Lattice energy-momentum tensor from the Yang-Mills gradient flow

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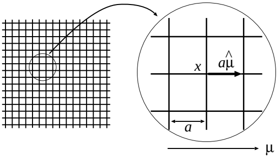
- H.S., Prog. Theor. Exp. Phys. (2013) 083B03 [arXiv:1304.0533 [hep-lat]].
- H.S., work in progress

- Lattice field theory



- best successful non-perturbative formulation of QFT; keeps internal gauge symmetries exactly
- ... but quite incompatible with spacetime symmetries (translation, rotation, SUSY, conformal, ...)
- Ward–Takahashi (WT) relation associated with translational invariance ($T_{\mu\nu}(x)$: energy-momentum tensor (EMT))

$$\langle \partial_\mu T_{\mu\nu}(x)\mathcal{O}(y)\mathcal{O}(z)\cdots\rangle = -\delta(x-y)\,\langle \partial_\nu \mathcal{O}(y)\mathcal{O}(z)\cdots\rangle + \cdots$$

- conservation law is a special case of this:

$$\langle \partial_\mu T_{\mu\nu}(x)\mathcal{O}(y)\mathcal{O}(z)\cdots\rangle = 0, \qquad \text{for } x \neq y,\ x \neq z,\ \ldots$$

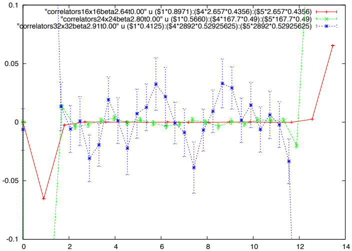
- can we construct lattice EMT which reproduces these relations in $a \to 0$?
- if this is possible, the application will be vast (thermodynamics, viscosities, conformal field theory, dilaton physics, vacuum energy, ...)

Naive construction will not work ...

- naive EMT for the pure Yang–Mills theory

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) - \frac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x)\right]$$

- a correct WT relation: $\langle\partial_\mu T_{\mu 1}(x)T_{01}(0)\rangle = \mathcal{C}\partial_0\delta(x)$
- Monte Carlo computation of LHS ($x = (x_0, 0, 0, 0)$)



- extremely noisy ...
- (although consistent with 0) it appears diverging as $a \to 0$
- after all, there is no guarantee that the naive expression is conserved for $a \to 0$, since lattice regularization breaks translational invariance

Possible approaches

- Invent somehow a lattice formulation that is invariant under the desired symmetry (in the present case, translation) as the lattice chiral symmetry on the basis of the Ginsparg–Wilson relation
- this is certainly ideal, but seems formidable for spacetime symmetries … (eventually, SLAC derivative?)

- What the general argument says is that a linear combination of dim. 4 operators being consistent with lattice symmetry

$$T_{\mu\nu}(x) = C_1 \left(\sum_{\rho} F^a_{\mu\rho} F^a_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu} \sum_{\rho\sigma} F^a_{\rho\sigma} F^a_{\rho\sigma} \right) + C_2 \delta_{\mu\nu} \sum_{\rho\sigma} F^a_{\rho\sigma} F^a_{\rho\sigma} + C_3 \delta_{\mu\nu} \sum_{\rho} F^a_{\mu\rho} F^a_{\nu\rho}$$

  is conserved in $a \to 0$; we may determine ratios of these coefficients by the conservation law (Caracciolo et al. (1989))
- overall normalization should be fixed separately (expectation value in a one-particle state? current algebra?)
- no one yet studied whether this construction generates correct translations on composite operators!
- approach on the basis of SUSY algebra and Ferrara–Zumino supermultiplet (H.S. (2012))

Our approach

- Use a UV finite quantity that can be related with EMT in a translationally invariant regularization
- Any regularization (including lattice) will produce the same number for such a UV finite quantity
- To define this UV finite quantity, we employ the so-called Yang–Mills gradient flow
- Yang–Mills gradient flow (a diffusion equation wrt a fictitious time $t \in \mathbb{R}$)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots, \qquad B_\mu(t=0,x) = A_\mu(x),$$

  where $G_{\mu\nu}$ is the field strength of the flowed gauge potential:

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)], \qquad D_\mu = \partial_\mu + [B_\mu, \cdot]$$

- Note: the mass dimension of t is -2

Yang–Mills gradient flow or the Wilson flow

- Yang–Mills gradient flow (continuum theory)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots, \qquad B_\mu(t=0,x) = A_\mu(x)$$

- Wilson flow (lattice theory)

$$\partial_t V(t,x,\mu) V(t,x,\mu)^{-1} = -g_0^2 \partial S_{\text{Wilson}}, \qquad V(t=0,x,\mu) = U(x,\mu)$$

- Applications (Lüscher):
  - definition of the topological charge
  - scale setting (just like the Sommer scale r_0)

    $$t^2 \left\langle E(t,x) \right\rangle \Big|_{t=t_0} = 0.3, \qquad \text{and set (for instance) } \sqrt{8t_0} = 0.5\,\text{fm}$$

    where

    $$E(t,x) \equiv \frac{1}{4} G_{\mu\nu}^a(t,x) G_{\mu\nu}^a(t,x)$$
  - define UV finite quantities ← Our usage here
  - computation of the chiral condensate

