Lattice energy-momentum tensor from the Yang-Mills gradient flow

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- H.S., work in progress
Lattice field theory and the energy-momentum tensor (EMT)

- Lattice field theory

best successful non-perturbative formulation of QFT; keeps internal gauge symmetries exactly

... but quite incompatible with spacetime symmetries (translation, rotation, SUSY, conformal, ...)

Ward–Takahashi (WT) relation associated with translational invariance ($T_{\mu\nu}(x)$: energy-momentum tensor (EMT))

$$\langle \partial_\mu T_{\mu\nu}(x)O(y)O(z)\cdots \rangle = -\delta(x-y) \langle \partial_\nu O(y)O(z)\cdots \rangle + \cdots$$

can we construct lattice EMT which reproduces these relations in $a \to 0$?

if this is possible, the application will be vast (thermodynamics, viscosities, conformal field theory, dilaton physics, vacuum energy, ...)

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Naive construction will not work . . .

- naive EMT for the pure Yang–Mills theory

\[
T_{\mu\nu}(x) = \frac{1}{g_0^2} \left[ F^a_{\mu\rho}(x) F^a_{\nu\sigma}(x) - \frac{1}{4} \delta_{\mu\nu} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x) \right]
\]

- a correct WT relation: \( \langle \partial_\mu T_{\mu 1}(x) T_{01}(0) \rangle = C \partial_0 \delta(x) \)
- Monte Carlo computation of LHS \((x = (x_0, 0, 0, 0))\)

- extremely noisy . . .
- (although consistent with 0) it appears diverging as \(a \to 0\)
- after all, there is no guarantee that the naive expression is conserved for \(a \to 0\), since lattice regularization breaks translational invariance
Possible approaches

- Invent somehow a lattice formulation that is invariant under the desired symmetry (in the present case, translation) as the lattice chiral symmetry on the basis of the Ginsparg–Wilson relation.

- This is certainly ideal, but seems formidable for spacetime symmetries . . . (eventually, SLAC derivative?)

- What the general argument says is that a linear combination of dim. 4 operators being consistent with lattice symmetry

\[ T_{\mu\nu}(x) = C_1 \left( \sum_{\rho} F_{\mu\rho}^a F_{\nu\rho}^a - \frac{1}{4} \delta_{\mu\nu} \sum_{\rho\sigma} F_{\rho\sigma}^a F_{\rho\sigma}^a \right) + C_2 \delta_{\mu\nu} \sum_{\rho\sigma} F_{\rho\sigma}^a F_{\rho\sigma}^a + C_3 \delta_{\mu\nu} \sum_{\rho} F_{\mu\rho}^a F_{\nu\rho}^a \]

is conserved in \( a \to 0 \); we may determine ratios of these coefficients by the conservation law (Caracciolo et al. (1989))

- Overall normalization should be fixed separately (expectation value in a one-particle state? current algebra?)

- No one yet studied whether this construction generates correct translations on composite operators!

- Approach on the basis of SUSY algebra and Ferrara–Zumino supermultiplet (H.S. (2012))
Our approach

- Use a UV finite quantity that can be related with EMT in a translationally invariant regularization
- Any regularization (including lattice) will produce the same number for such a UV finite quantity
- To define this UV finite quantity, we employ the so-called Yang–Mills gradient flow
- Yang–Mills gradient flow (a diffusion equation wrt a fictitious time $t \in \mathbb{R}$)

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu \mu}(t, x) = \Delta B_\mu(t, x) + \cdots, \quad B_\mu(t = 0, x) = A_\mu(x),$$

where $G_{\mu \nu}$ is the field strength of the flowed gauge potential:

$$G_{\mu \nu}(t, x) = \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) + [B_\mu(t, x), B_\nu(t, x)], \quad D_\mu = \partial_\mu + [B_\mu, \cdot]$$

- Note: the mass dimension of $t$ is $-2$
Yang–Mills gradient flow or the Wilson flow

- **Yang–Mills gradient flow (continuum theory)**

  \[ \partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) = \Delta B_\mu(t, x) + \cdots, \quad B_\mu(t=0, x) = A_\mu(x) \]

