Current Status of Exact Supersymmetry on the Lattice

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D'adda, Feo, Kanamori, Nagata, Saito,
Asaka, Kondo, Giguere

Why Lattice SUSY ?

Practical reason: discovery of SUSY particles ?

lattice SUSY ⟺ lattice QCD

N=D=4 lattice super Yang-Mills (string)

What makes exact lattice SUSY regularization difficult ?

exact lattice SUSY regularization possible ?

connection with lattice chiral fermion problem ?

clue for SUSY breaking mechanism ?

super gravity ?

Exact SUSY on the Lattice

A bit of history:

More than 30 years unsuccessful: *Dondi&Nicolai (1977)*

Many theoretical and numerical investigations

No realization of exact SUSY on the lattice untill 2003

Later developments:

Exact lattice SUSY was realized only for nilpotent super charge: $Q^2 = 0$

Kaplan, Katz, Unsal, Cohen (2003), Sugino, Catterall….

No-Go theorem for Leibniz rule of difference operator

Kato, Sakamoto and So (2008)

New approaches for exact SUSY on the lattice

A) Link approach: noncommutative

D'Adda, Kanamori, N.K. Nagata.(2005,6,8)

Hopf algebra invariance : D'adda, N.K. Saito (2010)

B) Super doubler approach: nonlocal

D'Adda, Feo, Kanamori, N.K. Saito (2011,12)

10 years of Sapporo-Torino collaboration

Exactness of $Q^2 = 0$ super charge (Kaplan, Sugino..)

Part of super charges of extended SUSY

$$\{Q_{\alpha i}, \overline{Q}_{\beta j}\} = 2\delta_{ij}(\gamma^\mu)_{\alpha\beta}P_\mu$$

Exact supersymmetry for all superchages
on the lattice

Two major difficulties for lattice SUSY

Let's consider the simplest lattice SUSY algebra:

$$\{Q, Q\} = 2H$$

$$Q^2 = i\partial \to i\hat{\partial} \qquad \hat{\partial}F(x) = \frac{1}{a}\{F(x+\frac{a}{2}) - F(x-\frac{a}{2})\}$$

$$(a : \text{lattice constant})$$

(0) Loss of Poincare invariance: discrete invariance ?

(1) Difference operator does not satisfy Leibniz rule.

(2) Species doublers of lattice chiral fermion copies appear:
unbalance of d.o.f. between bosons and fermions

difference operator

$$\hat{\partial} F(x) = \frac{1}{a}\{F(x+\tfrac{a}{2}) - F(x-\tfrac{a}{2})\}$$

symmetric

$$\hat{\partial}(F(x)G(x)) = \frac{1}{a}\left(F(x+\tfrac{a}{2})G(x+\tfrac{a}{2}) - F(x-\tfrac{a}{2})G(x-\tfrac{a}{2})\right)$$

$$= \frac{1}{a}\left(F(x+\tfrac{a}{2}) - F(x-\tfrac{a}{2})\right) G(x-\tfrac{a}{2}) + F(x+\tfrac{a}{2})\frac{1}{a}\left(G(x+\tfrac{a}{2}) - G(x-\tfrac{a}{2})\right)$$

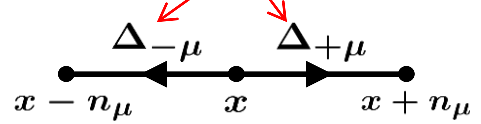
cancelation

$$= \hat{\partial} F(x) G(x-\tfrac{a}{2}) + F(x+\tfrac{a}{2})\hat{\partial} G(x)$$

$$= \hat{\partial} F(x) G(x+\tfrac{a}{2}) + F(x-\tfrac{a}{2})\hat{\partial} G(x)$$

breakdown of Leibniz rule

Link nature



Modified Leibniz rule

$$\left(\Delta_{+\mu} F\right)(x) = F(x+n_\mu) - F(x) \qquad (a=1)$$

forward

$$\left(\Delta_{+\mu} FG\right)(x) = (\Delta_{+\mu}F)(x)G(x) + F(x+n_\mu)(\Delta_{+\mu}G)(x)$$

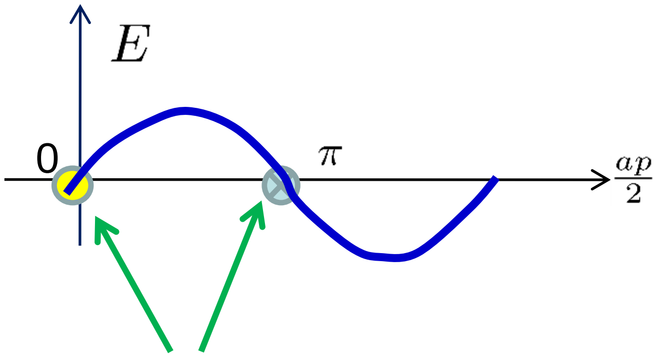
$$= (\Delta_{+\mu}F)(x)G(x+n_\mu) + F(x)(\Delta_{+\mu}G)(x)$$

(2) Species doublers of lattice chiral fermion copies appear: unbalance of d.o.f. between bosons and fermions

Massless fermion ➡ species doublers

$$\overline{\psi}(x) i\gamma_\mu \{\psi(x+\tfrac{a}{2}\widehat{\mu}) - \psi(x-\tfrac{a}{2}\widehat{\mu})\}/a$$

$$\frac{1}{\gamma^\mu sin\frac{a}{2}p_\mu} \qquad \frac{ap_\mu}{2} = 0, \pi$$



doubling of fermions

Continuum: $\frac{dE}{dp} = \frac{p}{\sqrt{p^2+m^2}} \xrightarrow{m\to 0} \frac{p}{|p|} = \pm 1 \ (\text{helicity})$

How do we solve these two fundamental problems ?

