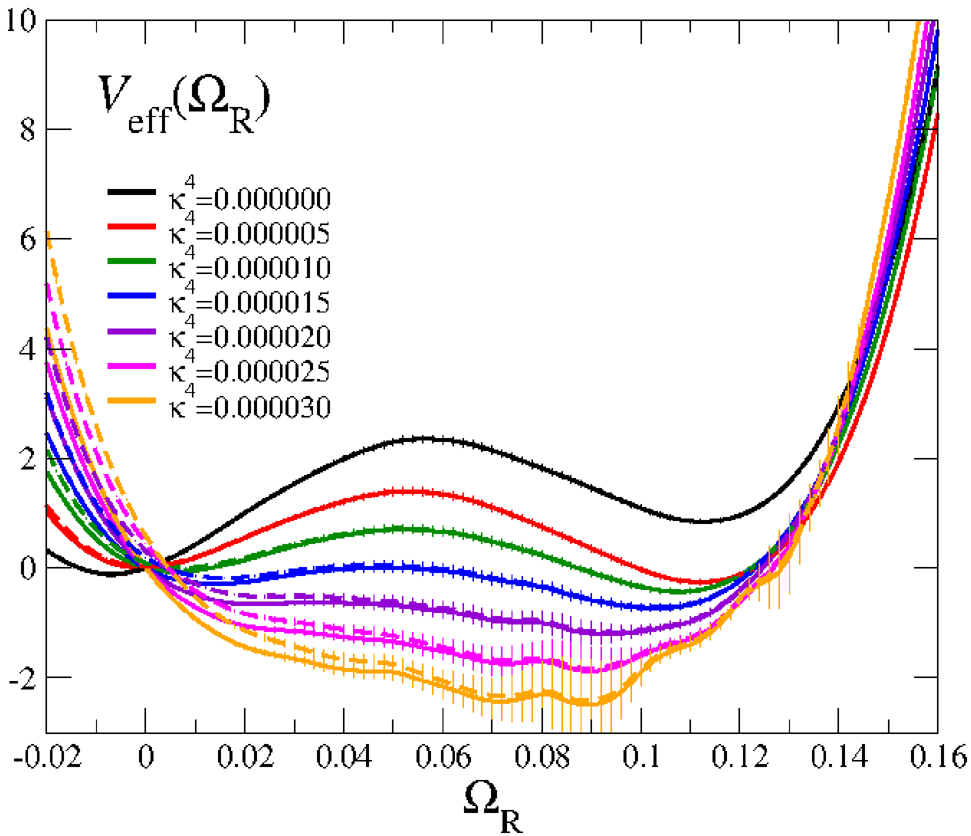
- At the critical point of phase-quenched part, the effect of higher order terms: small.

$$\kappa_{\rm cp}^{N_t}(0) = \kappa_{\rm cp}^{N_t}(\mu)\cosh(\mu/T) > \kappa_{\rm cp}^{N_t}(\mu)\sinh(\mu/T)$$

~0.00002

Effect from the complex phase factor (2-flavor)

- Polyakov loop effective potential at various $\kappa^{N_t}\cosh(\mu/T)$ at the transition point. (β^* is adjusted at the transition point.)
  - Solid lines: $\mu=0$, i.e., $\cosh(\mu/T)=1, \quad \tanh(\mu/T)=0$
  - Dashed lines: $\tanh(\mu/T)=1$



The effect from the complex phase factor is very small except near Ω_R=0.

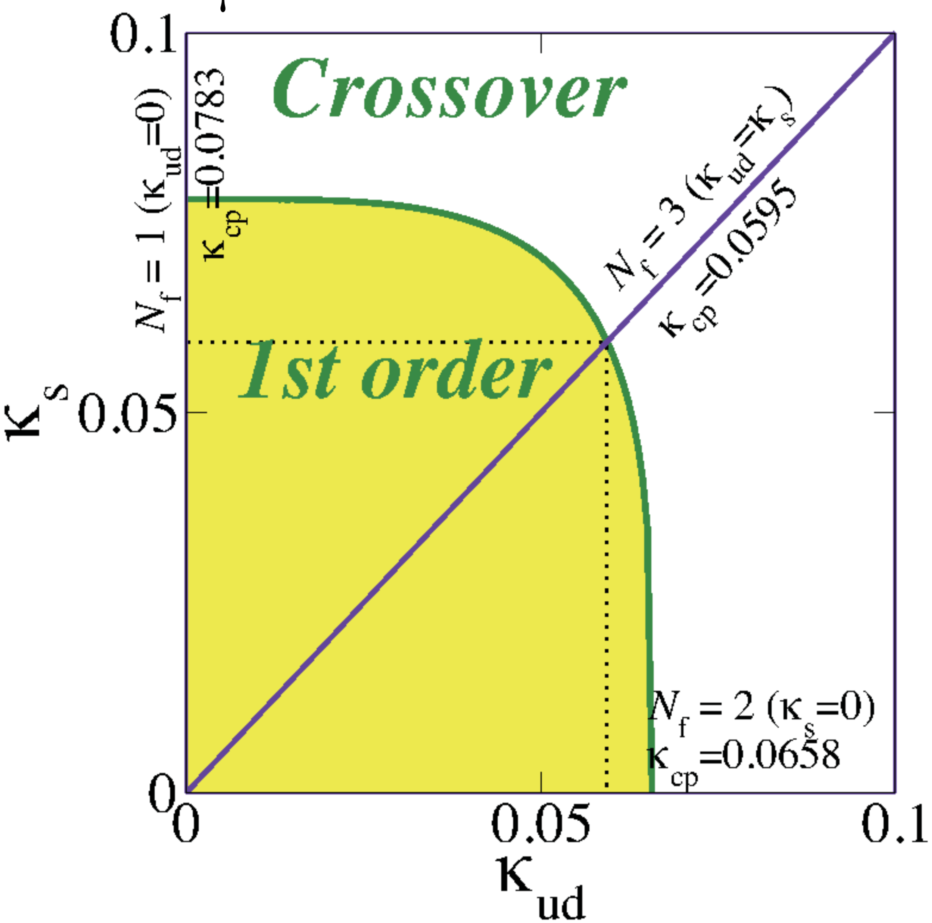
The nature of the phase transition is controlled only by

$$\prod_{f=1}^{N_f}\kappa_f^{N_t}\cosh(\mu_f/T)$$

Critical surface in the heavy quark region of (2+1)-flavor QCD

$\left(24^3 \times 4 \text{ lattice}\right)$

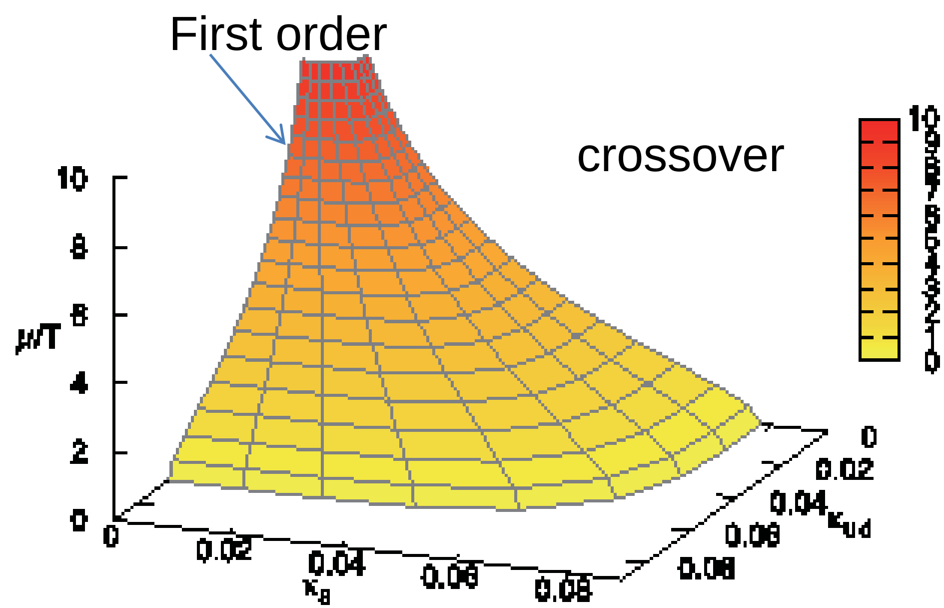
$\mu = 0$



$$\sum_{f=1}^{N_f} \kappa_{cp\,f}^{N_t} \cosh\left(\mu_f / T\right) = 4 \times 10^{-5}$$

$$\left(N_t = 4\right)$$

Critical surface at finite density



$\frac{T_c}{m_\pi} \approx 0.02$ at κ_{cp} for 2-flavor

arXiv:1309.2445 [hep-lat]

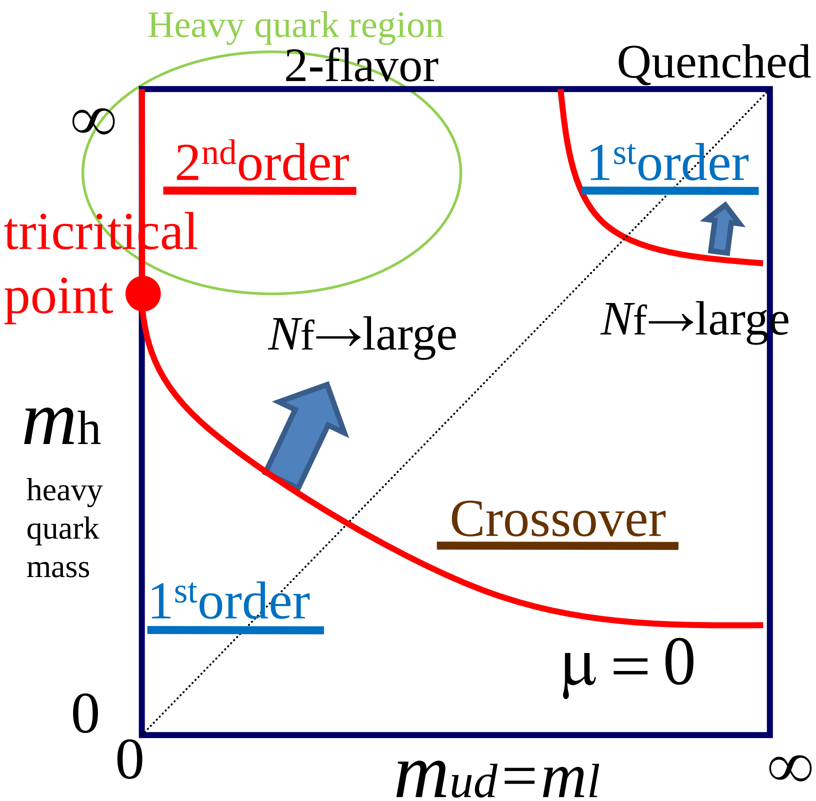
Phase transitions in many-flavor QCD

SE & Yamada, Phys. Rev. Lett. 110, 172001 (2013)

- 2 light quarks, many heavy quarks
- Technicolor model
- First order transition

  ⇒ Electro-weak baryogenesis
- Good test for (2+1)-flavor QCD

Nature of phase transition of 2+N_f-flavor QCD



- Assumption: N_f-flavors are heavy.
  - Hopping parameter κ expansion
- Parameter: $N_f \kappa^{N_t}$ $\Rightarrow$ $1/m_{h,ct} \sim \kappa_{ct} \propto 1/N_f^{1/N_t}$
- As increasing N_f, critical mass becomes larger.
- Tricritical scaling: the same as (2+1)-flavor QCD

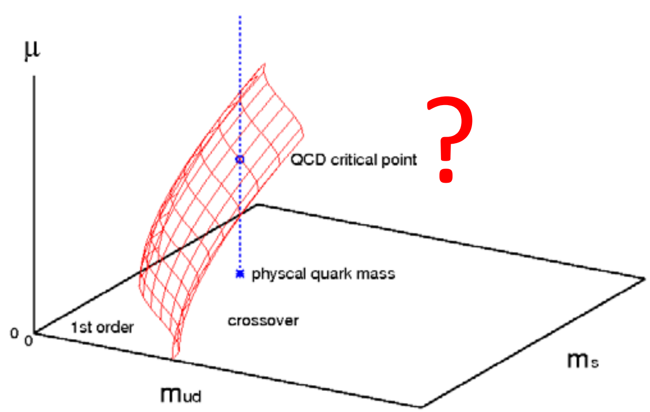
Tricritical point m_E :

$$m_{ud}^c \sim (m_E - m_h)^{5/2}$$

$$m_{ud}^c \sim \mu^5$$

Good test ground

At finite density?