- Yang–Mills gradient flow

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x),$$

where the second term in RHS was introduced to suppress the gauge mode; it can be seen that gauge invariant quantities are independent of α_0. This can be solved formally as

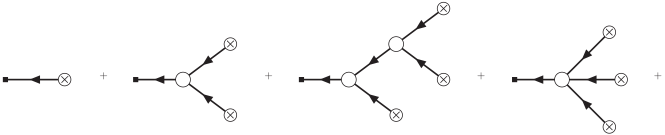
$$B_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R_\nu(s,y) \right],$$

where K is the heat kernel and R is non-linear terms

$$K_t(z)_{\mu\nu} = \int_p \frac{e^{ipz}}{p^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 tp^2} \right]$$

$$R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]]$$

Pictorially (cross: A_μ; open circle: flow vertex R),



Perturbative expansion of the gradient flow

- quantum correlation function of the flowed gauge field

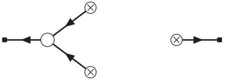
$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle ,$$

is obtained by taking the expectation value of the initial value $A_\mu(x)$. For example, the contraction of two A_μ's

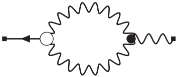
produces the propagator of the flowed field

$$\delta^{ab} g_0^2 \frac{1}{(p^2)^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-(t+s)p^2} + \frac{1}{\lambda_0} p_\mu p_\nu e^{-\alpha_0 (t+s) p^2} \right] ,$$

(where t and s are flow times at the end points; λ_0 is the conventional gauge parameter). Similarly, for



considering the contraction with the usual Yang–Mills vertex (the full circle)



Gauge invariance of the gradient flow

- Under the infinitesimal gauge transformation,

$$B_\mu(t,x) \to B_\mu(t,x) + D_\mu\omega(t,x),$$

  the flow equation

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x)$$

  changes to

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x) - D_\mu(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x)$$

- Therefore, by choosing $\omega(t,x)$ as the solution of

$$(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x) = -\delta\alpha_0 \partial_\nu B_\nu(t,x), \qquad \omega(t=0,x) = 0,$$

  α_0 can be changed as

$$\alpha_0 \to \alpha_0 + \delta\alpha_0$$

  That is, $B_\mu(t,x)$'s corresponding to different α_0's are related by a gauge transformation
- Also, by choosing $\omega(t,x)$ as the solution of

$$(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x) = 0, \qquad \omega(t=0,x) = \omega(x),$$

  the D dimensional gauge transformation $\omega(x)$ can be extended to a $D+1$ dimensional gauge transformation $\omega(t,x)$ that leaves the flow equation unchanged

- Correlation function of the flowed gauge field

$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle , \qquad t_1 > 0, \ldots, t_n > 0,$$

when expressed in terms of renormalized parameters, is UV finite without the wave function renormalization

- tree-level two-point function

$$\left\langle \tilde{B}^a_\mu(t, p) \tilde{B}^b_\nu(s, q) \right\rangle \sim \delta^{ab} g_0^2 \frac{1}{(p^2)^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-(t+s)p^2} + \frac{1}{\lambda_0} p_\mu p_\nu e^{-\alpha_0 (t+s)p^2} \right]$$

- 1-loop two point function (those containing only Yang–Mills vertices)

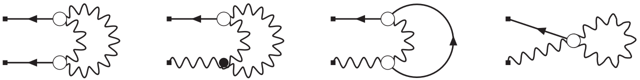
- The last counter term comes from rewriting to renormalized parameters

$$g_0^2 = \mu^{2\epsilon} g^2 Z, \qquad \lambda_0 = \lambda Z_3^{-1}$$

- Usually, this becomes UV finite only by taking the wave function renormalization factor into account . . .

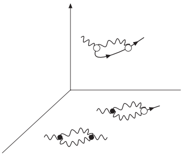
UV finiteness of the gradient flow I

- ... here, we have also diagrams containing flow vertices



which give rise to the precisely same effect as the wave function renormalization factor

- All order proof (Lüscher–Weisz (2011))



- when a loop contains a vertex in the bulk ($t > 0$), the loop integral contains the flow-time evolution factor

$$\sim e^{-t\ell^2}$$

which makes the loop integral finite; no bulk counterterm is necessary

- by using a BRS symmetry, it can be shown that all boundary ($t = 0$) counterterms are those of the Yang–Mills theory

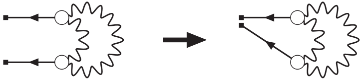
UV finiteness of the gradient flow II

- Correlation function of the flow gauge field

$$\langle B_{\mu_1}(t_1,x_1)B_{\mu_2}(t_2,x_2)\cdots B_{\mu_n}(t_n,x_n)\rangle\,,\qquad t_1>0,\ldots,t_n>0,$$

remains finite even for the equal-point product

$$t_1\longrightarrow t_2,\qquad x_1\longrightarrow x_2,$$



- the new loop always contains the flow-time evolution factor $\sim e^{-t\ell^2}$ and this makes integral finite; no new UV divergence arises
- This is an extremely powerful property!

$$B_\mu(t,x)B_\nu(t,x)|_{\text{Dimensional Regularization}}=B_\mu(t,x)B_\nu(t,x)|_{\text{Lattice}}$$

- On the other hand, the difficulty in the present problem comes from

$$(A_R)_\mu(x)(A_R)_\nu(x)|_{\text{Dimensional Regularization}}\neq(A_R)_\mu(x)(A_R)_\nu(x)|_{\text{lattice}}$$