Our proposals

A) Link Approach:

twisted SUSY,

shifted Leibniz rule for super charges

Dirac-Kaehler (Ivanenko-Landau) fermions

B) Super doubler approach:

lattice momentum conservation

Leibniz rule is satisfied under $\star$ product

non-local field theory

doublers = super partners for A) and B)

No chiral fermion problem

A) Link Approach:

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\delta_{ij}\gamma^{\mu}{}_{\alpha\beta}P_{\mu}$$

N=D=2 SUSY

$$Q_{\alpha i} = \left(\mathbf{1}s + \gamma^{\mu}s_{\mu} + \gamma^{5}\tilde{s}\right)_{\alpha i}$$

Dirac-Kaehler Twist

$$(\Psi)_{\alpha i} = (\chi + \chi_{\mu}\gamma^{\mu} + \chi_{\mu\nu}\gamma^{[\mu\nu]} + \cdots)_{\alpha i}$$

Dirac-Kaehler fermion

$$\{s, s_{\mu}\} = -i\partial_{\mu}, \quad \{\tilde{s}, s_{\mu}\} = i\epsilon_{\mu\nu}\partial^{\nu}$$

N=D=2 Twisted SUSY

$$s^2 = \{s, \tilde{s}\} = \tilde{s}^2 = \{s_{\mu}, s_{\nu}\} = 0,$$

Continuum ⟶ Lattice: $\partial_{\mu} \to \Delta_{\pm\mu}$

$$\{\boldsymbol{Q}, \boldsymbol{Q}_{\mu}\} = i\boldsymbol{\Delta}_{\pm\mu} \qquad \{\tilde{\boldsymbol{Q}}, \boldsymbol{Q}_{\mu}\} = -i\epsilon_{\mu\nu}\boldsymbol{\Delta}_{\pm\nu}$$

on a Lattice

$$\left(\boldsymbol{\Delta}_{\pm\mu}\boldsymbol{\Phi}\right)(\boldsymbol{x}) = \pm(\boldsymbol{\Phi}(\boldsymbol{x} \pm \boldsymbol{n}_{\mu}) - \boldsymbol{\Phi}(\boldsymbol{x}))$$

N=D=2 SUSY

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\delta_{ij}{\gamma^{\mu}}_{\alpha\beta}P_{\mu}$$

Dirac-Kaehler Twist

$$Q_{\alpha i} = \left(\mathbf{1}Q + \gamma^{\mu}Q_{\mu} + \gamma^{5}\tilde{Q}\right)_{\alpha i}$$

J' = J + R

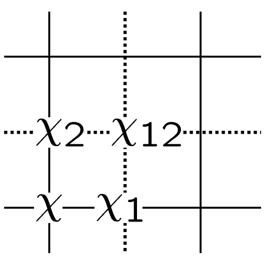
$$\{Q, Q_{\mu}\} = i\Delta_{\pm\mu} \qquad \{\tilde{Q}, Q_{\mu}\} = -i\epsilon_{\mu\nu}\Delta_{\pm\nu}$$

$$(\Psi)_{\alpha i} = (\chi + \chi_{\mu}\gamma^{\mu} + \chi_{\mu\nu}\gamma^{[\mu\nu]} + \cdots)_{\alpha i}$$

Dirac-Kaehler fermion

i : **flavour ?** ⟶ **Extended SUSY suffix**

$$\Phi = \chi + \chi_{\mu}dx^{\mu} + \chi_{\mu\nu}dx^{\mu}\wedge dx^{\nu} + \cdots$$



y ↔ 2x

$2^{d/2}$ super charges in d-dim.

2-dim. N=2
3-dim. N=4
4-dim. N=4

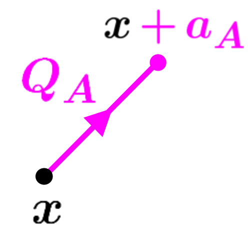
Dirac-Kaehler twisting

#boson = #fermion

New Ansatz:

$$\left(\Delta_{+\mu}\Phi\right)(x) = \Delta_{+\mu}\Phi(x) - \Phi(x+n_\mu)\Delta_{+\mu}$$

We need a modified Leibniz rule for Q_A too !