Reweighting method for plaquette distribution function

$$W(P,\beta,m,\mu) \equiv \int DU \delta(\hat{P}-P) \prod_{f=1}^{N_{\rm f}} \det M(m_f,\mu_f) \mathrm{e}^{6N_{\rm site}\beta\hat{P}}$$

$$S_g = -6N_{\rm site}\beta\hat{P}$$

$$(\beta = 6/g^2)$$

plaquette P (1x1 Wilson loop for the standard action)

$$R(P,\beta,\beta_0 m,m_0,\mu) \equiv W(P,\beta,m,\mu)/W(P,\beta_0,m_0,0)$$

(Reweight factor)

$$R(P) = \frac{\left\langle \delta(\hat{P}-P) e^{6N_{\rm site}(\beta-\beta_0)\hat{P}} \prod_f \frac{\det M(m_f,\mu_f)}{\det M(m_0,0)} \right\rangle_{(\beta_0,\mu=0)}}{\left\langle \delta(\hat{P}-P) \right\rangle_{(\beta_0,\mu=0)}} \equiv \left\langle e^{6N_{\rm site}(\beta-\beta_0)\hat{P}} \prod_f \frac{\det M(m_f,\mu_f)}{\det M(m_0,0)} \right\rangle_{P:\text{fixed}}$$

Effective potential:

$$V_{\rm eff}(P,\beta,m,\mu) = -\ln[W(P,\beta,m,\mu)] = V_{\rm eff}(P,\beta_0,m_0,0) - \ln R(P,\beta,\beta_0 m,m_0,\mu)$$

$$\ln R(P) = \underline{6N_{\rm site}(\beta-\beta_0)P} + \ln \left\langle \underline{\prod_f \frac{\det M(m_f,\mu_f)}{\det M(m_0,0)}} \right\rangle_{P:\text{fixed}}$$

First order transition point: two phases coexist

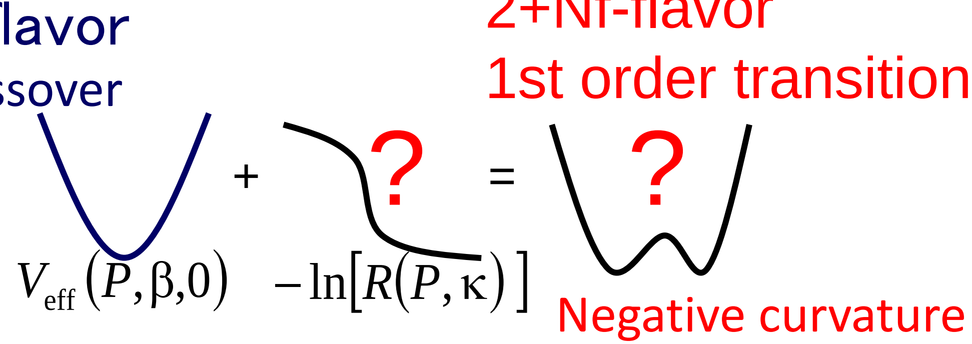
Plaquette distribution function

- Performing simulations of 2-flavor QCD,
- Dynamical effect of N_f-flavors are included by the reweighting.
- We assume N_f-flavors are heavy.
- Hopping parameter (κ) expansion（Wilson quark）

$$N_{\rm f} \ln\left(\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right) = N_{\rm f}\left(288 N_{\rm site}\kappa^4 P + 12\cdot 2^{N_t} N_s^3 \kappa^{N_t}\left(\cosh(\mu/T)\Omega_R + i\sinh(\mu/T)\Omega_I\right)+\cdots\right)$$

- Effective potential

$$V_{\rm eff}(P,\beta,\kappa) = -\ln\left[R(P,\kappa)W(P,\beta,0)\right] =$$



$$\ln R(P) = \ln\left\langle e^{6N_{\rm site}(\beta-\beta_0)\hat{P}} \prod_f \frac{\det M(\kappa_f,\mu_f)}{\det M(\kappa_0,0)}\right\rangle_{P:{\rm fixed}} \approx \ln\left\langle \exp\left(6hN_s^3\hat{\Omega}_R\right)\right\rangle_{P:{\rm fixed}}$$ +(linear term of P)

(degenerate mass case at μ=0)

Curvature of the effective potential

$$V_{\text{eff}}(P,\beta,h,\mu) = V_{\text{eff}}(P,\beta_0,0,0) - \ln \bar{R}(P,h,\mu) + \text{ (linear term of } P\text{)}$$

$$\bar{R}(P) = \left\langle \exp\left(6N_s^3 h \Omega_R\right)\right\rangle_{P:\text{fixed}} \qquad \text{(for the case of } \mu=0\text{)}$$

Wilson quark

$$h = 2N_{\text{f}}(2\kappa_{\text{h}})^{N_t}$$

Staggered quark

$$h = N_{\text{f}} \Big/ \left(4(2m_{\text{h}})^{N_t}\right)$$

- Linear term of P is irrelevant to the curvature
- β-dependence is only in the linear term.
- The curvature is independent of β.

χ_P: plaquette susceptibility

$$\frac{d^2 V_{\text{eff}}(0)}{dP^2} \approx \frac{6N_{\text{site}}}{\chi_P}$$

$$\frac{d^2 V_{\text{eff}}}{dP^2}(P,h,\mu) = \frac{d^2 V_{\text{eff}}}{dP^2}(P,0,0) - \frac{d^2 \ln \bar{R}}{dP^2}(P,h,\mu)$$

2-flavor

- If there exists the negative curvature region,

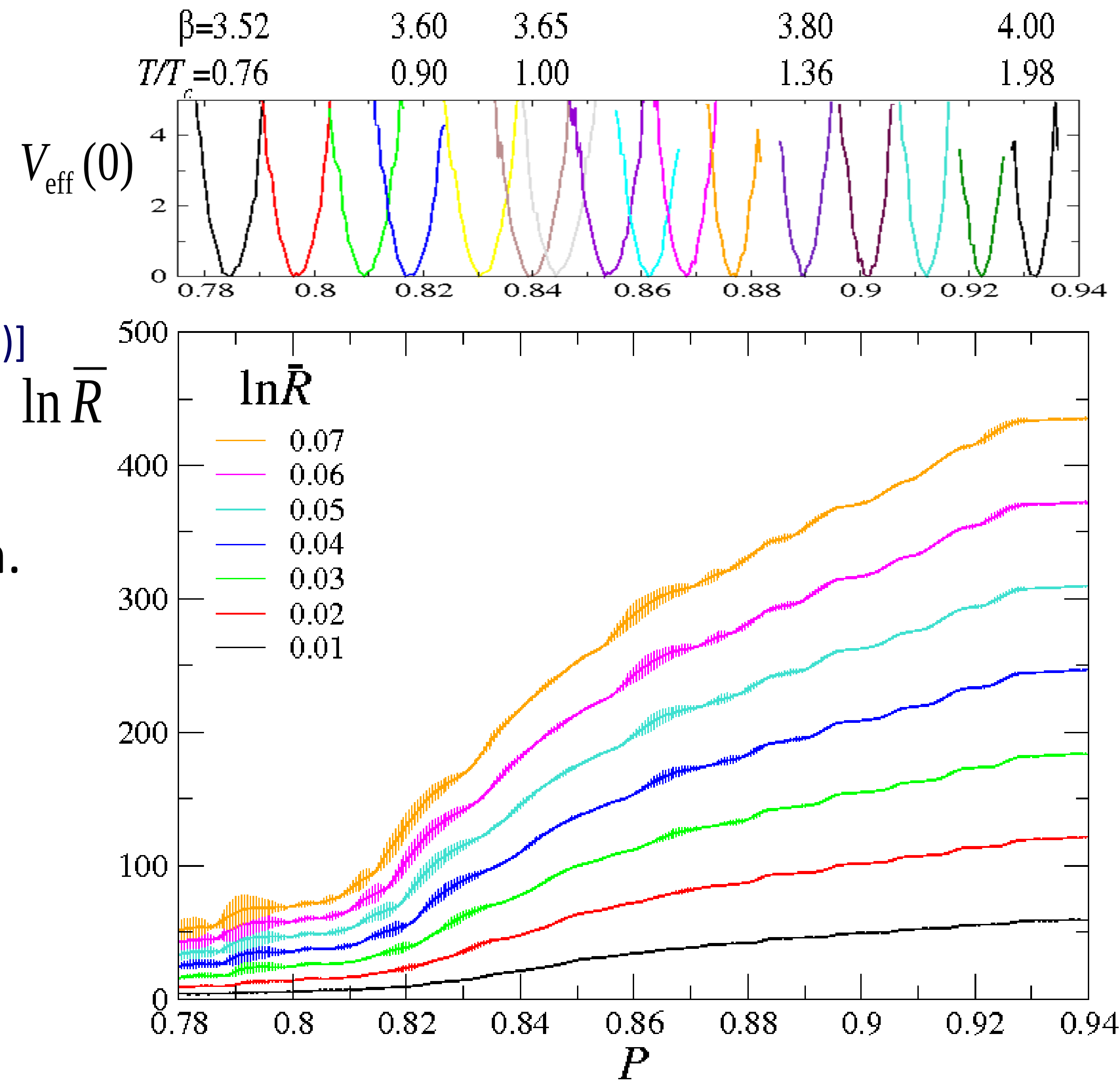
⇒ First order transition (double-well potential)

Effective potential at $h \neq 0$

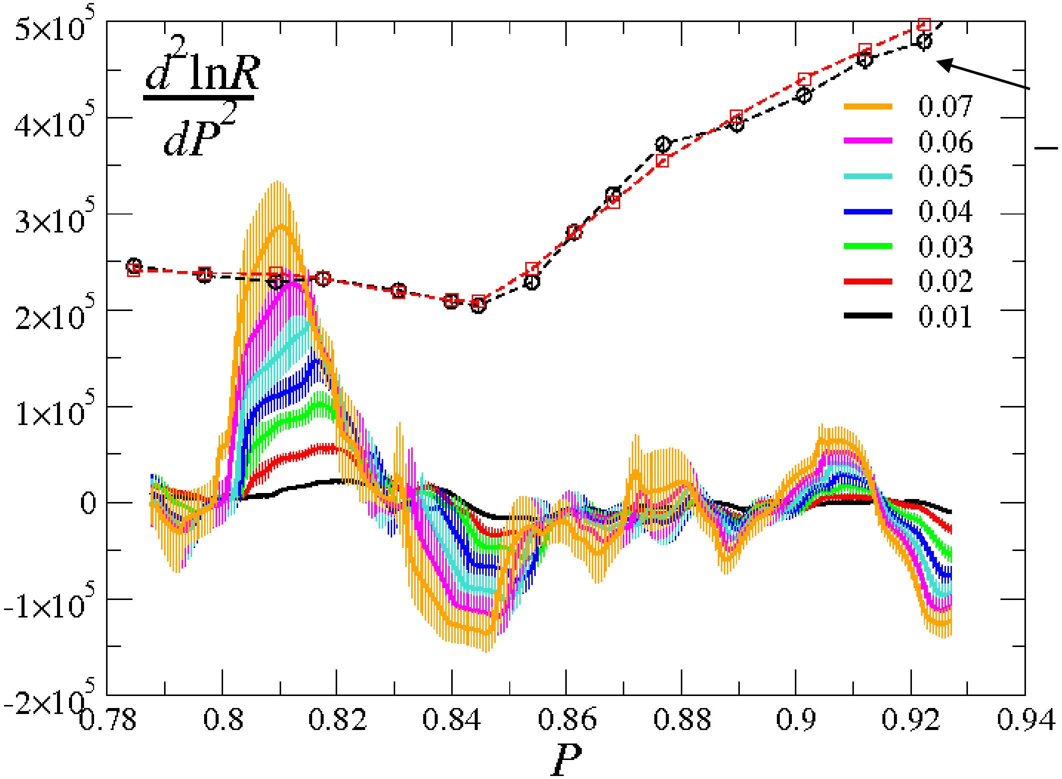
$$V_{\text{eff}}(P,\beta,h) = V_{\text{eff}}(P,\beta,0) - \ln R(P,h)$$

*N*f=2 p4-staggared,
$m_\pi/m_\rho \approx 0.7$
[data: Beilefeld-Swansea Collab.,PRD71,054508(2005)]

- det$M$: hopping parameter expansion.
- lnR increases as increasing h.
- The curvature increases with h.