- Using this property of the gradient flow, we relate a certain quantity defined by the gradient flow and EMT in the dimensional regularization

EMT in the dimensional regularization

- $SU(N)$ Yang–Mills theory in $D = 4 - 2\epsilon$ dimensions

$$S = \frac{1}{4g_0^2}\int d^Dx\, F_{\mu\nu}^a(x)F_{\mu\nu}^a(x)$$

- Assuming the dimensional regularization, since it preserves the translational invariance, the naive expression

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[F_{\mu\rho}^a(x)F_{\nu\rho}^a(x) - \frac{1}{4}\delta_{\mu\nu}F_{\rho\sigma}^a(x)F_{\rho\sigma}^a(x)\right]$$

  fulfills the correct WT relation

$$\langle\partial_\mu T_{\mu\nu}(x)\mathcal{O}(y)\mathcal{O}(z)\cdots\rangle = -\delta(x-y)\langle\partial_\nu\mathcal{O}(y)\mathcal{O}(z)\cdots\rangle + \cdots$$

  It follows from this that $T_{\mu\nu}(x)$ does not receive the multiplicative renormalization

- So, with dimensional regularization, we define a renormalized (finite) EMT by subtracting VEV,

$$\{T_{\mu\nu}\}_R(x) = T_{\mu\nu}(x) - \langle T_{\mu\nu}(x)\rangle$$

Local product in finite flow time and EMT

- We consider the following dim. 4 gauge invariant combinations

$$U_{\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) - \frac{1}{4}\delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x)$$

$$E(t,x) \equiv \frac{1}{4}G^a_{\mu\nu}(t,x)G^a_{\mu\nu}(t,x)$$

- These are quite similar to 4 dimensional EMT (Itou–Kitazawa, 2012 $\sim$), but can we make the relationship precise?
- The flow equation is a diffusion equation whose diffusion length is $\sim\sqrt{8t}$. So, in $t\to 0$ limit, $U_{\mu\nu}(t,x)$ and $E(t,x)$ can be regarded as local operators in D dimensional x space
- Moreover, from the UV finiteness of the gradient flow, these are UV finite
- From these facts, for $t\to 0$, above local products can be expressed by an asymptotic series of D dimensional renormalized operators (coefficients will be finite too):

$$U_{\mu\nu}(t,x) = \alpha_U(t)\left[\{T_{\mu\nu}\}_R(x) - \frac{1}{4}\delta_{\mu\nu}\{T_{\rho\rho}\}_R(x)\right] + O(t),$$

$$E(t,x) = \langle E(t,x)\rangle + \alpha_E(t)\{T_{\rho\rho}\}_R(x) + O(t),$$

Here, we have used the fact that $U_{\mu\nu}(x)$ is traceless for $D=4$. $O(t)$ is the contribution of operators with dim. 6 or higher

- By eliminating the trace part $\{T_{\rho\rho}\}_R(x)$ from the above expansion,

$$U_{\mu\nu}(t,x) = \alpha_U(t)\left[\{T_{\mu\nu}\}_R(x) - \frac{1}{4}\delta_{\mu\nu}\{T_{\rho\rho}\}_R(x)\right] + O(t),$$
$$E(t,x) = \langle E(t,x)\rangle + \alpha_E(t)\{T_{\rho\rho}\}_R(x) + O(t),$$

we have

$$\{T_{\mu\nu}\}_R(x) = \frac{1}{\alpha_U(t)}U_{\mu\nu}(t,x) + \frac{1}{4\alpha_E(t)}\delta_{\mu\nu}\left[E(t,x) - \langle E(t,x)\rangle\right] + O(t)$$

Therefore, if we know the $t \to 0$ behavior of the coefficients $\alpha_U(t)$ and $\alpha_E(t)$, the EMT can be obtained by $t \to 0$ limit of the combination in RHS

Renormalization group argument

- We apply

$$\left(\mu\frac{\partial}{\partial\mu}\right)_0, \qquad \mu\text{: renormalization scale, 0: bare quantities fixed}$$

to both sides of

$$U_{\mu\nu}(t,x) = \alpha_U(t)\left[\{T_{\mu\nu}\}_R(x) - \frac{1}{4}\delta_{\mu\nu}\{T_{\rho\rho}\}_R(x)\right] + O(t),$$
$$E(t,x) = \langle E(t,x)\rangle + \alpha_E(t)\{T_{\rho\rho}\}_R(x) + O(t)$$

- Expressed in terms of bare quantities, LHS does not contain μ. So,

$$\left(\mu\frac{\partial}{\partial\mu}\right)_0 \alpha_U(t)\left[\{T_{\mu\nu}\}_R(x) - \frac{1}{4}\delta_{\mu\nu}\{T_{\rho\rho}\}_R(x)\right] = 0,$$
$$\left(\mu\frac{\partial}{\partial\mu}\right)_0 \alpha_E(t)\{T_{\rho\rho}\}_R(x) = 0$$

- Further, the EMT is not renormalized,

$$\left(\mu\frac{\partial}{\partial\mu}\right)_0 \alpha_{U,E}(t) = 0$$

Renormalization group argument

- Introducing the β function by

$$\beta \equiv \left(\mu \frac{\partial}{\partial \mu}\right)_0 g$$

the above relation becomes

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g}\right) \alpha_{U,E}(t)(g;\mu) = 0$$