- **Wilson flow (lattice theory)**

  \[ \partial_t V(t, x, \mu)V(t, x, \mu)^{-1} = -g_0^2 \partial S_{\text{Wilson}}, \quad V(t=0, x, \mu) = U(x, \mu) \]

- **Applications (Lüscher):**
  - definition of the topological charge
  - scale setting (just like the Sommer scale \( r_0 \))

  \[ t^2 \langle E(t, x) \rangle \bigg|_{t=t_0} = 0.3, \quad \text{and set (for instance) } \sqrt{8t_0} = 0.5 \text{ fm} \]

  where

  \[ E(t, x) \equiv \frac{1}{4} G_{\mu\nu}^a(t, x) G_{\mu\nu}^a(t, x) \]

  - define UV finite quantities ← Our usage here
  - computation of the chiral condensate
Y Yang–Mills gradient flow

\[ \partial_t B_{\mu}(t, x) = D_\nu G_{\nu \mu}(t, x) + \alpha_0 D_\mu \partial_\nu B_\nu(t, x), \quad B_{\mu}(t = 0, x) = A_{\mu}(x), \]

where the second term in RHS was introduced to suppress the gauge mode; it can be seen that gauge invariant quantities are independent of \( \alpha_0 \). This can be solved formally as

\[ B_{\mu}(t, x) = \int d^D y \left[ K_t(x - y)_{\mu \nu} A_\nu(y) + \int_0^t ds K_{t-s}(x - y)_{\mu \nu} R_\nu(s, y) \right], \]

where \( K \) is the heat kernel and \( R \) is non-linear terms

\[ K_t(z)_{\mu \nu} = \int_p e^{ipz} \left[ (\delta_{\mu \nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 tp^2} \right] \]

\[ R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]] \]

Pictorially (cross: \( A_\mu \); open circle: flow vertex \( R \)),

\[ \text{Pictorial representation:} \]

\[ \text{Diagram 1:} \quad \text{Diagram 2:} \quad \text{Diagram 3:} \quad \text{Diagram 4:} \]

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Lattice energy-momentum tensor...
Perturbative expansion of the gradient flow

- quantum correlation function of the flowed gauge field

\[ \langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle, \]

is obtained by taking the expectation value of the initial value \( A_\mu(x) \). For example, the contraction of two \( A_\mu \)'s

\[ \langle \quad \otimes \quad \otimes \quad \rangle \quad = \quad \overrightarrow{\cdots} \overleftarrow{\cdots} \]

produces the propagator of the flowed field

\[ \delta^{ab} g_0^2 \frac{1}{(p^2)^2} \left[ (\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-(t+s)p^2} + \frac{1}{\lambda_0} p_\mu p_\nu e^{-\lambda_0 (t+s)p^2} \right], \]

(where \( t \) and \( s \) are flow times at the end points; \( \lambda_0 \) is the conventional gauge parameter). Similarly, for

considering the contraction with the usual Yang–Mills vertex (the full circle)
Gauge invariance of the gradient flow

- Under the infinitesimal gauge transformation,
  \[ B_\mu(t, x) \rightarrow B_\mu(t, x) + D_\mu \omega(t, x), \]
  the flow equation
  \[ \partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) + \alpha_0 D_\mu \partial_\nu B_\nu(t, x) \]
  changes to
  \[ \partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) + \alpha_0 D_\mu \partial_\nu B_\nu(t, x) - D_\mu (\partial_t - \alpha_0 D_\nu \partial_\nu) \omega(t, x) \]
- Therefore, by choosing \( \omega(t, x) \) as the solution of
  \[ (\partial_t - \alpha_0 D_\nu \partial_\nu) \omega(t, x) = -\delta \alpha_0 \partial_\nu B_\nu(t, x), \quad \omega(t = 0, x) = 0, \]
  \( \alpha_0 \) can be changed as
  \[ \alpha_0 \rightarrow \alpha_0 + \delta \alpha_0 \]
  That is, \( B_\mu(t, x) \)'s corresponding to different \( \alpha_0 \)'s are related by a gauge transformation
- Also, by choosing \( \omega(t, x) \) as the solution of
  \[ (\partial_t - \alpha_0 D_\nu \partial_\nu) \omega(t, x) = 0, \quad \omega(t = 0, x) = \omega(x), \]
  the \( D \) dimensional gauge transformation \( \omega(x) \) can be extended to a \( D + 1 \) dimensional gauge transformation \( \omega(t, x) \) that leaves the flow equation unchanged
Correlation function of the flowed gauge field

\[ \langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle, \quad t_1 > 0, \ldots, t_n > 0, \]

when expressed in terms of renormalized parameters, is UV finite without the wave function renormalization