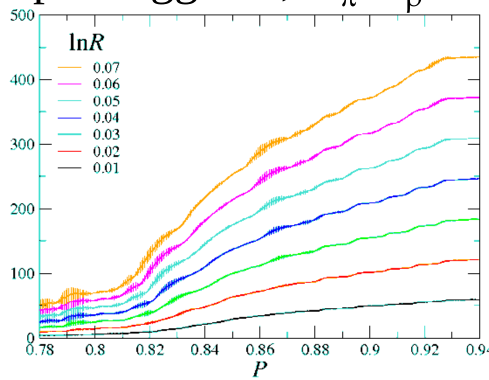
Curvature of the effective potential

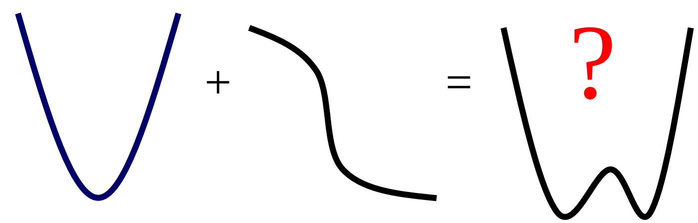


$-\frac{d^2 \ln W}{dP^2}$

at h=0

Nf=2 p4-staggared, $m_\pi/m_\rho \approx 0.7$





First order transition: $$\frac{d^2 V_{\text{eff}}(P,\beta,h)}{dP^2} = \frac{d^2 V_{\text{eff}}(P,\beta,0)}{dP^2} - \frac{d^2 \ln \overline{R}(P,h)}{dP^2} < 0$$

$$h = 2N_{\text{f}}(2\kappa_{\text{h}})^{N_t}$$

(Wilson quarks)

- First order transition for h > 0.06

Critical value: $h_c = 0.0614(69)$

N_f–dependence of the critical mass

$$h_c = 0.0614(69)$$

- Critical mass increases as N_f increases.

$$h = 2N_\mathrm{f}\left(2\kappa_\mathrm{h}\right)^{N_t} \Rightarrow \kappa_h^c = \frac{1}{2}\left(\frac{h_c}{2N_\mathrm{f}}\right)^{1/N_t}$$

  – When N_f is large, κ is small. Then, the hopping parameter (κ) expansion is good.

  – On the hand, when N_f is small, the κ-expansion is bad.

- In a quenched simulation with N_t=4, the first and second terms becomes comparable around κ=0.18.
- For N_f=10, N_t=4, $h_c = 0.0614(69)$ $\Rightarrow$ $\kappa_h^c \approx 0.118$
  – It may be applicable for N_f~10.

Curvature of the effective potential at finite μ

$$\frac{d^2 V_{\mathrm{eff}}}{dP^2}(P,h,\mu) = \frac{d^2 V_{\mathrm{eff}}}{dP^2}(P,0,0) - \frac{d^2 \ln \bar{R}}{dP^2}(P,h,\mu)$$

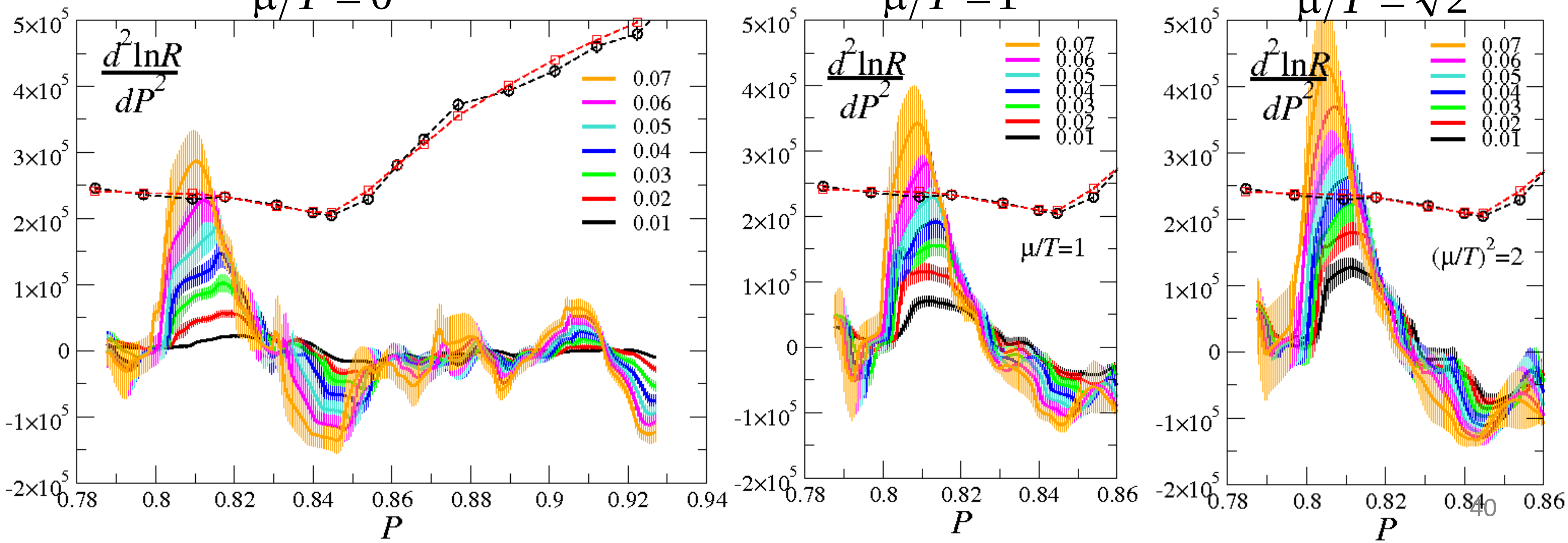
$$h = 2N_{\mathrm{f}}\left(2\kappa_{\mathrm{h}}\right)^{N_t}$$ for Wilson quarks

$$\ln R(P) = \ln \left\langle \left(\frac{\det M(m,\mu)}{\det M(m,0)} \right)^2 \left(\frac{\det M(h,\mu_h)}{\det M(0,0)} \right)^{N_{\mathrm{f}}} \right\rangle_{P:\mathrm{fixed}}$$

- Calculations of detM: Taylor expansion up to O(μ^6)
- Distribution function of the complex phase of detM: approximated by a Gaussian function

$$\mu_h / T = 0$$

$\mu/T = 0$ $\qquad$ $\mu/T = 1$ $\qquad$ $\mu/T = \sqrt{2}$



Critical line at finite density

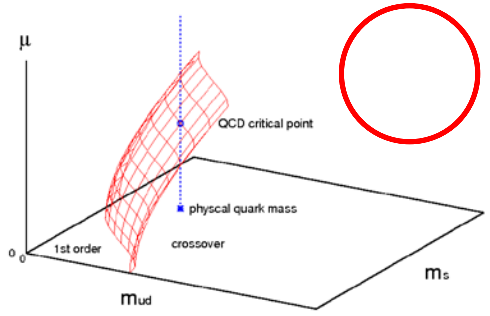
$$h = 2N_f(2\kappa_h)^{N_t}$$

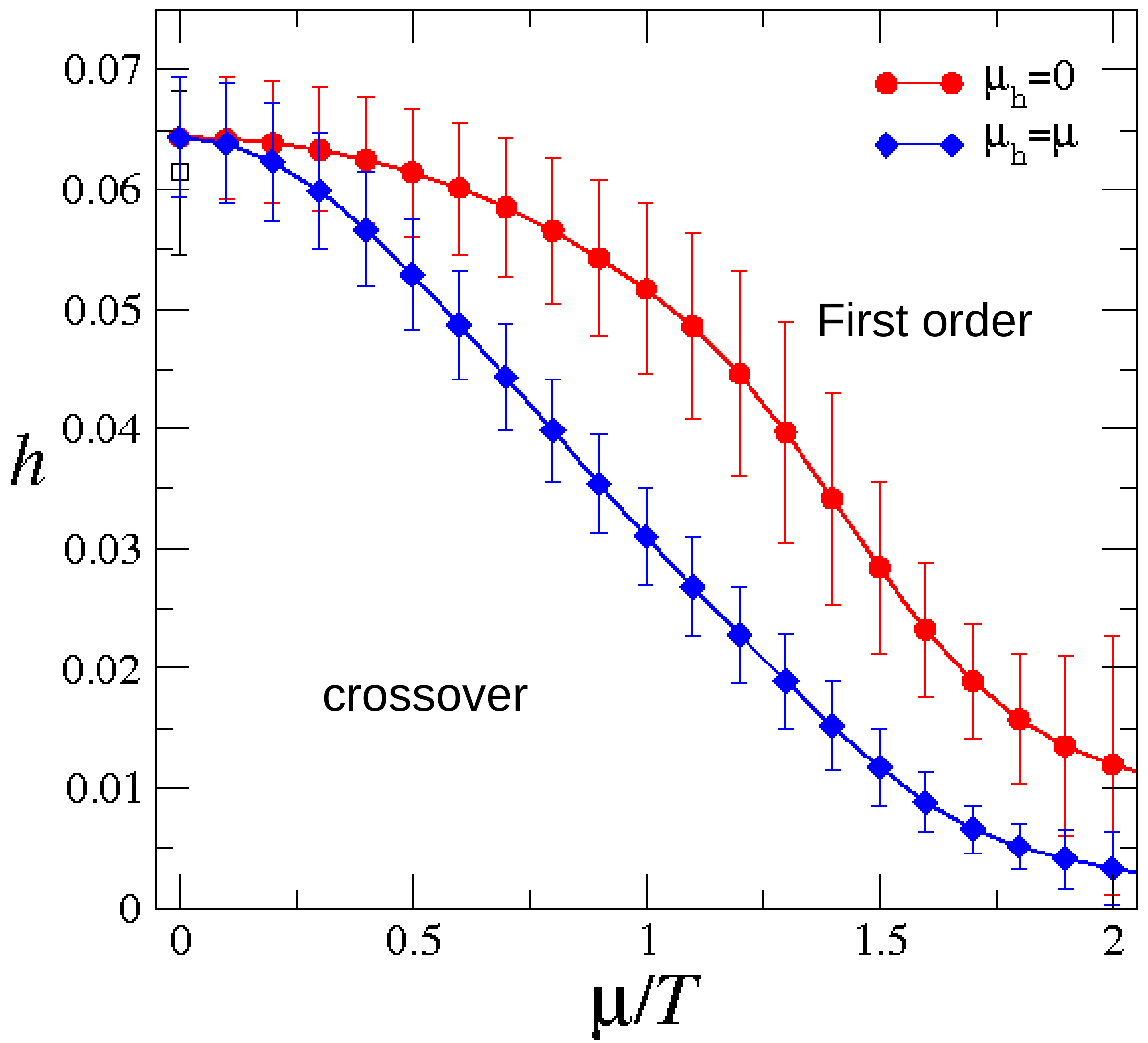
for Wilson quarks

$$h = N_f \big/ \left(4(2m_h)^{N_t}\right)$$

for staggered quarks

- Calculations of detM: Taylor expansion up to O(μ6)
- Distribution function of the complex phase of detM: approximated by a Gaussian function