- This implies that using the running coupling $\bar{g}$ defined by

$$q \frac{d\bar{g}(q)}{dq} = \beta\left(\bar{g}(q)\right), \qquad \bar{g}(q=\mu) = g,$$

the coefficients do not depend on the renormalization scale:

$$\alpha_{U,E}(t)(\bar{g}(q);q) = \alpha_{U,E}(t)(\bar{g}(q');q').$$

- So, we may set

$$q = \mu, \qquad q' = \frac{1}{\sqrt{8t}}$$

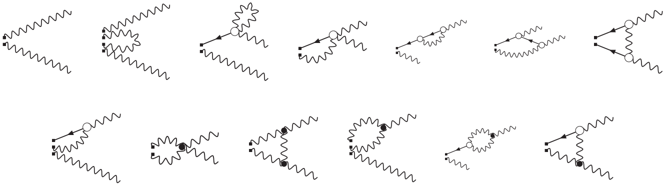
and then

$$\alpha_{U,E}(t)(g;\mu) = \alpha_{U,E}(t)(\bar{g}(1/\sqrt{8t}); 1/\sqrt{8t})$$

- Because of the asymptotic freedom, $\bar{g}(1/\sqrt{8t}) \to 0$ for $t \to 0$ and coefficients can evaluated by the perturbation theory! (a sort of factorization)

Perturbative calculation of coefficients

- To 1-loop, we have to evaluate following flow-line Feynman diagrams



- In terms of the renormalized gauge coupling in the MS scheme,

$$\alpha_U(t)(g;\mu) = g^2 \left\{ 1 + 2b_0 \left[\ln(\sqrt{8t}\mu) + s_1 \right] g^2 + O(g^4) \right\},$$

$$\alpha_E(t)(g;\mu) = \frac{1}{2b_0} \left\{ 1 + 2b_0 s_2 g^2 + O(g^4) \right\},$$

where

$$s_1 = \ln\sqrt{\pi} + \frac{7}{16} \simeq 1.00986, \qquad s_2 = \frac{109}{176} - \frac{b_1}{2b_0^2} \simeq 0.197831,$$

and $b_0 = 11N/(48\pi^2)$ and $b_1 = 17N^2/(384\pi^4)$ are the first two coefficients of the β function; we see that $\alpha_{U,E}(t)$ are actually UV finite

- By the above RG argument,

$$\alpha_U(t) = \bar{g}(1/\sqrt{8t})^2 \left\{1 + 2b_0 s_1 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4)\right\},$$
$$\alpha_E(t) = \frac{1}{2b_0} \left\{1 + 2b_0 s_2 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4)\right\},$$

where

$$\bar{g}(q)^2 = \frac{1}{b_0 \ln(q^2/\Lambda^2)} - \frac{b_1 \ln[\ln(q^2/\Lambda^2)]}{b_0^3 \ln^2(q^2/\Lambda^2)} + O\left(\frac{\ln^2[\ln(q^2/\Lambda^2)]}{\ln^3(q^2/\Lambda^2)}\right)$$

- Therefore,

$$\frac{1}{\alpha_U(t)} = \frac{1}{\bar{g}(1/\sqrt{8t})^2} - 2b_0 s_1 + O(\bar{g}^2),$$

and

$$\frac{1}{4\alpha_E(t)} = \frac{b_0}{2}\left[1 - 2b_0 s_2 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4)\right]$$

Master formula

- Gathering all the above arguments,

$$\{T_{\mu\nu}\}_R(x) \overset{t\to 0^+}{\sim} \left\{ \left[\frac{1}{\bar{g}(1/\sqrt{8t})^2} - 2b_0 s_1 \right] U_{\mu\nu}(t,x) \right. \\ \left. + \frac{b_0}{2} \left[1 - 2b_0 s_2 \bar{g}(1/\sqrt{8t})^2 \right] \delta_{\mu\nu} \left[E(t,x) - \langle E(t,x) \rangle \right] \right\},$$

  and we obtained a formula that extracts a correctly normalized conserved EMT from local products defined through the gradient flow

- Correlation functions of the quantities in RHS can (in principle) be computed non-perturbatively by using lattice regularization

- Practically, we have to take sufficiently small t in the window

$$a \ll \sqrt{8t} \ll \frac{1}{\Lambda}$$

  and the applicability is not quite obvious . . .

- Configuration: Wilson plaquette action, pseudo-heat bath ($+$ overrelaxation)
- Wilson flow: 3rd order Runge–Kutta method (Lüscher), $\epsilon = \Delta t/a^2 = 0.01$, $t/a^2 \in [0, 6]$
- field strength $G^a_{\mu\nu}(x)$ is clover-type, symmetric difference is used
- simulation parameters

lattice	β	N_{config}	$a/\sqrt{t_0}$	
16^4	2.64	100	0.8971(63)	
24^4	2.80	100	0.5660(48)	$a \simeq 0.036\,\text{fm}$ for $\sqrt{\sigma} = 440\,\text{MeV}$
32^4	2.91	100	0.4125(40)	

- Here, we have introduced a reference flow time t_0 by using the expectation value of the "energy density"