Tree-level two-point function

\[ \langle \tilde{B}^a_\mu(t, p) \tilde{B}^b_\nu(s, q) \rangle \sim \delta^{ab} g_0^2 \frac{1}{(p^2)^2} \left[ (\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-(t+s)p^2} + \frac{1}{\lambda_0} p_\mu p_\nu e^{-\alpha_0(t+s)p^2} \right] \]

1-loop two point function (those containing only Yang–Mills vertices)

The last counter term comes from rewriting to renormalized parameters

\[ g_0^2 = \mu^{2\epsilon} g^2 Z, \quad \lambda_0 = \lambda Z_3^{-1} \]

Usually, this becomes UV finite only by taking the wave function renormalization factor into account . . .
... here, we have also diagrams containing flow vertices

which give rise to the precisely same effect as the wave function renormalization factor

- All order proof (Lüscher–Weisz (2011))

- when a loop contains a vertex in the bulk \((t > 0)\), the loop integral contains the flow-time evolution factor

  \[ \sim e^{-t\ell^2} \]

  which makes the loop integral finite; no bulk counterterm is necessary

- by using a BRS symmetry, it can be shown that all boundary \((t = 0)\) counterterms are those of the Yang–Mills theory
Correlation function of the flow gauge field

\[ \langle B_{\mu_1}(t_1, x_1) B_{\mu_2}(t_2, x_2) \cdots B_{\mu_n}(t_n, x_n) \rangle, \quad t_1 > 0, \ldots, t_n > 0, \]

remains finite even for the equal-point product

\[ t_1 \to t_2, \quad x_1 \to x_2, \]

the new loop always contains the flow-time evolution factor \( e^{-t_\ell^2} \) and this makes integral finite; no new UV divergence arises

This is an extremely powerful property!

\[ B_\mu(t, x) B_\nu(t, x) \bigg|_{\text{Dimensional Regularization}} = B_\mu(t, x) B_\nu(t, x) \bigg|_{\text{Lattice}} \]

On the other hand, the difficulty in the present problem comes from

\[ (A_R)_\mu(x) (A_R)_\nu(x) \bigg|_{\text{Dimensional Regularization}} \neq (A_R)_\mu(x) (A_R)_\nu(x) \bigg|_{\text{lattice}} \]

Using this property of the gradient flow, we relate a certain quantity defined by the gradient flow and EMT in the dimensional regularization.
**EMT in the dimensional regularization**

- **SU(N) Yang–Mills theory in** \( D = 4 - 2\epsilon \) **dimensions**

\[
S = \frac{1}{4g_0^2} \int d^D x \, F^a_{\mu\nu}(x) F^a_{\mu\nu}(x)
\]

- Assuming the dimensional regularization, since it preserves the translational invariance, the naive expression

\[
T_{\mu\nu}(x) = \frac{1}{g_0^2} \left[ F^a_{\mu\rho}(x) F^a_{\nu\rho}(x) - \frac{1}{4} \delta_{\mu\nu} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x) \right]
\]

fulfills the correct WT relation

\[
\langle \partial_\mu T_{\mu\nu}(x) \mathcal{O}(y) \mathcal{O}(z) \cdots \rangle = -\delta(x - y) \langle \partial_\nu \mathcal{O}(y) \mathcal{O}(z) \cdots \rangle + \cdots
\]

It follows from this that \( T_{\mu\nu}(x) \) does not receive the multiplicative renormalization

- So, with dimensional regularization, we define a renormalized (finite) EMT by subtracting VEV,

\[
\{ T_{\mu\nu} \}_R(x) = T_{\mu\nu}(x) - \langle T_{\mu\nu}(x) \rangle
\]
We consider the following dim. 4 gauge invariant combinations

\[ U_{\mu\nu}(t, x) \equiv G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4}\delta_{\mu\nu}G_{\rho\sigma}(t, x)G_{\rho\sigma}(t, x) \]

\[ E(t, x) \equiv \frac{1}{4}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \]

These are quite similar to 4 dimensional EMT (Itou–Kitazawa, 2012 ~), but can we make the relationship precise?