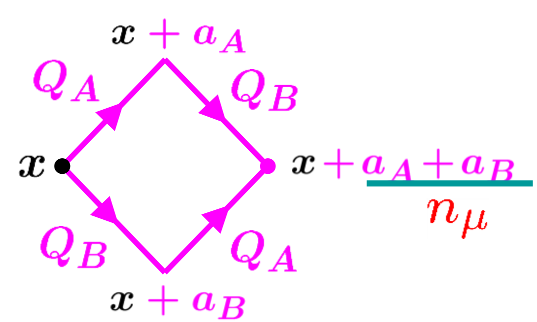
$$s_A\Phi(x,\theta) = Q_A\Phi(x,\theta) - \Phi(x+a_A,\theta)Q_A$$

$$\{Q, Q_\mu\} = i\Delta_{+\mu} \qquad \{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{-\nu}$$

$$a + a_\mu = +n_\mu \qquad \tilde{a} + a_\mu = -|\epsilon_{\mu\nu}|n_\nu$$

Compatibility of Shifts

$$(\{Q_A, Q_B\}\Phi)(x) = \{Q_A, Q_B\}\Phi(x) - \Phi(x+a_A+a_B)\{Q_A, Q_B\}$$



Cond. for Twisted N=D=2

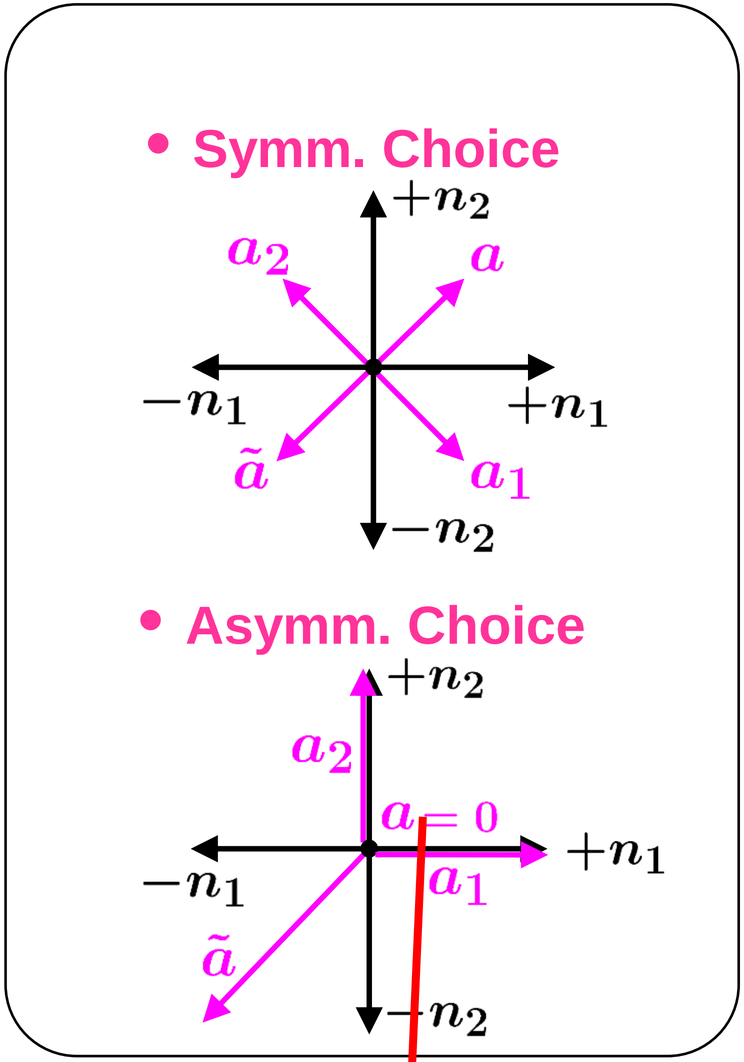
$$a + a_\mu = +n_\mu$$
$$\tilde{a} + a_\mu = -|\epsilon_{\mu\nu}| n_\nu$$

Solutions

$$a = (arbitrary)$$
$$a_\mu = +n_\mu - a$$
$$\tilde{a} = -n_1 - n_2 + a$$
$$a + a_1 + a_2 + \tilde{a} = 0$$

Twisted N=D=2
Lattice SUSY Algebra

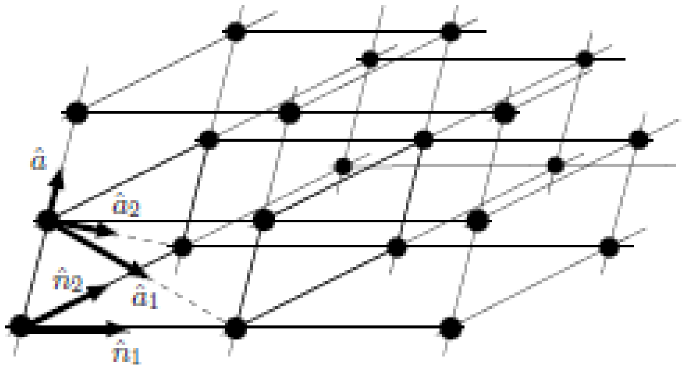
$$\{Q, Q_\mu\} = +i\Delta_{+\mu}$$
$$\{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{-\nu}$$



Equivalent to orbifold construction: $Q^2 = 0$
by Kaplan et.al.