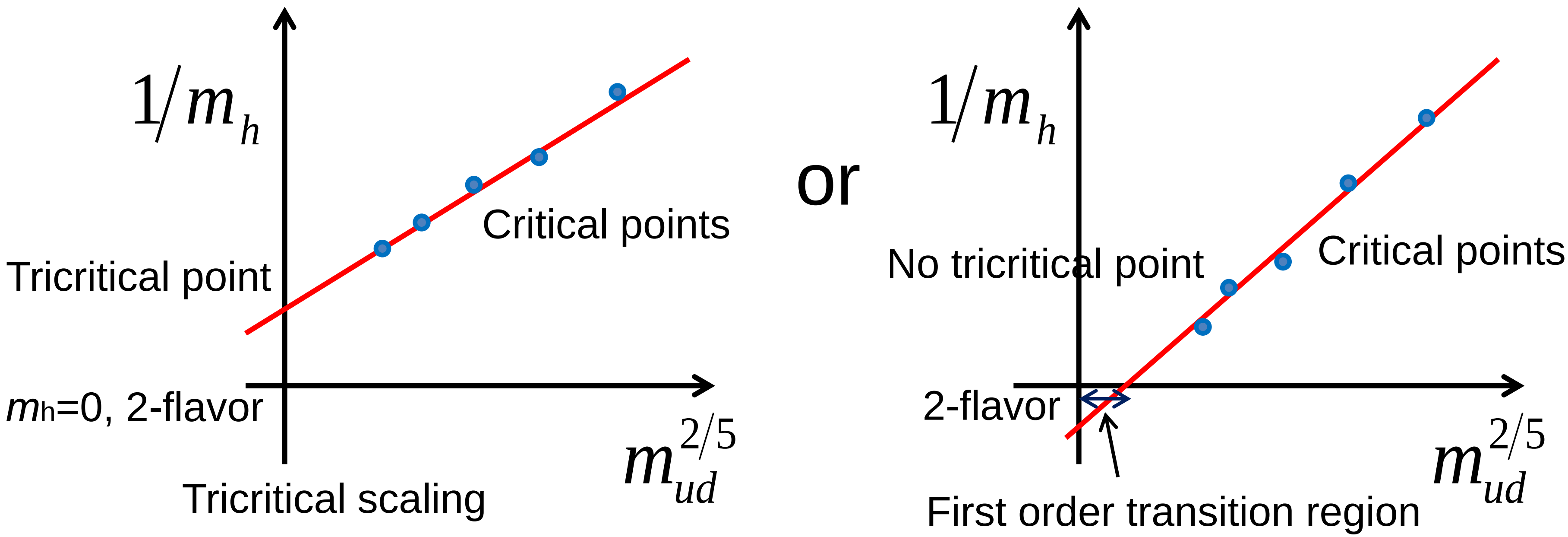
Phys. Rev. Lett. 110, 172001 (2013)

Phase structure of (2+many)-flavor QCD using Wilson quark action

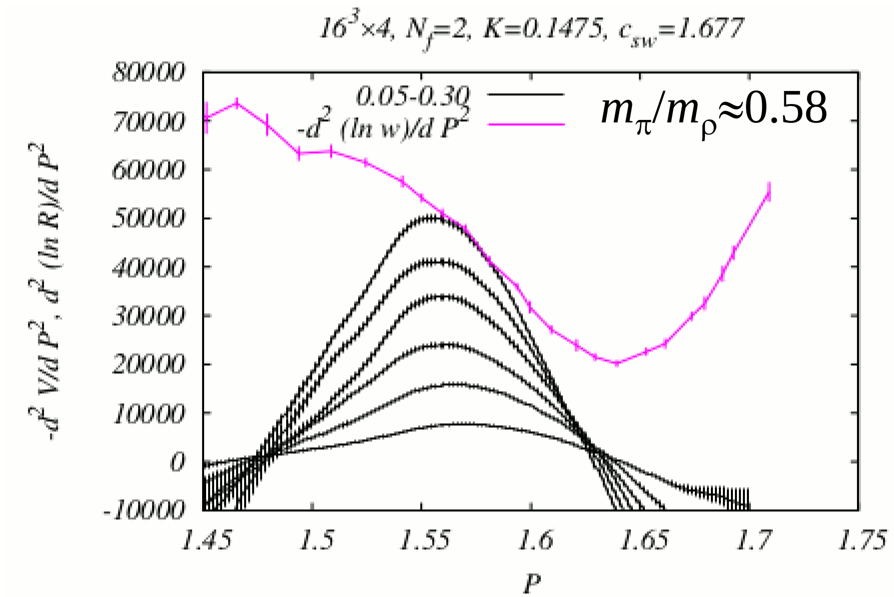
2-flavor QCD simulations + reweighting

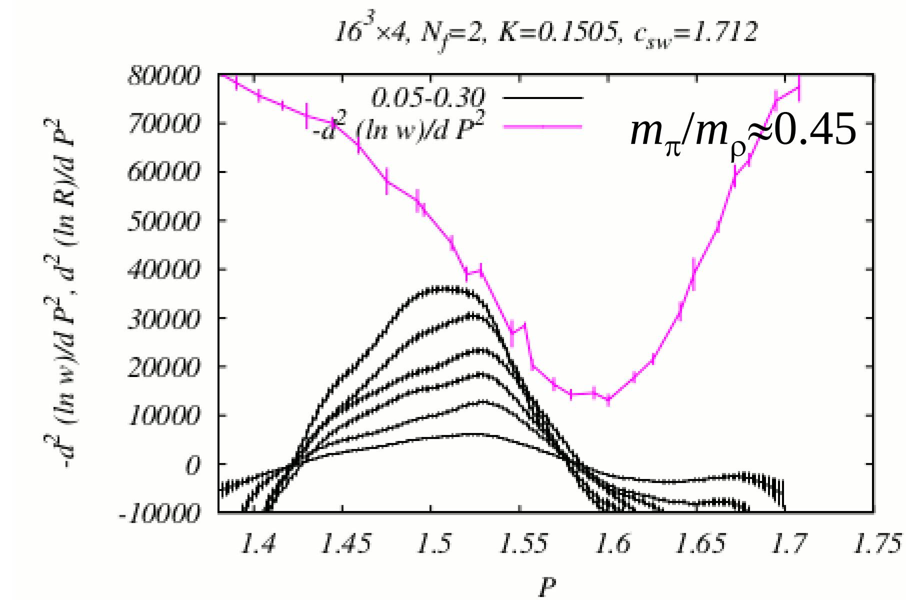
Light quark mass dependence of the critical line

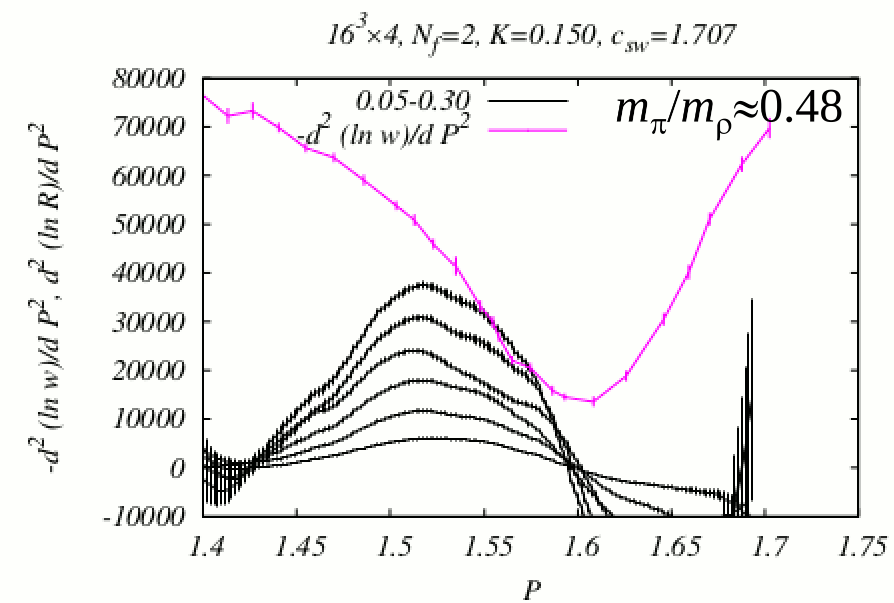
- Trictitical scaling behavior?
- Is there a first order transition region in 2-flavor QCD?



Light quark mass dependence (preliminary)







h=0.05, 0.10, 0.15, 0.20, 0.25, 0.30

- Critical point: light quark mass dependence is small in this mass region.
- In progress

Distribution function in the light quark region

WHOT-QCD Collaboration, in preparation,
(Nakagawa et al., arXiv:1111.2116)

- Perform phase quenched simulations
- Add the effect of the complex phase by the reweighting.
- Calculate the probability distribution function.

- Goal
  - The critical point
  - The equation of state

    Pressure, Energy density, Quark number density, Quark number susceptibility, Speed of sound, etc.

Probability distribution function by phase quenched simulation

- We perform phase quenched simulations with the weight:

$$\left|\det M(m,\mu)\right|^{N_\mathrm{f}} e^{-S_g}$$

$$W(P',F',\beta,m,\mu)=\int DU\,\delta(\hat{P}-P')\delta(\hat{F}-F')(\det M(m,\mu))^{N_\mathrm{f}}\,e^{-S_g}$$

$$=\int DU\,\delta(\hat{P}-P')\delta(\hat{F}-F')e^{i\theta}\left|\det M(m,\mu)\right|^{N_\mathrm{f}}\,e^{-S_g}$$

$$=\underbrace{\left\langle e^{i\theta}\right\rangle_{P',F'}}_{\text{expectation value with fixed } P,F}\times\underbrace{W_0(P',F',\beta,m,\mu)}_{\text{histogram}}$$