The flow equation is a diffusion equation whose diffusion length is \( \sim \sqrt{8t} \). So, in \( t \to 0 \) limit, \( U_{\mu\nu}(t, x) \) and \( E(t, x) \) can be regarded as local operators in \( D \) dimensional \( x \) space.

Moreover, from the UV finiteness of the gradient flow, these are UV finite.

From these facts, for \( t \to 0 \), above local products can be expressed by an asymptotic series of \( D \) dimensional renormalized operators (coefficients will be finite too):

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ \{T_{\mu\nu}\}_R(x) - \frac{1}{4}\delta_{\mu\nu}\{T_{\rho\rho}\}_R(x) \right] + O(t), \]

\[ E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) \{T_{\rho\rho}\}_R(x) + O(t), \]

Here, we have used the fact that \( U_{\mu\nu}(x) \) is traceless for \( D = 4 \). \( O(t) \) is the contribution of operators with dim. 6 or higher.
By eliminating the trace part $\{ T_{\rho \rho} \}_R(x)$ from the above expansion,

\[
U_{\mu \nu}(t, x) = \alpha_U(t) \left[ \{ T_{\mu \nu} \}_R(x) - \frac{1}{4} \delta_{\mu \nu} \{ T_{\rho \rho} \}_R(x) \right] + O(t),
\]

\[
E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) \{ T_{\rho \rho} \}_R(x) + O(t),
\]

we have

\[
\{ T_{\mu \nu} \}_R(x) = \frac{1}{\alpha_U(t)} U_{\mu \nu}(t, x) + \frac{1}{4\alpha_E(t)} \delta_{\mu \nu} [E(t, x) - \langle E(t, x) \rangle] + O(t)
\]

Therefore, if we know the $t \to 0$ behavior of the coefficients $\alpha_U(t)$ and $\alpha_E(t)$, the EMT can be obtained by $t \to 0$ limit of the combination in RHS.
Renormalization group argument

- We apply
  \[ \left( \mu \frac{\partial}{\partial \mu} \right)_0 \], \( \mu \): renormalization scale, 0: bare quantities fixed
  to both sides of
  \[
  U_{\mu\nu}(t, x) = \alpha_U(t) \left\{ T_{\mu\nu} \right\}_R (x) - \frac{1}{4} \delta_{\mu\nu} \left\{ T_{\rho\rho} \right\}_R (x) \right] + O(t),
  \]
  \[
  E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) \left\{ T_{\rho\rho} \right\}_R (x) + O(t)
  \]
- Expressed in terms of bare quantities, LHS does not contain \( \mu \). So,
  \[
  \left( \mu \frac{\partial}{\partial \mu} \right)_0 \alpha_U(t) \left\{ T_{\mu\nu} \right\}_R (x) - \frac{1}{4} \delta_{\mu\nu} \left\{ T_{\rho\rho} \right\}_R (x) \right] = 0,
  \]
  \[
  \left( \mu \frac{\partial}{\partial \mu} \right)_0 \alpha_E(t) \left\{ T_{\rho\rho} \right\}_R (x) = 0
  \]
- Further, the EMT is not renormalized,
  \[
  \left( \mu \frac{\partial}{\partial \mu} \right)_0 \alpha_{U,E}(t) = 0
  \]
Renormalization group argument

- Introducing the $\beta$ function by
  \[ \beta \equiv \left( \mu \frac{\partial}{\partial \mu} \right)_0 g \]
  the above relation becomes
  \[ \left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) \alpha_{U,E}(t)(g; \mu) = 0 \]

- This implies that using the running coupling $\bar{g}$ defined by
  \[ q \frac{d \bar{g}(q)}{dq} = \beta (\bar{g}(q)) , \quad \bar{g}(q = \mu) = g, \]
  the coefficients do not depend on the renormalization scale:
  \[ \alpha_{U,E}(t)(\bar{g}(q); q) = \alpha_{U,E}(t)(\bar{g}(q'); q'). \]