This solution has three dimensional nature even if it is two dimensional lattice formulation.

$$
\begin{aligned}
a &= (arbitrary) \\
a_\mu &= +n_\mu - a \\
\tilde{a} &= -n_1 - n_2 + a \\
a + a_1 + a_2 + \tilde{a} &= 0
\end{aligned}
$$



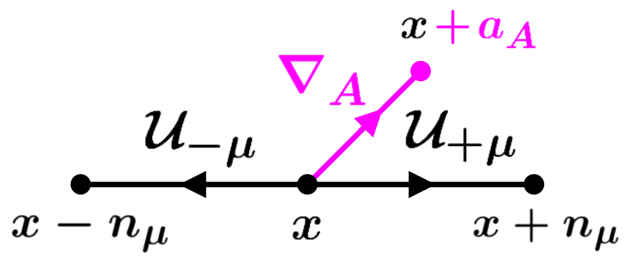
N=D=2 Twisted Super Yang-Mills

Introduce Bosonic & Fermionic Link variables

$$(\Delta_{\pm\mu})_{x\pm n_\mu,x} \rightarrow \mp(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x}$$
$$(Q_A)_{x+a_A,x} \rightarrow (\nabla_A)_{x+a_A,x}$$

Gauge trans.

$$(\mathcal{U}_{\pm\mu})' = G_{x\pm n_\mu}\,(\mathcal{U}_{\pm\mu})\,G_x^{-1}$$
$$(\nabla_A)' = G_{x+a_A}\,(\nabla_A)\,G_x^{-1}$$



- $(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x} = (e^{\pm i(A_\mu \pm i\phi^{(\mu)})})_{x\pm n_\mu,x},$

  $\phi^{(\mu)}\ (\mu=1,2)$: Scalar fields in SYM multiplet

- $\mathcal{U}_{+\mu}\mathcal{U}_{-\mu} \neq 1$

N=D=2 Twisted Lattice SUSY Algebra for SYM

$$\{\nabla, \nabla_\mu\}_{x+a+a_\mu,x} = +i(\mathcal{U}_{+\mu})_{x+n_\mu,x}$$

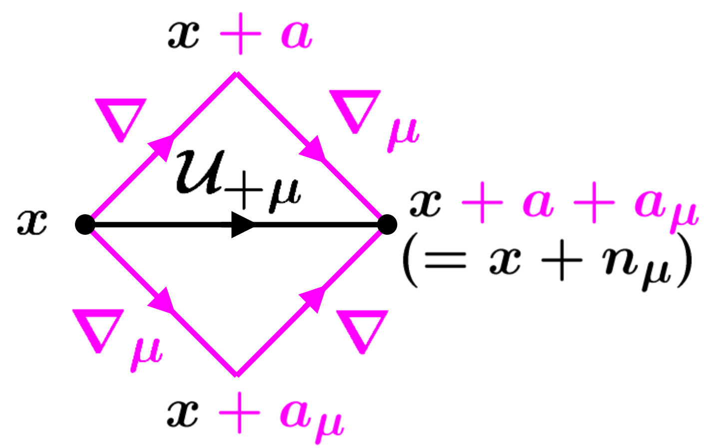
$$\{\tilde{\nabla}, \nabla_\mu\}_{x+\tilde{a}+a_\mu,x} = +i\epsilon_{\mu\nu}\ (\mathcal{U}_{-\nu})_{x-n_\nu,x}$$

$$\{others\} = 0$$

$$\{\nabla, \nabla_\mu\}_{x+a+a_\mu,x}$$

$$\equiv (\nabla)_{x+a+a_\mu,x+a_\mu}(\nabla_\mu)_{x+a_\mu,x}$$

$$+(\nabla_\mu)_{x+a+a_\mu,x+a}(\nabla)_{x+a,x}$$



"Shifted" Anti-commutator

$$\left(\because\ a+a_\mu = +n_\mu \ \vdots \right)$$

Jacobi Identities

$$[\nabla_\mu\{\nabla_\nu, \nabla\}]_{x+a_\mu+n_\nu,x} + (cyclic) = 0,$$

$$[\nabla_\mu, \mathcal{U}_{+\nu}]_{x+a_\mu+n_\nu,x} + [\nabla_\nu, \mathcal{U}_{+\mu}]_{x+a_\nu+n_\mu,x} \ = \ 0,$$

$\vdots$

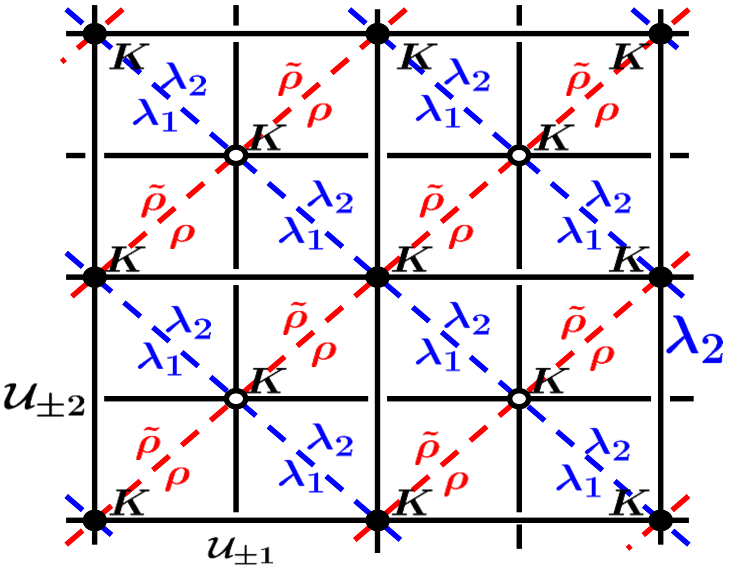
Define fermionic link components

$$[\nabla_\mu, \mathcal{U}_{+\nu}]_{x+a_\mu+n_\nu,x} \equiv -\epsilon_{\mu\nu}(\tilde{\rho})_{x-\tilde{a},x} \ ,$$