P: plaquette

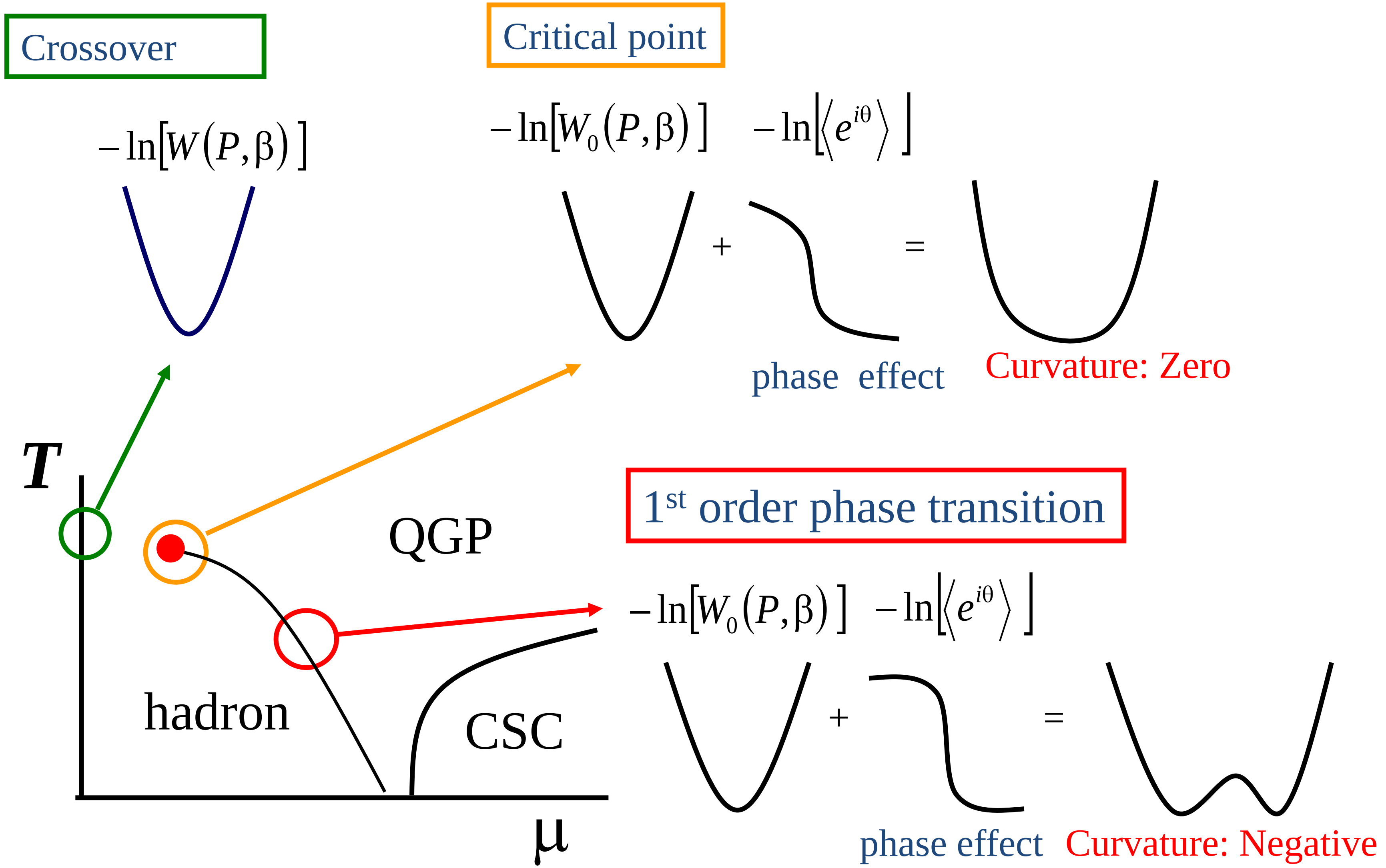
$$F(\mu)=\frac{N_\mathrm{f}}{N_\mathrm{site}}\ln\left|\frac{\det M(\mu)}{\det M(0)}\right| \qquad \theta\equiv N_\mathrm{f}\,\mathrm{Im}\ln\det M$$

Distribution function of the phase quenched.

$$W_0(P',F')=\int DU\,\delta(\hat{P}-P')\delta(\hat{F}-F')\left|\det M\right|^{N_\mathrm{f}}\,e^{6N_\mathrm{site}\beta\hat{P}}$$

μ-dependence of the effective potential

Curvature of the effective potential



Curvature of the effective potential

- If the distribution is Gaussian,

$$W_0(P,F) \approx \sqrt{\frac{6N_{\text{site}}}{2\pi\chi_P}} \exp\left[-\frac{6N_{\text{site}}}{2\chi_P}\left(P-\langle P\rangle\right)^2\right] \times \sqrt{\frac{N_{\text{site}}}{2\pi\chi_F}} \exp\left[-\frac{N_{\text{site}}}{2\chi_F}\left(F-\langle F\rangle\right)^2\right]$$

$$\chi_P = 6N_{\text{site}}\left\langle \left(P-\langle P\rangle\right)^2 \right\rangle \qquad \chi_F = N_{\text{site}}\left\langle \left(F-\langle F\rangle\right)^2 \right\rangle$$

$$\boxed{\frac{\partial^2(-\ln W_0)}{\partial P^2}\left(\langle P\rangle,\langle F\rangle\right) = \frac{6N_{\text{site}}}{\chi_P}} \qquad \boxed{\frac{\partial^2(-\ln W_0)}{\partial F^2}\left(\langle P\rangle,\langle F\rangle\right) \approx \frac{N_{\text{site}}}{\chi_F}}$$

at the peak of the distribution

Avoiding the sign problem

(SE, Phys.Rev.D77,014508(2008), WHOT-QCD, Phys.Rev.D82, 014508(2010))

θ: complex phase $\theta \equiv \mathrm{Im} \ln \det M$

- Sign problem: If $e^{i\theta}$ changes its sign,

$$\left\langle e^{i\theta} \right\rangle_{P,F \text{ fixed}} \ll \text{(statistical error)}$$

- Cumulant expansion

$<..>_{P,F}$: expectation values fixed F and P.

$$\left\langle e^{i\theta} \right\rangle_{P,F} = \exp\left[\cancel{i \langle \theta \rangle_C}_{\to 0} - \frac{1}{2} \langle \theta^2 \rangle_C - \cancel{\frac{i}{3!} \langle \theta^3 \rangle_C}_{\to 0} + \frac{1}{4!} \langle \theta^4 \rangle_C + \cancel{\cdots} \right]$$

cumulants

$$\langle \theta \rangle_C = \langle \theta \rangle_{P,F}, \quad \langle \theta^2 \rangle_C = \langle \theta^2 \rangle_{P,F} - \langle \theta \rangle_{P,F}^2, \quad \langle \theta^3 \rangle_C = \langle \theta^3 \rangle_{P,F} - 3 \langle \theta^2 \rangle_{P,F} \langle \theta \rangle_{P,F} + 2 \langle \theta \rangle_{P,F}^3, \quad \langle \theta^4 \rangle_C = \cdots$$

– Odd terms vanish from a symmetry under $\mu \leftrightarrow -\mu$ ($\theta \leftrightarrow -\theta$)

Source of the complex phase

If the cumulant expansion converges, No sign problem.

Convergence in the large volume (V) limit

The cumulant expansion is good in the following situations.

- If the phase is given by $\theta = \sum_x \theta_x$
  - No correlation between θ_x.

$$\left\langle e^{i\theta} \right\rangle_{P,F} = \left\langle e^{i\sum_x \theta_x} \right\rangle_{P,F} \approx \prod_x \left\langle e^{i\theta_x} \right\rangle_{P,F} = \exp\left[\sum_x \sum_n \frac{i^n}{n!} \left\langle \theta_x^n \right\rangle_C \right]$$

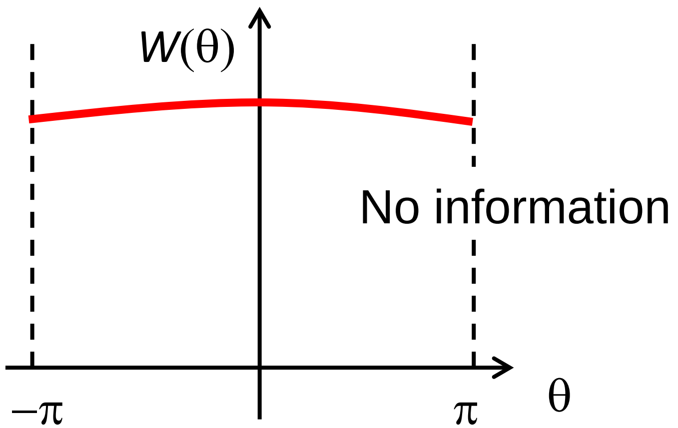
$$\left\langle e^{i\theta} \right\rangle_{P,F} = \exp\left[\sum_n \frac{i^n}{n!} \left\langle \theta^n \right\rangle_C \right] \Rightarrow \left\langle \theta^n \right\rangle_C \approx \sum_x \left\langle \theta_x^n \right\rangle_C \sim O(V)$$

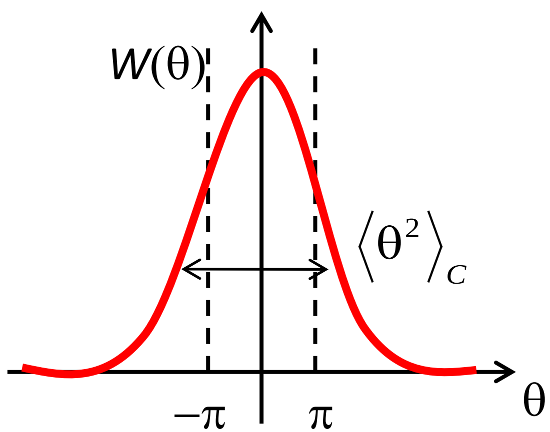
  - Ratios of cumulants do not change in the large V limit.
  - Convergence property is independent of V, although the phase fluctuation becomes larger as V increases.
  - The application range of μ can be measured on a small lattice.

Complex phase distribution

- We should not define the complex phase in the range from -π to π.
- When the distribution of θ is perfectly Gaussian, the average of the complex phase is give by the second order (variance),

$$\left\langle e^{i\theta}\right\rangle_{P,F} = \exp\left[-\frac{1}{2}\left\langle \theta^2\right\rangle_C\right]$$





- Gaussian distribution → The cumulant expansion is good.
- We define the phase

$$\theta(\mu) = N_f \,\mathrm{Im}\left(\ln\frac{\det M(\mu)}{\det M(0)}\right) = N_f \int_0^{\mu/T} \mathrm{Im}\left[\frac{\partial \ln\det M}{\partial(\mu/T)}\right]_{\bar{\mu}} d\left(\frac{\bar{\mu}}{T}\right)$$

  – The range of θ is from -∞ to ∞.

$$\frac{\partial \ln\det M}{\partial(\mu/T)} = Tr\left[M^{-1}\frac{\partial \det M}{\partial(\mu/T)}\right] = \sum_i \left[M^{-1}\frac{\partial \det M}{\partial(\mu/T)}\right]_{ii}$$

Integral method for the calculation of the quark determinant

$$\frac{\partial \ln \det M}{\partial(\mu/T)}$$

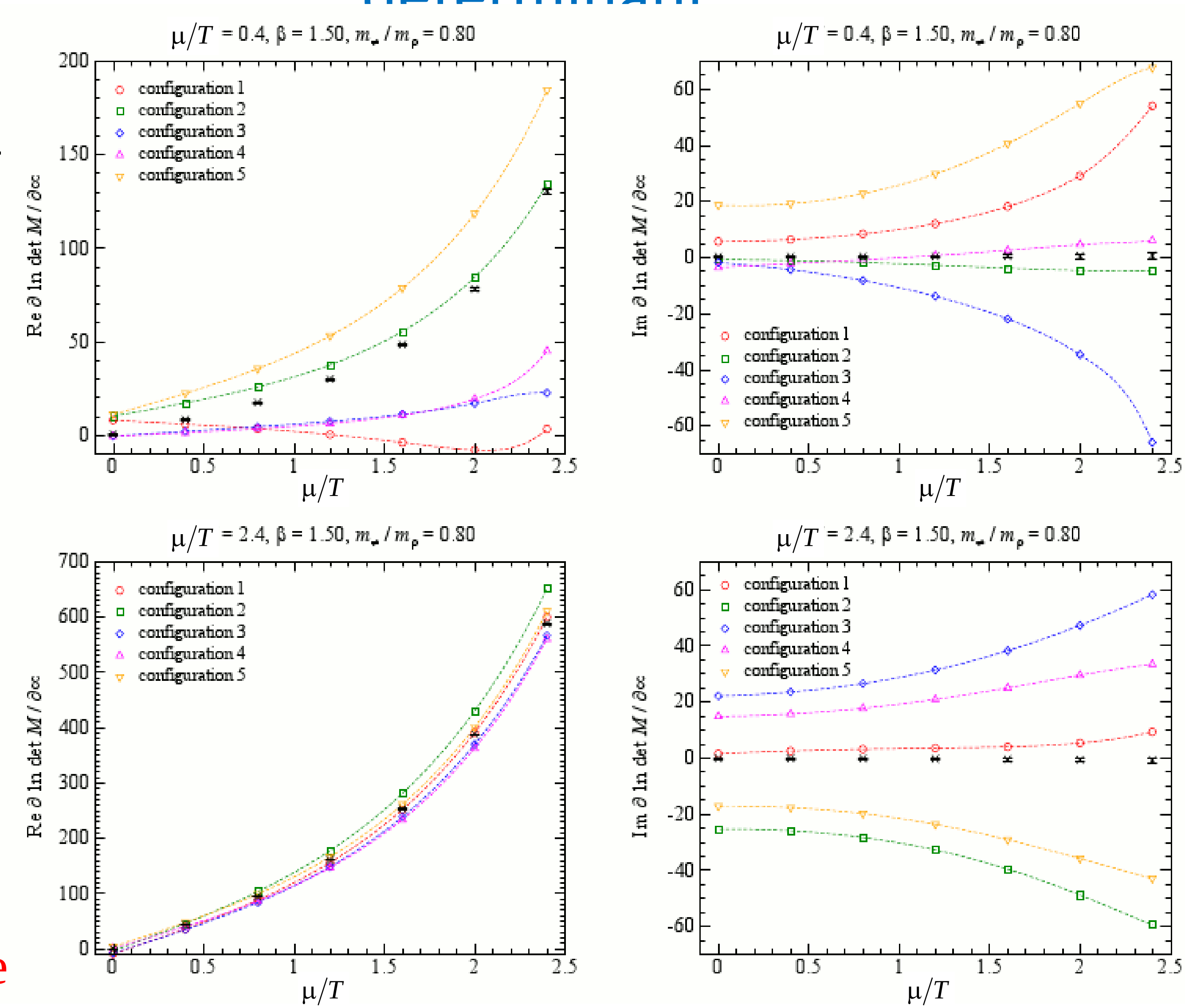
$8^3 \times 4$ lattice

$\beta = 1.50$

$m_\pi / m_\rho = 0.8$

2-flavor QCD
Iwasaki gauge
+ clover Wison
quark action

Random noise method is used.