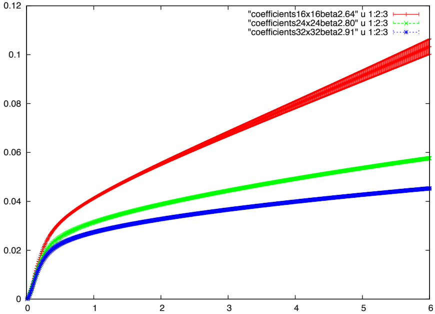
$$E(t,x) = \frac{1}{4} G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x)$$

as

$$t^2 \left\langle E(t,x) \right\rangle \Big|_{t=t_0} = 0.045$$

Study of the feasibility by numerical experiment

- $t^2 \langle E(t,x) \rangle$ as a function of t/a^2



- From these values and the perturbative calculation (Lüscher (2010))

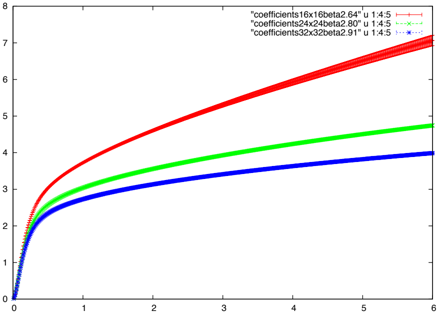
$$t^2 \langle E(t,x) \rangle \overset{t\to 0^+}{\sim} \frac{3(N^2-1)}{128\pi^2} \bar{g}(1/\sqrt{8t})^2 \left[1 + 2b_0 c \bar{g}(1/\sqrt{8t})^2\right],$$

where (in the MS scheme)

$$c \equiv \ln(2\sqrt{\pi}) + \frac{26}{33} - \frac{9}{22}\ln 3 \simeq 1.60396,$$

we estimate the perturbative running coupling $\bar{g}(1/\sqrt{8t})^2$

- Perturbative running coupling $\bar{g}(1/\sqrt{8t})^2$

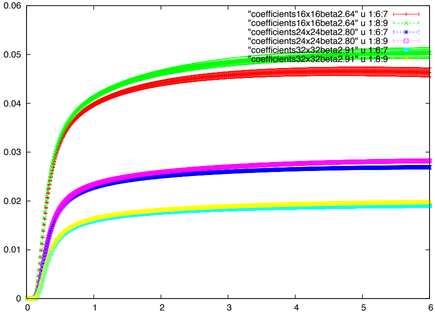


- We may trust this perturbative computation of $\bar{g}(1/\sqrt{8t})^2$ for the region of t in which the following "effective Λ parameter" is (almost) constant

$$\Lambda(t) \equiv \frac{1}{\sqrt{8t}} \left[b_0 \bar{g}(1/\sqrt{8t})^2 \right]^{-b_1/(2b_0^2)} e^{-1/[2b_0\bar{g}(1/\sqrt{8t})^2]} \leftarrow \text{2-loop}$$

$$\times \exp\left[-\frac{-b_1^2 + b_0 b_2}{2b_0^3} \bar{g}(1/\sqrt{8t})^2 - \frac{b_1^3 - 2b_0 b_1 b_2 + b_0^2 b_3}{4b_0^4} \bar{g}(1/\sqrt{8t})^4 \right] \leftarrow \text{4-loop}$$

- $a\Lambda(t)$ as a function of t/a^2



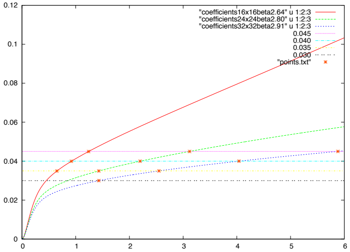
(Ideally) we should use t/a^2 in the almost-flat region

Study of the feasibility by numerical experiment

- The order of the limits is important
- First, while keeping the flow time t fixed in physical units, take the continuum limit $a \to 0$. This gives flowed values in the continuum Yang–Mills theory
- Then, to extract EMT, take a small flow time limit $t \to 0$
- We may fix t in physical units by setting

$$t^2 \langle E(t,x) \rangle = \text{const.}$$

- We considered 10 combinations, $t^2\langle E(t,x)\rangle = 0.045, 0.040, 0.035$ with 3 different lattice spacings and $t^2\langle E(t,x)\rangle = 0.030$ on 32^4 lattice

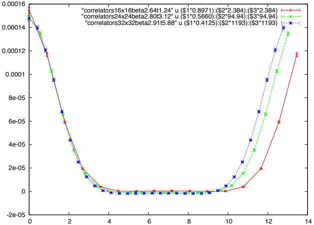


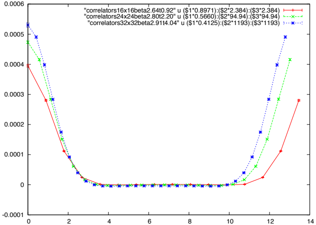
Study of the feasibility by numerical experiment

- Example of the correlation function

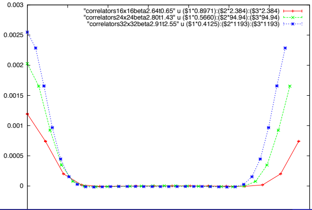
$$\langle U_{01}(t,x)U_{01}(t,0)\rangle \qquad x = (x_0,0,0,0)$$