$\vdots$

Auxiliary Field

$$K = \frac{1}{2}\{\nabla_\mu, \lambda_\mu\}$$



Twisted N=2 Lattice SUSY Transformation

Shifts of Fields

$$s_A(\varphi) \equiv [\nabla_A, \varphi\}_{x+a_A+a_\varphi,x}$$

∇	$\tilde{\nabla}$	∇_μ	$\mathcal{U}_{\pm\mu}$	ρ	$\tilde{\rho}$	λ_μ	K
a	$\tilde{a}$	a_μ	$\pm n_\mu$	$-a$	$-\tilde{a}$	$-a_\mu$	0

	s	$\tilde{s}$	s_μ
$\mathcal{U}_{+\nu}$	0	$+\epsilon_{\nu\rho}\lambda_\rho$	$-\epsilon_{\mu\nu}\tilde{\rho}$
$\mathcal{U}_{-\nu}$	$-\lambda_\nu$	0	$-\delta_{\mu\nu}\rho$
λ_ν	0	0	$-i[\mathcal{U}_{+\mu}, \mathcal{U}_{-\nu}] + \delta_{\mu\nu}(K + \frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}])$
ρ	$-\frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}] - K$	$+\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho}, \mathcal{U}_{-\sigma}]$	0
$\tilde{\rho}$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{+\rho}, \mathcal{U}_{+\sigma}]$	$+\frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}] - K$	0
K	$+\frac{i}{2}[\mathcal{U}_{+\rho}, \lambda_\rho]$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho}, \lambda_\sigma]$	$-\frac{i}{2}[\mathcal{U}_{+\mu}, \rho] - \frac{i}{2}\epsilon_{\mu\nu}[\mathcal{U}_{-\nu}, \tilde{\rho}]$

Twisted SUSY Algebra closes off-shell

$$\begin{aligned}
\{s, s_\mu\}(\varphi)_{x+a_\varphi,x} &= +i[\mathcal{U}_{+\mu}, \varphi]_{x+a_\varphi+n_\mu,x} \\
\{\tilde{s}, s_\mu\}(\varphi)_{x+a_\varphi,x} &= +i\epsilon_{\mu\nu}[\mathcal{U}_{-\nu}, \varphi]_{x+a_\varphi-n_\nu,x} \\
s^2(\varphi)_{x+a_\varphi,x} &= \tilde{s}^2(\varphi)_{x+a_\varphi,x} = 0 \\
\{s, \tilde{s}\}(\varphi)_{x+a_\varphi,x} &= \{s_\mu, s_\nu\}(\varphi)_{x+a_\varphi,x} = 0
\end{aligned}$$

Twisted D=N=2 Super Yang-Mills Action

Action has twisted SUSY exact form. → **Off-shell SUSY invariance for all twisted super charges.**

$$S \equiv \frac{1}{2}\sum_x \mathrm{Tr}\ s\tilde{s}s_1 s_2\ \mathcal{U}_{+\mu}\mathcal{U}_{-\mu}$$

$$= S_B + S_F$$

$$S_B = \sum_x \mathrm{Tr}\Big[\frac{1}{4}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x,x} + K^2_{x,x}$$

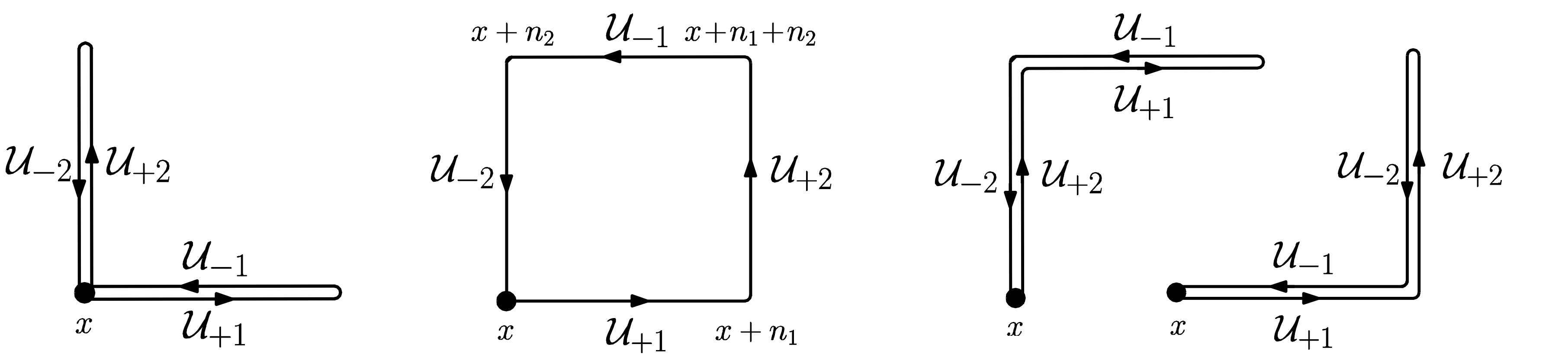
$$-\frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x-n_\mu-n_\nu}[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]_{x-n_\rho-n_\sigma,x}\Big]$$