- So, we may set
  \[ q = \mu, \quad q' = \frac{1}{\sqrt{8t}} \]
  and then
  \[ \alpha_{U,E}(t)(g; \mu) = \alpha_{U,E}(t)(\bar{g}(1/\sqrt{8t}); 1/\sqrt{8t}) \]

- Because of the asymptotic freedom, $\bar{g}(1/\sqrt{8t}) \to 0$ for $t \to 0$ and coefficients can evaluated by the perturbation theory! (a sort of factorization)
Perturbative calculation of coefficients

- To 1-loop, we have to evaluate following flow-line Feynman diagrams

- In terms of the renormalized gauge coupling in the MS scheme,

\[
\alpha_U(t)(g; \mu) = g^2 \left\{ 1 + 2b_0 \left[ \ln(\sqrt{8t\mu}) + s_1 \right] g^2 + O(g^4) \right\},
\]

\[
\alpha_E(t)(g; \mu) = \frac{1}{2b_0} \left\{ 1 + 2b_0 s_2 g^2 + O(g^4) \right\},
\]

where

\[
s_1 = \ln \sqrt{\pi} + \frac{7}{16} \approx 1.00986, \quad s_2 = \frac{109}{176} - \frac{b_1}{2b_0^2} \approx 0.197831,
\]

and \(b_0 = 11N/(48\pi^2)\) and \(b_1 = 17N^2/(384\pi^4)\) are the first two coefficients of the \(\beta\) function; we see that \(\alpha_{U,E}(t)\) are actually UV finite.
By the above RG argument,

\[
\alpha_U(t) = \bar{g}(1/\sqrt{8t})^2 \left\{ 1 + 2b_0 s_1 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4) \right\},
\]

\[
\alpha_E(t) = \frac{1}{2b_0} \left\{ 1 + 2b_0 s_2 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4) \right\},
\]

where

\[
\bar{g}(q)^2 = \frac{1}{b_0 \ln(q^2/\Lambda^2)} - \frac{b_1 \ln[\ln(q^2/\Lambda^2)]}{b_0^3 \ln^2(q^2/\Lambda^2)} + O\left( \frac{\ln^2[\ln(q^2/\Lambda^2)]}{\ln^3(q^2/\Lambda^2)} \right).
\]

Therefore,

\[
\frac{1}{\alpha_U(t)} = \frac{1}{\bar{g}(1/\sqrt{8t})^2} - 2b_0 s_1 + O(\bar{g}^2),
\]

and

\[
\frac{1}{4\alpha_E(t)} = \frac{b_0}{2} \left[ 1 - 2b_0 s_2 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4) \right].
\]
Master formula

- Gathering all the above arguments,
  \[
  \{ T_{\mu\nu} \}_R (x) \xrightarrow{t \to 0^+} \left\{ \frac{1}{\bar{g}(1/\sqrt{8t})^2} - 2b_0s_1 \right\} U_{\mu\nu}(t, x) \\
  \quad + \frac{b_0}{2} \left[ 1 - 2b_0s_2\bar{g}(1/\sqrt{8t})^2 \right] \delta_{\mu\nu} [E(t, x) - \langle E(t, x) \rangle] \right\},
  \]
  and we obtained a formula that extracts a correctly normalized conserved EMT from local products defined through the gradient flow.

- Correlation functions of the quantities in RHS can (in principle) be computed non-perturbatively by using lattice regularization.

- Practically, we have to take sufficiently small \( t \) in the window
  \[
  a \ll \sqrt{8t} \ll \frac{1}{\Lambda}
  \]
  and the applicability is not quite obvious . . .
Study of the feasibility by numerical experiment (gauge group $SU(2)$)

- Configuration: Wilson plaquette action, pseudo-heat bath (+ overrelaxation)
- Wilson flow: 3rd order Runge–Kutta method (Lüscher), $\epsilon = \Delta t/a^2 = 0.01$, $t/a^2 \in [0, 6]$
- Field strength $G^a_{\mu\nu}(x)$ is clover-type, symmetric difference is used
- Simulation parameters