- For $t^2\langle E(t,x)\rangle = 0.045$ (the upper horizontal line) and for $t^2\langle E(t,x)\rangle = 0.040$ (the 2nd horizontal line)





- For $t^2\langle E(t,x)\rangle = 0.035$ (the 3rd horizontal line)

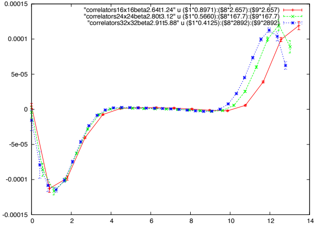


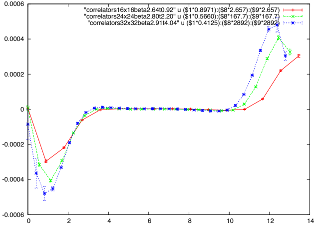
Study of the feasibility by numerical experiment

- Example of the correlation function

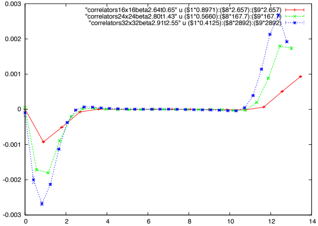
$$\langle \partial_\mu U_{\mu 0}(t,x) E(t,0) \rangle \qquad x = (x_0, 0, 0, 0)$$

- For $t^2\langle E(t,x)\rangle = 0.045$ and for $t^2\langle E(t,x)\rangle = 0.040$





- For $t^2\langle E(t,x)\rangle = 0.035$

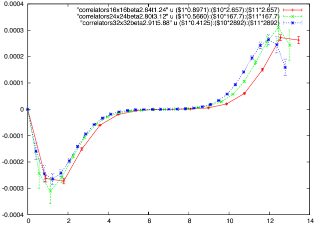


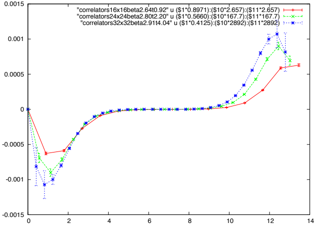
Study of the feasibility by numerical experiment

- Example of the correlation function

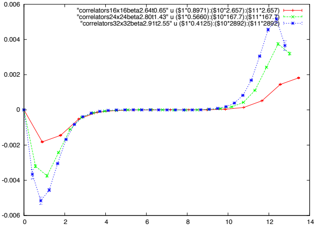
$$\langle \partial_\mu \delta_{\mu 0} E(t,x) E(t,0) \rangle \qquad x = (x_0, 0, 0, 0)$$

- For $t^2 \langle E(t,x) \rangle = 0.045$ and for $t^2 \langle E(t,x) \rangle = 0.040$





- For $t^2 \langle E(t,x) \rangle = 0.035$



Study of the feasibility by numerical experiment

- Although next we should make an extrapolation to $a \to 0$, here we proceed by regarding the 32^4 results are sufficiently close to the continuum
- Remembering the $t \to 0$ behavior

$$U_{\mu\nu}(t,x) = \alpha_U(t)\left[\{T_{\mu\nu}\}_R(x) - \frac{1}{4}\delta_{\mu\nu}\{T_{\rho\rho}\}_R(x)\right] + O(t),$$
$$E(t,x) = \langle E(t,x)\rangle + \alpha_E(t)\{T_{\rho\rho}\}_R(x) + O(t),$$

  for example,

$$\frac{1}{\alpha_U(t)\alpha_E(t)}\langle \partial_\mu U_{\mu0}(t,x)E(t,0)\rangle \stackrel{t\to0+}{\longrightarrow} \left\langle \partial_\mu \left[\{T_{\mu0}\}_R(x) - \frac{1}{4}\delta_{\mu0}\{T_{\rho\rho}\}_R(x)\right]\{T_{\rho\rho}\}_R(0)\right.$$

  and RHS is obtained as an $t \to 0$ extrapolation of LHS (Thanks, Aoki-san!)
- Similarly,

$$\frac{1}{4}\frac{1}{\alpha_E(t)^2}\langle \partial_\mu \delta_{\mu0} E(t,x)E(t,0)\rangle \stackrel{t\to0+}{\longrightarrow} \frac{1}{4}\left\langle \partial_\mu \delta_{\mu0}\{T_{\rho\rho}\}_R(x)\{T_{\rho\rho}\}_R(0)\right\rangle$$

- Sum of these two is

$$\stackrel{t\to0+}{\longrightarrow} \left\langle \partial_\mu \{T_{\mu0}\}_R(x)\{T_{\rho\rho}\}_R(0)\right\rangle$$

  and should be 0 for $x \neq 0$ (conservation law of EMT!)