$\mu/T = 0.4$

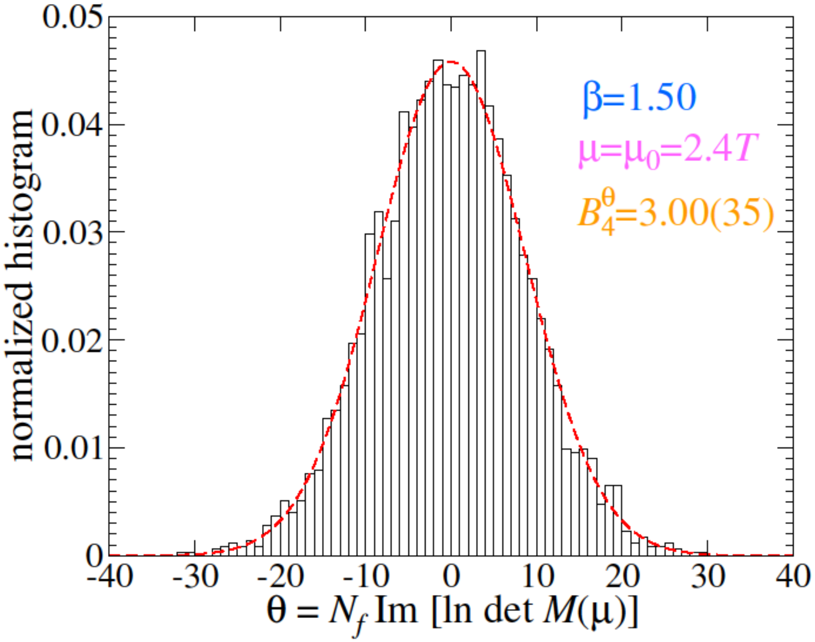
$\mu/T = 2.4$

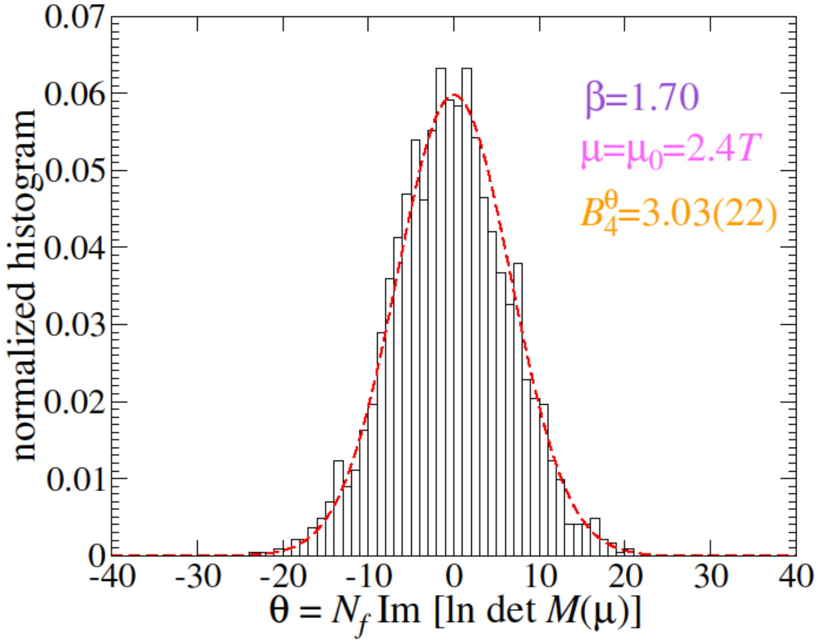
Real part

Imaginary

part

Distribution of the complex phase





- Well approximated by a Gaussian function.
- Convergence of the cumulant expansion: good.

$$\left\langle e^{i\theta} \right\rangle_{P,F} \approx \exp\left[-\frac{1}{2} \left\langle \theta^2 \right\rangle \right] \qquad \frac{1}{2} \left\langle \theta^2 \right\rangle_{\langle P \rangle, \langle F \rangle} \approx \frac{1}{2} \left\langle \theta^2 \right\rangle_{\beta_0, \mu_0}$$

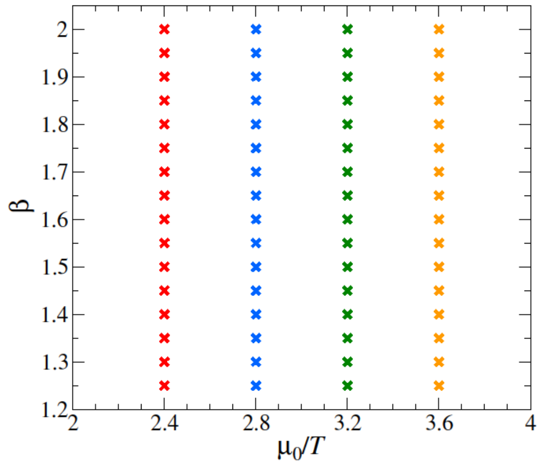
at the peak of W_0 in each simulation

Simulations

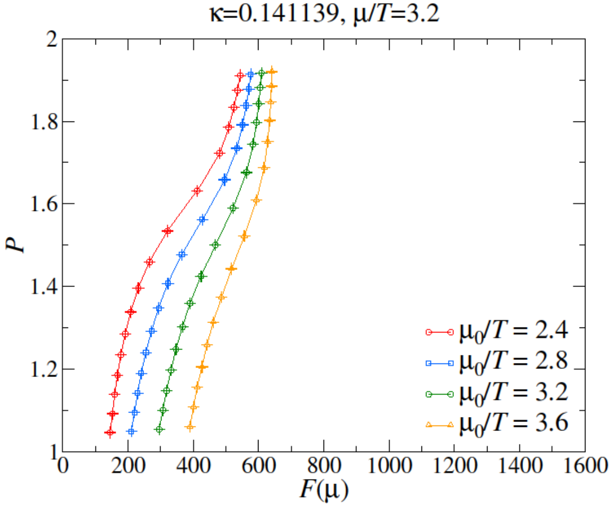
2-flavor QCD Iwasaki gauge
+ clover Wilson quark action
Random noise method is used.

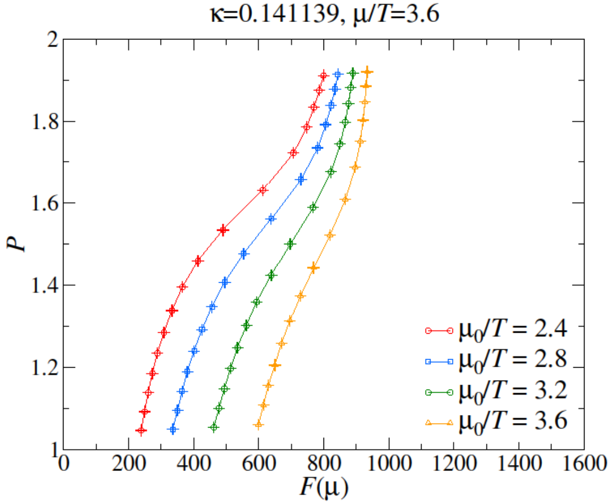
$8^3 \times 4$ lattice $\quad m_\pi / m_\rho \approx 0.8$

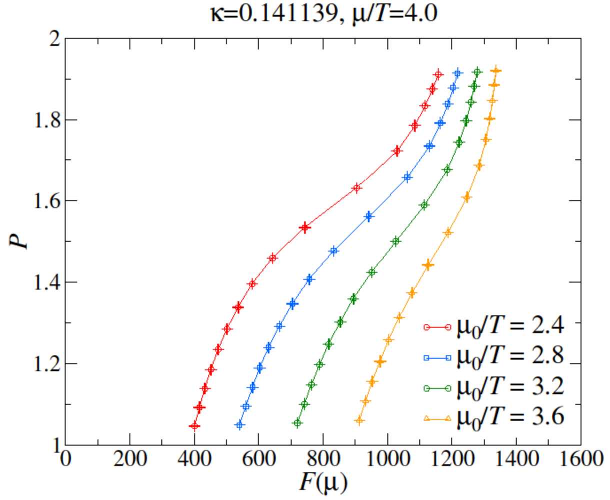
- Simulation point in the (β, μ_0/T)