<table>
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<tr>
<th>lattice</th>
<th>$\beta$</th>
<th>$N_{\text{config}}$</th>
<th>$a/\sqrt{t_0}$</th>
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<tr>
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<td>100</td>
<td>0.8971(63)</td>
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<tr>
<td>$24^4$</td>
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<td>100</td>
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<td>$32^4$</td>
<td>2.91</td>
<td>100</td>
<td>0.4125(40)</td>
</tr>
</tbody>
</table>

- Here, we have introduced a reference flow time $t_0$ by using the expectation value of the “energy density”

$$E(t, x) = \frac{1}{4} G^a_{\mu\nu}(t, x) G^a_{\mu\nu}(t, x)$$

as

$$t^2 \langle E(t, x) \rangle \bigg|_{t=t_0} = 0.045$$
$t^2 \langle E(t, x) \rangle$ as a function of $t/a^2$

From these values and the perturbative calculation (Lüscher (2010))

\[
t^2 \langle E(t, x) \rangle \overset{t \to 0^+}{\sim} \frac{3(N^2 - 1)}{128 \pi^2} \bar{g}(1/\sqrt{8t})^2 \left[ 1 + 2b_0 c \bar{g}(1/\sqrt{8t})^2 \right],
\]

where (in the MS scheme)

\[
c \equiv \ln(2\sqrt{\pi}) + \frac{26}{33} - \frac{9}{22} \ln 3 \approx 1.60396,
\]

we estimate the perturbative running coupling $\bar{g}(1/\sqrt{8t})^2$. 
Perturbative running coupling $\bar{g}(1/\sqrt{8t})^2$

We may trust this perturbative computation of $\bar{g}(1/\sqrt{8t})^2$ for the region of $t$ in which the following “effective $\Lambda$ parameter” is (almost) constant

$$\Lambda(t) \equiv \frac{1}{\sqrt{8t}} \left[ b_0 \bar{g}(1/\sqrt{8t})^2 \right]^{-b_1/(2b_0^2)} e^{-1/[2b_0 \bar{g}(1/\sqrt{8t})^2]} \leftarrow 2\text{-loop}$$

$$\times \exp \left[ -\frac{-b_1^2 + b_0 b_2}{2b_0^3} \bar{g}(1/\sqrt{8t})^2 - \frac{b_3^3 - 2b_0 b_1 b_2 + b_0^2 b_3}{4b_0^4} \bar{g}(1/\sqrt{8t})^4 \right] \leftarrow 4\text{-loop}$$
Study of the feasibility by numerical experiment

- $a\Lambda(t)$ as a function of $t/a^2$

(Ideally) we should use $t/a^2$ in the almost-flat region
The order of the limits is important.

First, while keeping the flow time $t$ fixed in physical units, take the continuum limit $a \to 0$. This gives flowed values in the continuum Yang–Mills theory.

Then, to extract EMT, take a small flow time limit $t \to 0$.

We may fix $t$ in physical units by setting

$$t^2 \langle E(t, x) \rangle = \text{const.}$$

We considered 10 combinations, $t^2 \langle E(t, x) \rangle = 0.045, 0.040, 0.035$ with 3 different lattice spacings and $t^2 \langle E(t, x) \rangle = 0.030$ on $32^4$ lattice.

![Graph showing lattice energy-momentum tensor variations](attachment:points.txt)
Study of the feasibility by numerical experiment

- Example of the correlation function
  \[ \langle U_{01}(t, x)U_{01}(t, 0) \rangle \quad x = (x_0, 0, 0, 0) \]

- For \( t^2 \langle E(t, x) \rangle = 0.045 \) (the upper horizontal line) and for \( t^2 \langle E(t, x) \rangle = 0.040 \) (the 2nd horizontal line)

- For \( t^2 \langle E(t, x) \rangle = 0.035 \) (the 3rd horizontal line)
Study of the feasibility by numerical experiment

- Example of the correlation function

\[ \langle \partial_\mu U_{\mu 0}(t, x) E(t, 0) \rangle \quad x = (x_0, 0, 0, 0) \]

- For \( t^2 \langle E(t, x) \rangle = 0.045 \) and for \( t^2 \langle E(t, x) \rangle = 0.040 \)

- For \( t^2 \langle E(t, x) \rangle = 0.035 \)
Study of the feasibility by numerical experiment

- Example of the correlation function

\[ \langle \partial_\mu \delta_{\mu 0} E(t, x) E(t, 0) \rangle \quad x = (x_0, 0, 0, 0) \]

- For \( t^2 \langle E(t, x) \rangle = 0.045 \) and for \( t^2 \langle E(t, x) \rangle = 0.040 \)

- For \( t^2 \langle E(t, x) \rangle = 0.035 \)
Although next we should make an extrapolation to \( a \rightarrow 0 \), here we proceed by regarding the \( 32^4 \) results are sufficiently close to the continuum.