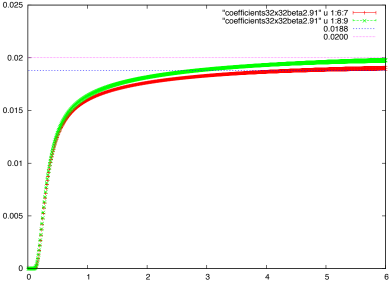
Study of the feasibility by numerical experiment

- To obtain,

$$\alpha_U(t) = \bar{g}(1/\sqrt{8t})^2 \left\{ 1 + 2b_0 s_1 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4) \right\},$$

$$\alpha_E(t) = \frac{1}{2b_0} \left\{ 1 + 2b_0 s_2 \bar{g}(1/\sqrt{8t})^2 + O(\bar{g}^4) \right\}$$

we have to know Λ that corresponds to the initial value of the running coupling $\bar{g}(1/\sqrt{8t})$



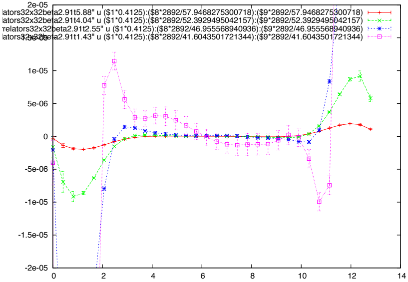
Here, as a rough estimate,

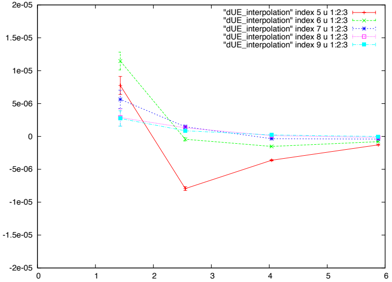
$$0.0188 \leq a\Lambda \leq 0.0200$$

and used the 2-loop running formula

- (in what follows, only show plots with $a\Lambda = 0.0188$; not much difference for $a\Lambda = 0.0200$)

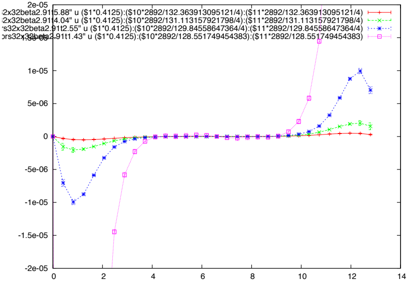
$$\frac{1}{\alpha_U(t)\alpha_E(t)} \langle \partial_\mu U_{\mu 0}(t,x) E(t,0) \rangle \overset{t\to 0+}{\longrightarrow} \left\langle \partial_\mu \left[\{T_{\mu 0}\}_R(x) - \frac{1}{4}\delta_{\mu 0}\{T_{\rho\rho}\}_R(x) \right] \{T_{\rho\rho}\}_R(0) \right.$$

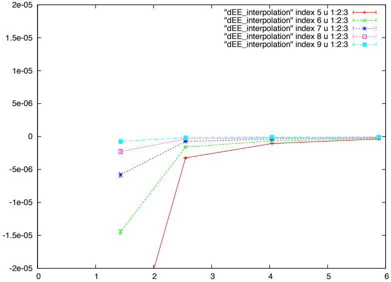




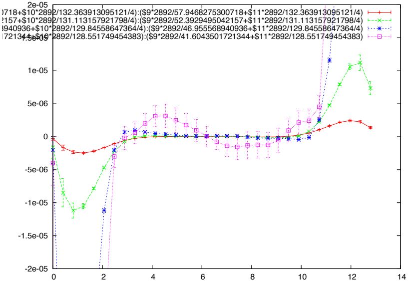
Study of the feasibility by numerical experiment

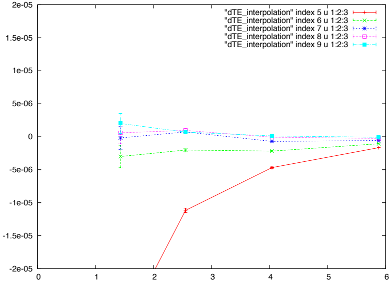
- $$\frac{1}{4}\frac{1}{\alpha_E(t)^2}\left\langle\partial_\mu\delta_{\mu0}E(t,x)E(t,0)\right\rangle \xrightarrow{t\to0+} \frac{1}{4}\left\langle\partial_\mu\delta_{\mu0}\left\{T_{\rho\rho}\right\}_R(x)\left\{T_{\rho\rho}\right\}_R(0)\right\rangle$$





- $$\xrightarrow{t\to 0+} \left\langle \partial_\mu \left\{ T_{\mu 0} \right\}_R (x) \left\{ T_{\rho\rho} \right\}_R (0) \right\rangle$$

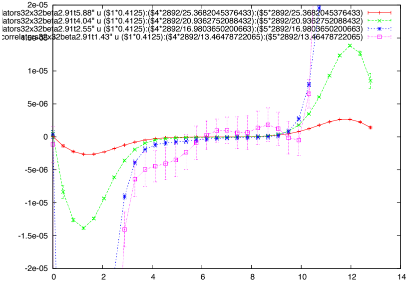


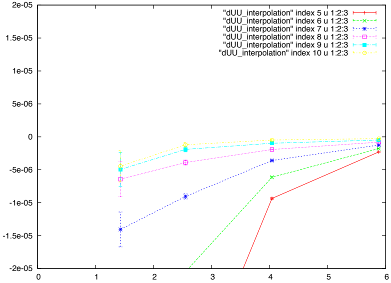


- Good indication for the EMT conservation?!!

- But the situation is not so clear for ...