$$S_F = \sum_x \mathrm{Tr}\Big[-i[\mathcal{U}_{+\mu},\lambda_\mu]_{x,x-a}(\rho)_{x-a,x}$$

$$- i(\tilde{\rho})_{x,x+\tilde{a}}\epsilon_{\mu\nu}[\mathcal{U}_{-\mu},\lambda_\nu]_{x+\tilde{a},x}\Big]$$

Bosonic part of the Action

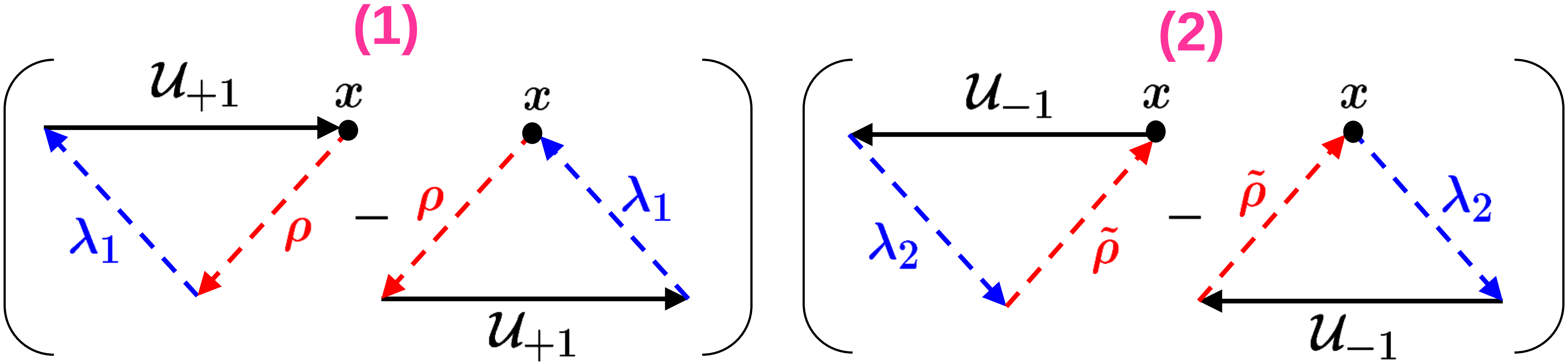
$$\begin{aligned} S_B &= \sum_x \mathrm{Tr}\Big[\frac{1}{4}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x,x} + K_{x,x}^2 \\ &\quad -\frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x-n_\mu-n_\nu}[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]_{x-n_\rho-n_\sigma,x}\Big] \end{aligned}$$



Fermionic part of the Action

$$S_F = \sum_x \mathrm{Tr}\Big[-i[\mathcal{U}_{+\mu}, \lambda_\mu]_{x,x-a} (\rho)_{x-a,x} \qquad \cdots (1)$$

$$- i(\tilde{\rho})_{x,x+\tilde{a}} \epsilon_{\mu\nu} [\mathcal{U}_{-\mu}, \lambda_\nu]_{x+\tilde{a},x} \Big] \qquad \cdots (2)$$



Higher dimensional extension is possible:

Twisted D=3,N=4 Super Yang-Mills Action

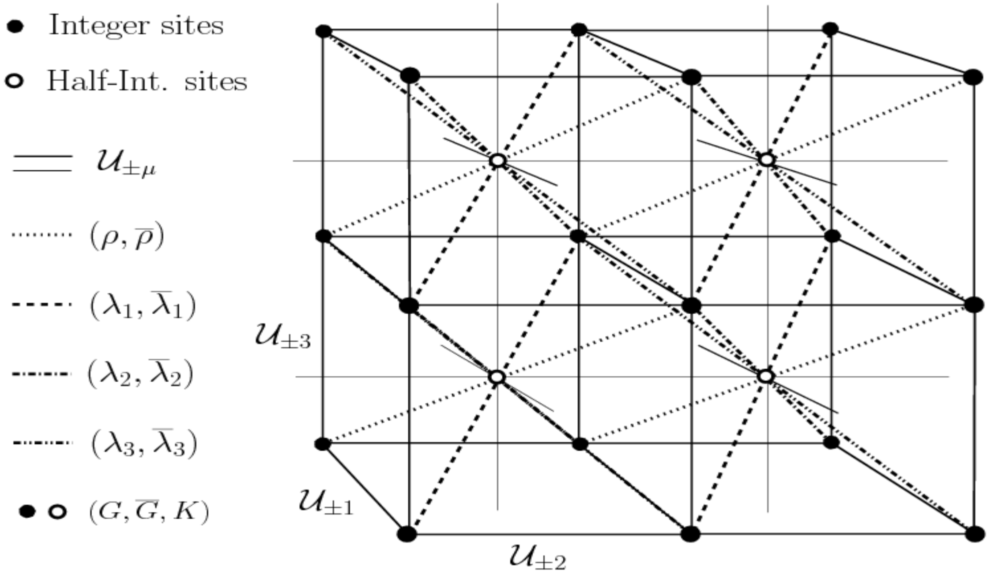
$$S = +\sum_x \frac{1}{2}\bar{s}_1\bar{s}_2 s_1 s_2 \; tr \; U_{+3} \; U_{+3} \;\; = \;\; -\sum_x \frac{1}{2} s\bar{s}_3\bar{s}s_3 \; tr \; U_{-3} \; U_{-3}$$