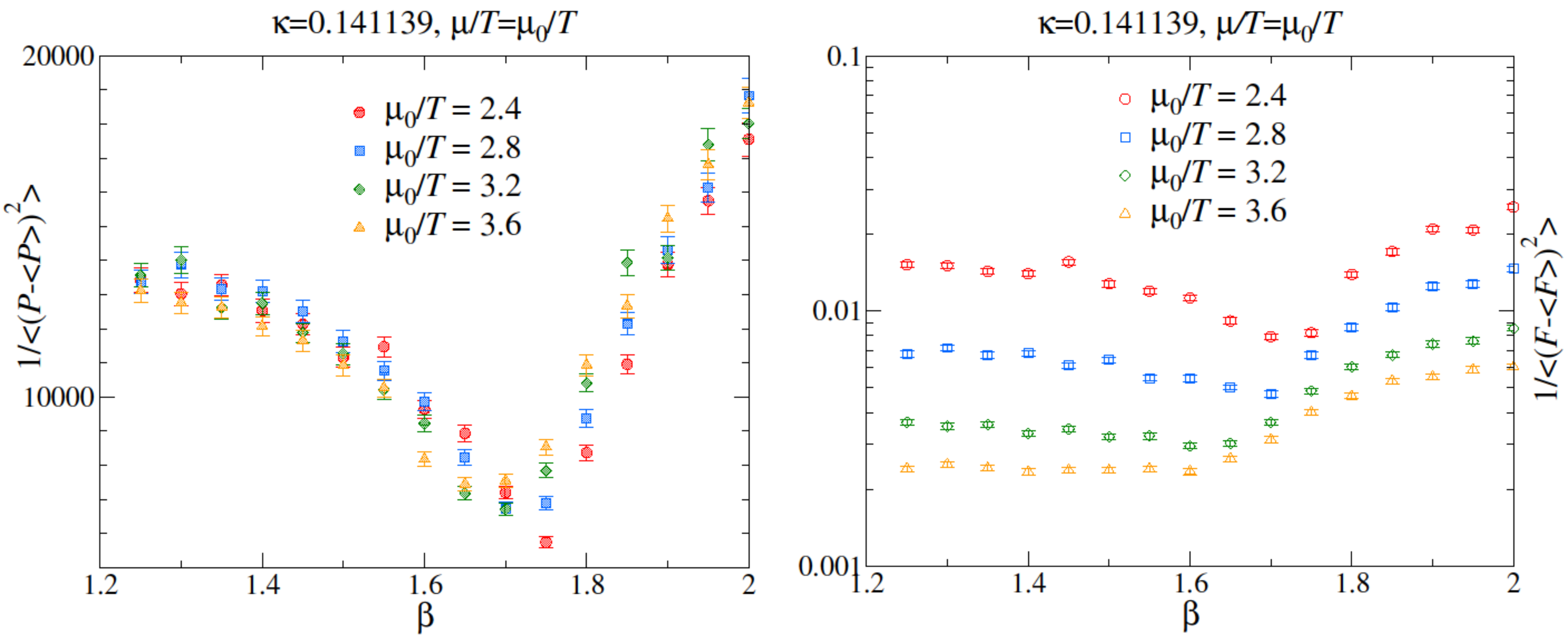
- Peak of $W_0(P,F)$ for each μ





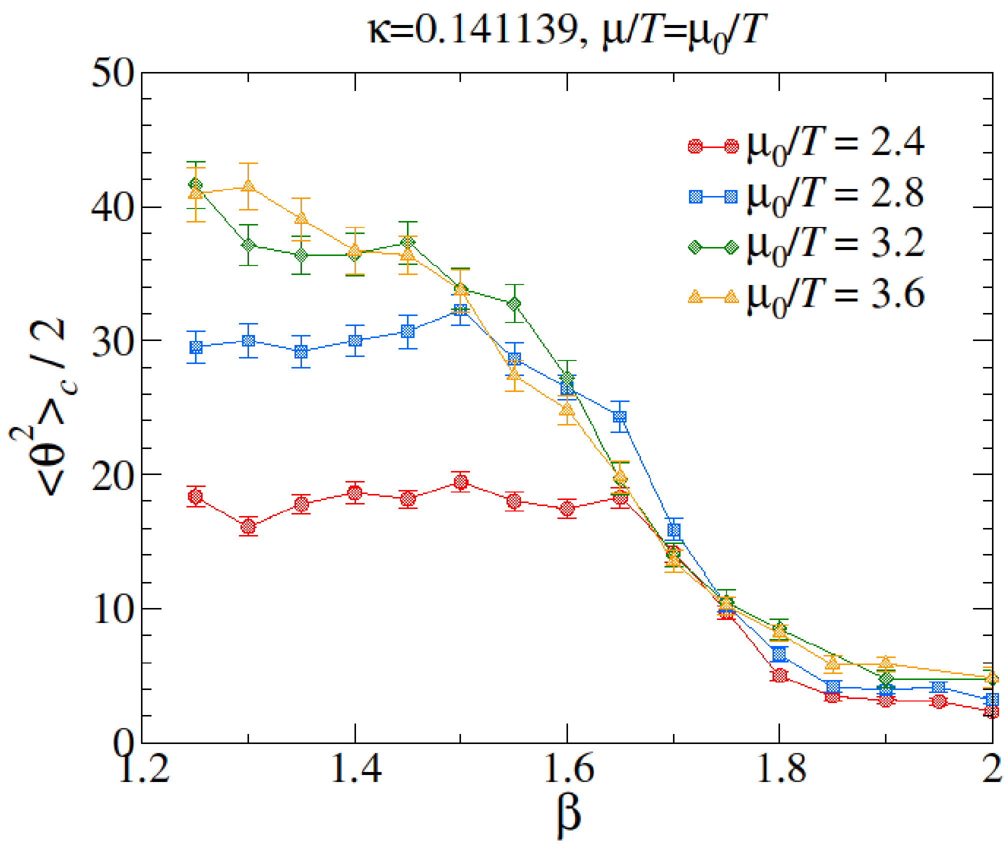


Curvature of the effective potential $-\ln W_0$



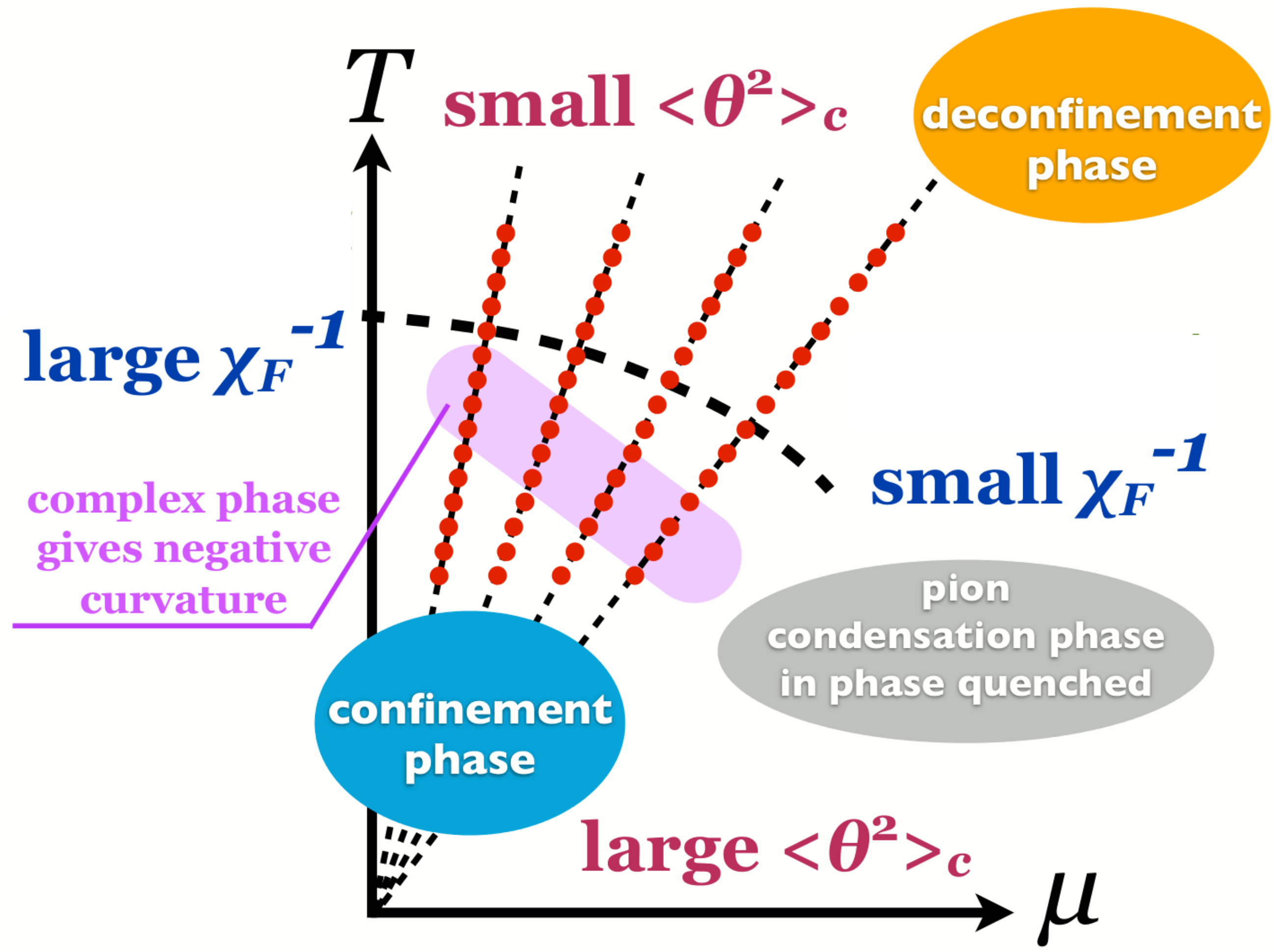
- The curvature for F decreases as μ increases.

Effect from the complex phase



- Rapidly changes around the pseudo-critical point.

Critical point at finite μ



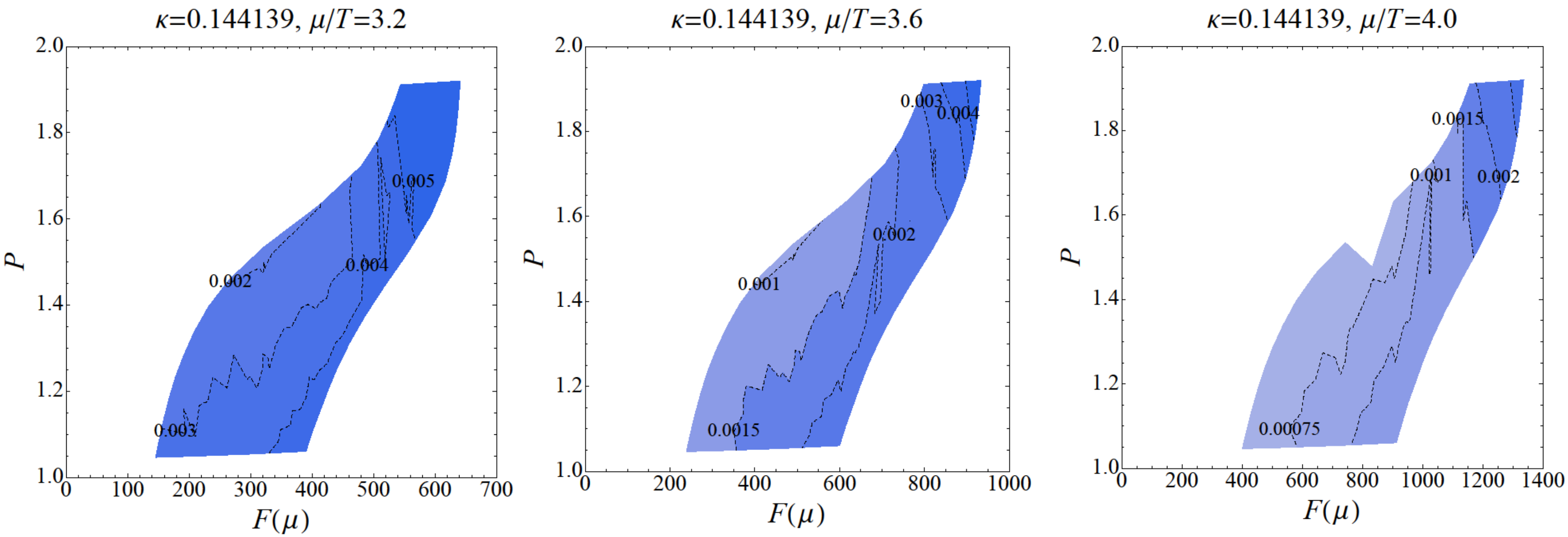
- zero curvature: expected at a large μ.