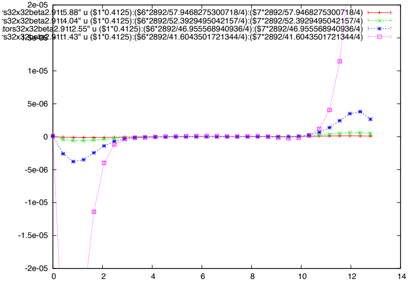
$$\frac{1}{\alpha_U(t)^2}\left\langle\partial_\mu U_{\mu1}(t,x)U_{01}(t,0)\right\rangle \overset{t\to0+}{\longrightarrow} \left\langle\partial_\mu\left[\{T_{\mu1}\}_R(x)-\frac{1}{4}\delta_{\mu1}\{T_{\rho\rho}\}_R(x)\right]\{T_{01}\}_R(0)\right\rangle$$

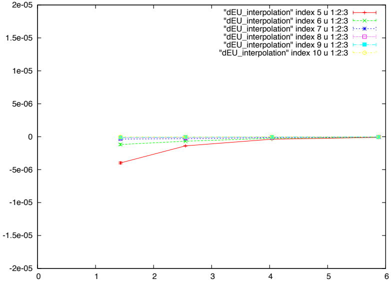




- … and

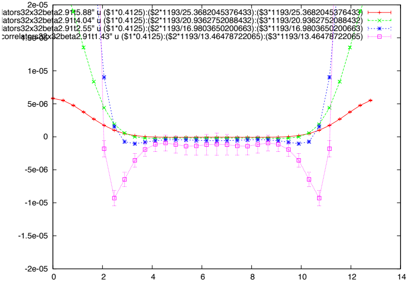
$$\frac{1}{4}\frac{1}{\alpha_E(t)\alpha_U(t)}\left\langle \partial_\mu \delta_{\mu 1} E(t,x) U_{01}(t,0) \right\rangle \overset{t\to 0+}{\longrightarrow} \frac{1}{4}\left\langle \partial_\mu \delta_{\mu 1}\left\{T_{\rho\rho}\right\}_R(x)\left\{T_{01}\right\}_R(0) \right\rangle$$

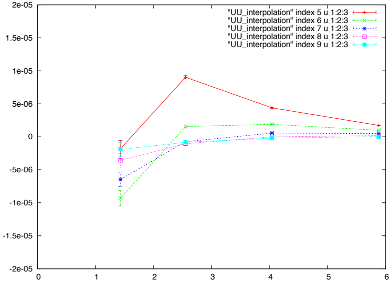




- An example of 2 point correlation function (relevant for the shear viscosity)

$$\frac{1}{\alpha_U(t)^2}\left\langle U_{01}(t,x)U_{01}(t,0)\right\rangle \stackrel{t\to 0+}{\longrightarrow} \left\langle \{T_{01}\}_R(x)\{T_{01}\}_R(0)\right\rangle$$





Future direction: numerical experiment

- It seems that we had a good indication (!) although we still have to carry out . . .
- systematic extrapolation to the continuum $a \to 0$
- systematic extrapolation to $t \to 0$ (hopefully) using data with smaller flow times
- clear demonstration of the conservation of EMT
- "$O(t)$ improvement" might be useful

$$G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) \to G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) - t\left[D_\sigma D_\sigma G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) + G^a_{\mu\rho}(t,x)D_\sigma D_\sigma G^a_{\nu\rho}(t,x)\right]$$

  this replacement removes $O(t)$ terms in the tree level
- also 1-loop improvement will not be impossible (presumably)
- step size scaling for small t?

Future direction: theoretical aspects

- Inclusion of matter fields: flowed matter field requires the wave function renormalization (Lüscher (2013))

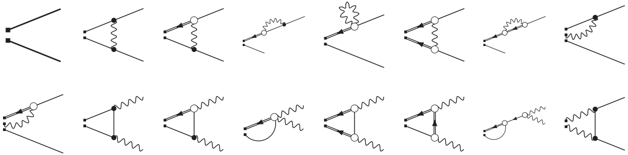
$$\chi(t,x) = Z_\chi^{-1/2}\chi_R(t,x), \qquad \bar{\chi}(t,x) = Z_\chi^{-1/2}\bar{\chi}_R(t,x)$$

  To avoid the determination of Z_χ in lattice/continuum theory, we may define an operator by normalizing it by the "condensation" as, for example,

$$\frac{\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)}{t_0^{3/2}\langle\bar{\chi}(t_0,x)\chi(t_0,x)\rangle}$$

  where t_0 is an arbitrary fixed flow time. This is a dim. 4 UV finite quantity to which our argument is applied

- 1-loop mixing coefficients (to be computed)



- Non-perturbative determination of mixing coefficients? (Del Debbio–Patella–Rago, arXiv:1306.1173 [hep-th])

Future direction:

- Physical application? (thermodynamics, viscosities, conformal field theory, dilaton physics, vacuum energy, . . .)
- For bulk thermodynamical quantities, for instance, for the so-called “trace anomaly”

$$\langle \varepsilon - 3p \rangle_T = \left\langle - \{T_{\mu\mu}\}_R(x) \right\rangle_T ,$$

  or for the entropy density

$$\langle \varepsilon + p \rangle_T = \left\langle - \{T_{00}\}_R(x) + \frac{1}{3} \{T_{ii}\}_R(x) \right\rangle_T ,$$

  our definition should coincide with the traditional one (Engels–Karsch–Scheideler, (1982)) in the continuum limit
- This is the case also for other off-diagonal components (Giusti–Meyer (2013))?
- Can we define the chiral current and/or SUSY current from the gradient flow?