$$= \sum_x tr \Big[\; \frac{1}{4}[U_{+\mu}, U_{-\mu}]_{x,x}[U_{+\nu}, U_{-\nu}]_{x,x} + K^2_{x,x}$$

$$-\frac{1}{2}[U_{+\mu}, U_{+\nu}]_{x,x-n_\mu-n_\nu}[U_{-\mu}, U_{-\nu}]_{x-n_\mu-n_\nu,x} + G_{x,x+\bar{a}-a}\overline{G}_{x+\bar{a}-a,x}$$

$$+i(\overline{\lambda}_\mu)_{x,x+\bar{a}_\mu}[U_{+\mu}, \rho]_{x+\bar{a}_\mu,x} + i(\lambda_\mu)_{x,x+a_\mu}[U_{+\mu}, \overline{\rho}]_{x+a_\mu,x} + \epsilon_{\mu\nu\rho}(\lambda_\mu)_{x,x+a_\mu}[U_{-\nu}, \overline{\lambda}_\rho]_{x+a_\mu,x} \Big]$$

Higer dimensional extension is possible:



3 dimensions

3-dim. N=4 super Yang-Mills

Noncommutativity needed for link approach

$$Q_A(\phi_1(x)\phi_2(x)) = (Q_A\phi_1(x))\phi_2(x) + \phi_1(x+a_A)Q_A\phi_2(x)$$

$$Q_A(\phi_2(x)\phi_1(x)) = (Q_A\phi_2(x))\phi_1(x) + \phi_2(x+a_A)Q_A\phi_1(x)$$

Bruckmann
Kok

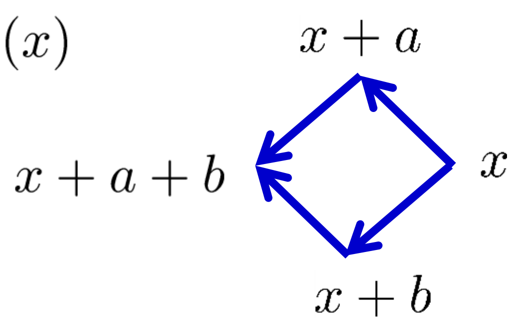
When $\phi_1(x)\phi_2(x) = \phi_2(x)\phi_1(x)$ "inconsistency ?"

but if we introduce the following "mild non-commutativity":

$$(Q_A\phi_i(x))\phi_j(x) = \phi_j(x+a_A)(Q_A\phi_i(x)) \quad i,j = 1,2$$

then $$Q_A(\phi_1(x)\phi_2(x)) = Q_A(\phi_2(x)\phi_1(x))$$

In general $$\phi_a(x+b)\phi_b(x) = \phi_b(x+a)\phi_a(x)$$



Algebraic consistency of Link Approach

1) Modified Leibniz rule:

$$\left(\Delta_{+\mu}\Phi\right)(x) = \Delta_{+\mu}\Phi(x) - \Phi(x+n_{\mu})\Delta_{+\mu}$$

$$s_A\Phi(x,\theta) = Q_A\Phi(x,\theta) - \Phi(x+a_A,\theta)Q_A$$

2) Shifted anti-commutators

$$\{\nabla, \nabla_{\mu}\}_{x+a+a_{\mu},x} \equiv (\nabla)_{x+a+a_{\mu},x+a_{\mu}}(\nabla_{\mu})_{x+a_{\mu},x} + (\nabla_{\mu})_{x+a+a_{\mu},x+a}(\nabla)_{x+a,x}$$

3) non-commutativity

$$(Q_A\phi_i(x))\phi_j(x) = \phi_j(x+a_A)(Q_A\phi_i(x)) \quad i,j=1,2$$

Hopf algebraic consistency

(D'Adda, N.K., Saito, 2009)

Lattice super algebra as Hopf Algebra

Anzats: How do we look at modified Leibniz rule ?

$$Q_A(\phi_1(x)\varphi_2(x)) = Q_A\varphi_1(x)\varphi_2(x+a_A) + (-1)^{|\varphi|}\varphi_1(x+a_A)Q_A\varphi_2(x)$$

multiplication $\quad m(\varphi_1(x) \times \varphi_2(x)) = \varphi_1(x)\cdot\varphi_2(x)$

operation $\quad Q_A| > \varphi(x) = (Q_A\varphi)(x)$

co derivative $\quad \Delta(Q_A) = Q_A \times T_{a_A} + (-1)^F T_{a_A} \times Q_A$

$(T_{a_A}\varphi(x) = \varphi(x+a_A))$

braiding $\quad \Psi(\phi_1 \times \phi_2) = \phi_2 \times \phi_1$

$$Q_A| > (\varphi_1(x)\cdot\varphi_2(x)) = m(\Delta(Q_A)| > (\varphi_1(x) \times \varphi_2(x))$$

Summary of Link Approach

1) For D=N=2, three dimensiona space is suggested for the lattice space. In fact fermionic links are not in the space.