Curvature of the effective potential

- Without the complex phase effect

$$\boxed{\frac{\partial^2(-\ln W_0)}{\partial F^2}(\langle P\rangle,\langle F\rangle) \approx \frac{N_{\text{site}}}{\chi_F}}$$

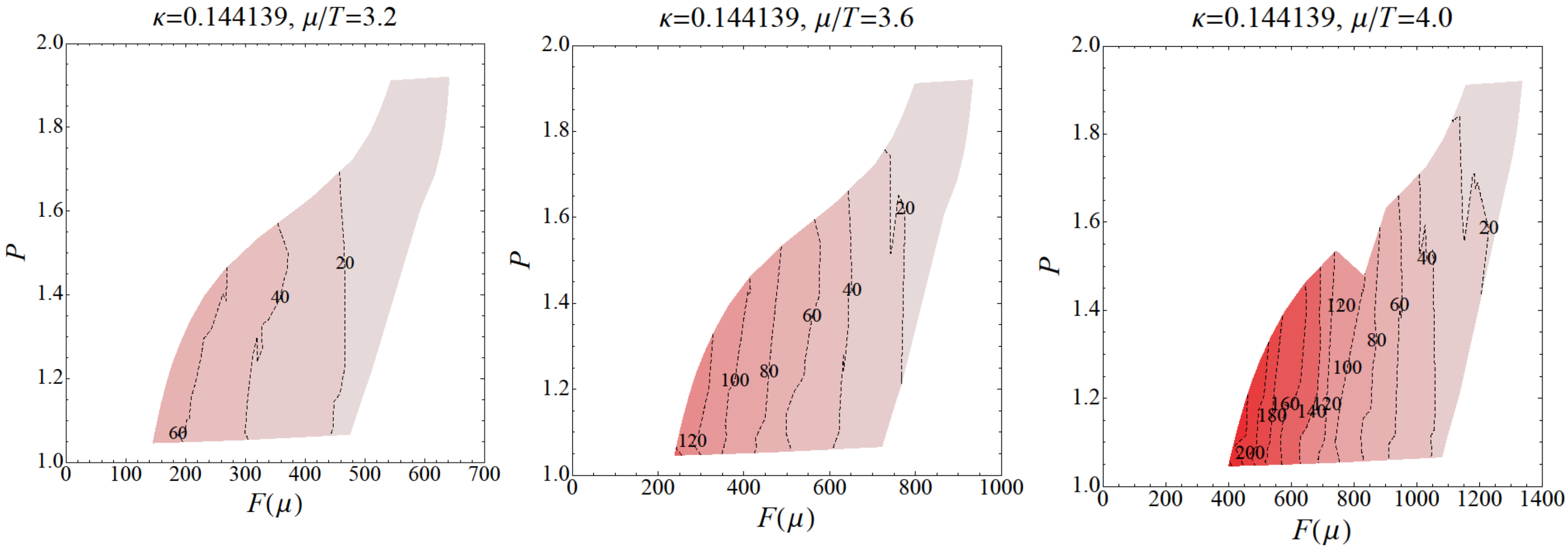
$$\chi_F = N_{\text{site}}\left\langle \left(F - \langle F\rangle\right)^2 \right\rangle$$



Phase average

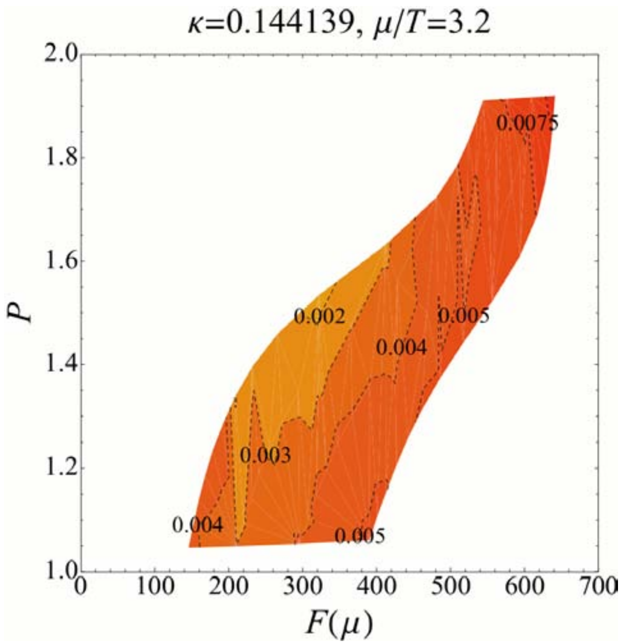
- 2nd order cumulant

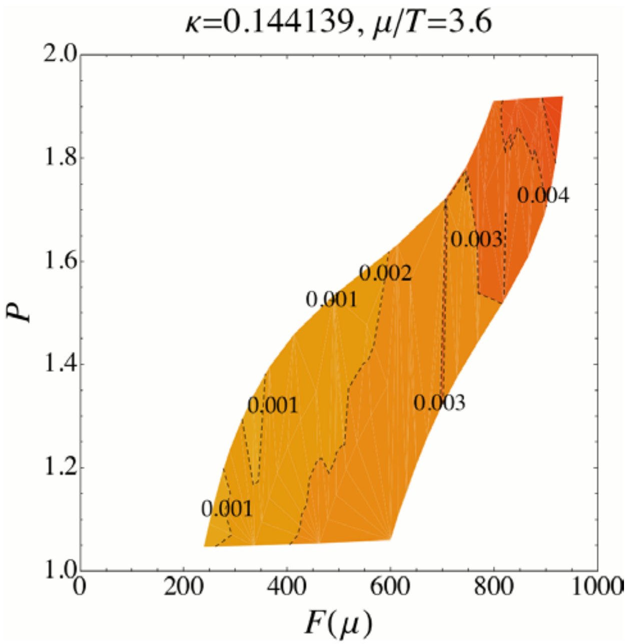
$$\ln\left\langle e^{i\theta}\right\rangle_{P,F} \approx -\frac{1}{2}\left\langle \theta^2\right\rangle_{P,F} \qquad \frac{1}{2}\left\langle \theta^2\right\rangle_{\langle P\rangle,\langle F\rangle} \approx \frac{1}{2}\left\langle \theta^2\right\rangle_{\beta_0,\mu_0}$$

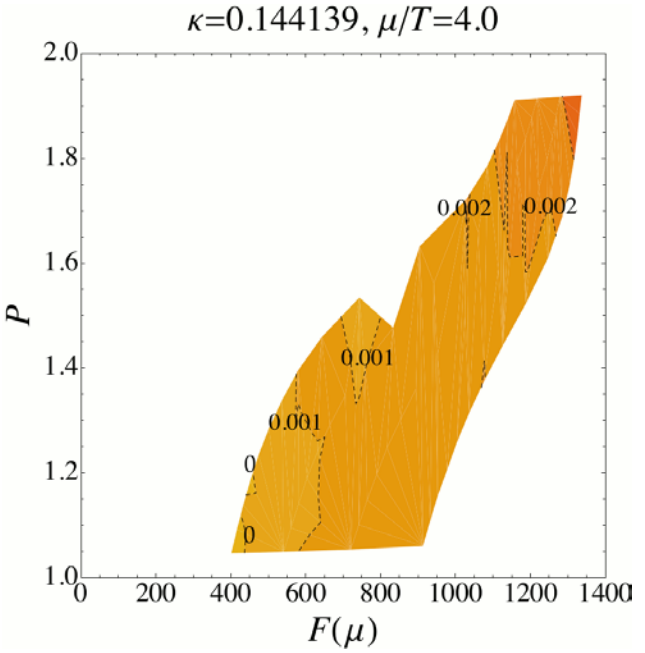


Curvature of the effective potential

- The effect of the phase incruded.





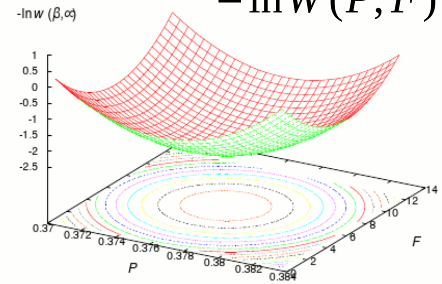


zero curvature
Critical point

Peak position of $W(P,F)$

- The slopes are zero at the peak of $W(P,F)$. $\quad \dfrac{\partial \ln W}{\partial P} = 0, \quad \dfrac{\partial \ln W}{\partial F} = 0$

$-\ln W(P,F)$



$$\left(R(P,F,\mu,\mu_0) = \frac{W_0(P,F,\beta,\mu)}{W_0(P,F,\beta,\mu_0)} \right)$$

$$\frac{\partial \ln W}{\partial P}(P,F,\beta,\mu) = \frac{\partial \ln W_0}{\partial P}(P,F,\beta,\mu) + \frac{\partial \ln \langle e^{i\theta} \rangle_{P,F}}{\partial P}$$

$$= \frac{\partial \ln W_0}{\partial P}(P,F,\beta_0,\mu_0) + \underbrace{6N_{site}(\beta-\beta_0) + \frac{\partial \ln R}{\partial P}(P,F,\mu,\mu_0) + \frac{\partial \ln \langle e^{i\theta} \rangle_{P,F}}{\partial P}}_{=0}$$

$$\frac{\partial \ln W}{\partial F}(P,F,\beta,\mu) = \frac{\partial \ln W_0}{\partial F}(P,F,\beta,\mu) + \frac{\partial \ln \langle e^{i\theta} \rangle_{P,F}}{\partial F}$$

$$= \frac{\partial \ln W_0}{\partial F}(P,F,\beta_0,\mu_0) + \underbrace{\frac{\partial \ln R}{\partial F}(P,F,\mu,\mu_0) + \frac{\partial \ln \langle e^{i\theta} \rangle_{P,F}}{\partial F}}_{=0}$$

If these terms are canceled,

$$\boxed{W(P,F,\beta,\mu) \approx W_0(P,F,\beta_0,\mu_0) \times (\text{const.})}$$

- $W(\beta, \mu)$ can be computed by simulations around (β_0,μ_0).

Phase quenched simulation

$$\boxed{W(P,F,\beta,m,\mu)=\underline{\left\langle e^{i\theta}\right\rangle_{P,F}}\times W_0(P,F,\beta,m,\mu)}$$

$$\det M(K,-\mu)=[\det M(K,\mu)]^*,\qquad \left|\det M(K,\mu)\right|^2=\det M(K,\mu)\det M(K,-\mu)$$

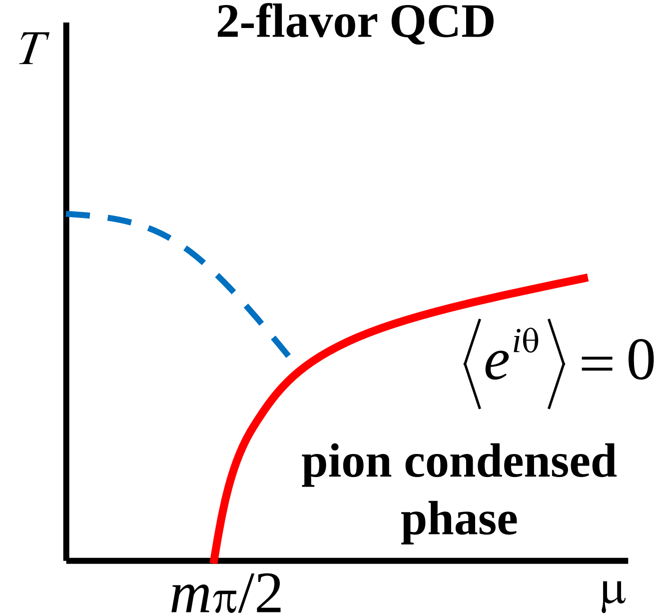
- When $\mu_u=-\mu_d$, pion condensation occurs.
- $\left\langle e^{i\theta}\right\rangle=0$ is suggested in the pion condensed phase by phenomenological studies. [Han-Stephanov '08, Sakai et al. '10]

➡ No overlap between $W(\mu)$ and $W_0(\mu)$.

- Near the phase boundary,
  – large fluctuations in θ: expected.

$$\left\langle e^{i\theta}\right\rangle_{P,F}\to 0\quad\left(\ln\left\langle e^{i\theta}\right\rangle_{P,F}\to-\infty\right)$$

Phase structure of the phase quenched 2-flavor QCD



Summary

- We discussed the QCD phase transition in the heavy quark region.

  WHOT-QCD(H. Saito, S. Ejiri, S. Aoki, K. Kanaya, Y. Nakagawa, H. Ohno, K. Okuno, and T. Umeda), arXiv:1309.2445

  S. Ejiri, Euro. Phys. J. A 49, 86 (2013) [arXiv:1306.0295]

  – The critical surface in the heavy quark region of (2+1)-flavor QCD is computed.
- We investigated the phase structure of (2+Nf)-flavor QCD.

  S. Ejiri & N. Yamada, Phys. Rev. Lett. 110, 172001 (2013) [arXiv:1212.5899]

  – This model is interesting for the feasibility study of the electroweak baryogenesis in the technicolor scenario.

  – An appearance of a first order phase transition at finite temperature is required for the baryogenesis.

  – Applying the reweighting method, we determine the critical mass of heavy flavors terminating the first order region.
    - The critical mass becomes larger with N_f.
    - The first order region becomes wider as increasing μ.
- This may be a good test for the determination of boundary of the first order region in (2+1)-flavor QCD at finite density.