Remembering the \( t \rightarrow 0 \) behavior

\[
U_{\mu\nu}(t, x) = \alpha_U(t) \left[ \{ T_{\mu\nu} \}_R(x) - \frac{1}{4} \delta_{\mu\nu} \{ T_{\rho\rho} \}_R(x) \right] + O(t),
\]

\[
E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) \{ T_{\rho\rho} \}_R(x) + O(t),
\]

for example,

\[
\frac{1}{\alpha_U(t)\alpha_E(t)} \langle \partial_\mu U_{\mu0}(t, x)E(t, 0) \rangle \xrightarrow{t \to 0^+} \left\langle \partial_\mu \left[ \{ T_{\mu0} \}_R(x) - \frac{1}{4} \delta_{\mu0} \{ T_{\rho\rho} \}_R(x) \right] \{ T_{\rho\rho} \}_R(0) \right\rangle
\]

and RHS is obtained as an \( t \rightarrow 0 \) extrapolation of LHS (Thanks, Aoki-san!)

Similarly,

\[
\frac{1}{4} \frac{1}{\alpha_E(t)^2} \langle \partial_\mu \delta_{\mu0}E(t, x)E(t, 0) \rangle \xrightarrow{t \to 0^+} \frac{1}{4} \left\langle \partial_\mu \delta_{\mu0} \{ T_{\rho\rho} \}_R(x) \{ T_{\rho\rho} \}_R(0) \right\rangle
\]

Sum of these two is

\[
\xrightarrow{t \to 0^+} \left\langle \partial_\mu \{ T_{\mu0} \}_R(x) \{ T_{\rho\rho} \}_R(0) \right\rangle
\]

and should be 0 for \( x \neq 0 \) (conservation law of EMT!)
Study of the feasibility by numerical experiment

To obtain,

\[ \alpha_U(t) = \bar{g}(1/\sqrt{8t})^2 \left\{ 1 + 2b_0 s_1 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4) \right\}, \]

\[ \alpha_E(t) = \frac{1}{2b_0} \left\{ 1 + 2b_0 s_2 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4) \right\} \]

we have to know \( \Lambda \) that corresponds to the initial value of the running coupling \( \bar{g}(1/\sqrt{8t}) \)

Here, as a rough estimate,

\[ 0.0188 \leq a\Lambda \leq 0.0200 \]

and used the 2-loop running formula
Study of the feasibility by numerical experiment

(in what follows, only show plots with $a\Lambda = 0.0188$; not much difference for $a\Lambda = 0.0200$)

$$\frac{1}{\alpha_U(t)\alpha_E(t)} \left\langle \partial_\mu U_{\mu 0}(t, x) E(t, 0) \right\rangle \xrightarrow{t \to 0^+} \left\langle \partial_\mu \left[ \{ T_{\mu 0} \}_R(x) - \frac{1}{4} \delta_{\mu 0} \{ T_{\rho \rho} \}_R(x) \right] \right\rangle \{ T_{\rho \rho} \}_R(0)$$
Study of the feasibility by numerical experiment

\[
\frac{1}{4} \frac{1}{\alpha E(t)^2} \left\langle \partial_\mu \delta_\mu_0 E(t, x) E(t, 0) \right\rangle \xrightarrow{t \to 0+} \frac{1}{4} \left\langle \partial_\mu \delta_\mu_0 \left\{ T_{\rho \rho} \right\}_R (x) \left\{ T_{\rho \rho} \right\}_R (0) \right\rangle
\]
Study of the feasibility by numerical experiment

\[ t \rightarrow 0^{+} \quad \langle \partial_{\mu} \{ T_{\mu 0} \}_R (x) \{ T_{\rho \rho} \}_R (0) \rangle \]

- Good indication for the EMT conservation?!!
Study of the feasibility by numerical experiment

But the situation is not so clear for . . .

\[
\frac{1}{\alpha U(t)^2} \langle \partial_\mu U_{\mu 1}(t, x) U_{0 1}(t, 0) \rangle \xrightarrow{t \to 0^+} \left\langle \partial_\mu \left[ \left\{ T_{\mu 1}\right\}_R(x) - \frac{1}{4} \delta_{\mu 1} \left\{ T_{\rho \rho}\right\}_R(x) \right] \left\{ T_{0 1}\right\}_R(0) \right\rangle
\]
Study of the feasibility by numerical experiment

... and

\[
\frac{1}{4} \frac{1}{\alpha E(t) \alpha U(t)} \left\langle \partial_\mu \delta_{\mu 1} E(t, x) U_{01}(t, 0) \right\rangle \xrightarrow{t \to 0} \frac{1}{4} \left\langle \partial_\mu \delta_{\mu 1} \{ T_{\rho \rho} \}_R(x) \{ T_{01} \}_R(0) \right\rangle
\]

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Lattice energy-momentum tensor ...
An example of 2 point correlation function (relevant for the shear viscosity)

\[
\frac{1}{\alpha U(t)^2} \left\langle U_{01}(t, x) U_{01}(t, 0) \right\rangle \xrightarrow{t \to 0^+} \left\langle \left\{ T_{01} \right\}_R(x) \left\{ T_{01} \right\}_R(0) \right\rangle
\]
Future direction: numerical experiment

- It seems that we had a good indication (!) although we still have to carry out . . .
- systematic extrapolation to the continuum $a \rightarrow 0$
- systematic extrapolation to $t \rightarrow 0$ (hopefully) using data with smaller flow times
- clear demonstration of the conservation of EMT
- "$O(t)$ improvement" might be useful

$$G^{a}_{\mu \rho}(t, x)G^{a}_{\nu \rho}(t, x) \rightarrow G^{a}_{\mu \rho}(t, x)G^{a}_{\nu \rho}(t, x) - t \left[ D_{\sigma} D_{\sigma} G^{a}_{\mu \rho}(t, x)G^{a}_{\nu \rho}(t, x) + G^{a}_{\mu \rho}(t, x)D_{\sigma} D_{\sigma} G^{a}_{\nu \rho}(t, x) \right]$$

this replacement removes $O(t)$ terms in the tree level

- also 1-loop improvement will not be impossible (presumably)
- step size scaling for small $t$?
Inclusion of matter fields: flowed matter field requires the wave function renormalization (Lüscher (2013))

\[
\chi(t, x) = Z_{\chi}^{-1/2} \chi_R(t, x), \quad \bar{\chi}(t, x) = Z_{\chi}^{-1/2} \bar{\chi}_R(t, x)
\]

To avoid the determination of \(Z_{\chi}\) in lattice/continuum theory, we may define an operator by normalizing it by the “condensation” as, for example,

\[
\frac{\bar{\chi}(t, x) D_{\chi}(t, x)}{t_0^{3/2} \langle \bar{\chi}(t_0, x) \chi(t_0, x) \rangle}
\]

where \(t_0\) is an arbitrary fixed flow time. This is a dim. 4 UV finite quantity to which our argument is applied.

1-loop mixing coefficients (to be computed)

Future direction:

- Physical application? (thermodynamics, viscosities, conformal field theory, dilaton physics, vacuum energy, ...)
- For bulk thermodynamical quantities, for instance, for the so-called “trace anomaly”

\[
\langle \varepsilon - 3p \rangle_T = \left\langle - \{ T_{\mu\mu} \}_R (x) \right\rangle_T,
\]

or for the entropy density

\[
\langle \varepsilon + p \rangle_T = \left\langle - \{ T_{00} \}_R (x) + \frac{1}{3} \{ T_{ii} \}_R (x) \right\rangle_T,
\]

our definition should coincide with the traditional one (Engels–Karsch–Scheideler, (1982)) in the continuum limit
- This is the case also for other off-diagonal components (Giusti–Meyer (2013))?
- Can we define the chiral current and/or SUSY current from the gradient flow?