2) Totally different derivation of Kaplan's exact lattice SUSY (a=0). No orbifold condition used.

3) Hopf algebraic exact lattice SUSY invariance is realized.

Remaining one problem in this approach

After ∇_A operation to a gauge invariant quantity, gauge variant terms appear ?

$$\nabla_A(\phi_1 \cdots \phi_n)_{x,x} = (\nabla_A \phi_1 \cdots \phi_n)_{x,x+a_A}$$

gauge variant ?

$$(\mathcal{U}_{\pm\mu})' = G_{x\pm n_\mu}(\mathcal{U}_{\pm\mu})G_x^{-1}$$

$$((\nabla_A)' = G_{x\pm n_\mu}(\nabla_A)G_x^{-1})$$

$${\nabla_A}' = \nabla_A$$

No-gauge invariance in the fermionic direction

"since it is the extra dimention ? "

B) Super doubler approach

Difficulties

Solutions

(1) No Leibniz rule
in coordinate space

$$Q^2 = \hat{P} = i\hat{\partial}$$

→

algebraic construction
with lattice momentum

$$Q^2 = \frac{2}{a}\sin\frac{ap}{2} = \hat{p}$$

$$\delta(\hat{p_1} + \hat{p_2} \cdots)$$

new $*$ product
Leibniz rule on $*$ product

(2) doublers of chiral fermion → Doublers as
super partners

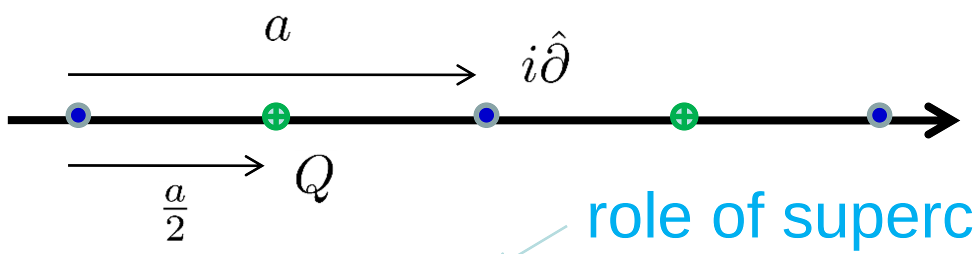
No chiral fermion problem !

Basic Idea

The simplest example (D=N=1)

$Q^2 = i\hat{\partial}$ → translation generator of a

↓ half translation generator $\frac{a}{2}$



$$x = n\frac{a}{2} \leftrightarrow \frac{2x}{a} = n$$

role of supercoordinate

$$\Phi(x) = \varphi(x) + \frac{\sqrt{a}}{2}\boxed{(-1)^{\frac{2x}{a}}}\,\psi(x) = \begin{cases} \varphi(x) & (x = \frac{na}{2}) \\ \frac{\sqrt{a}}{2}(-1)^{\frac{2x}{a}}\psi(x) & (x = \frac{na}{2} + \frac{a}{4}) \end{cases}$$

$$\delta\Phi(x) = a^{-\frac{1}{2}}\alpha(-1)^{\frac{2x}{a}}\left\{\Phi(x+\tfrac{a}{4}) - \Phi(x-\tfrac{a}{4})\right\} = \delta\varphi(x) + \frac{\sqrt{a}}{2}(-1)^{\frac{2x}{a}}\delta\psi(x)$$

$$\delta\varphi(x) = \frac{i\alpha}{2}\left[\psi(x+\tfrac{a}{4}) + \psi(x-\tfrac{a}{4})\right] \rightarrow i\alpha\psi(x)$$

$$\delta\psi(x) = 2a^{-1}\alpha\left[\varphi(x+\tfrac{a}{4}) - \varphi(x-\tfrac{a}{4})\right] \rightarrow \alpha\frac{\partial\varphi(x)}{\partial x}$$

D=1 N=2 Lattice SUSY

$$e^{i\frac{2\pi}{a}x}$$

$$\frac{1}{2}\sum_{x=\frac{na}{2}+\frac{a}{4}} e^{ipx}(-1)^{\frac{2x}{a}}\psi(x) = \psi\left(p+\frac{2\pi}{a}\right) = -\psi\left(p-\frac{2\pi}{a}\right)$$

alternating sign → species doubler

$$\delta_1\Phi(p) = i\cos\frac{ap}{4}\alpha\Psi(p)$$

$$\delta_1\Psi(p) = -4i\sin\frac{ap}{4}\alpha\Phi(p)$$

$$\Psi(p) \rightarrow -i\Psi\left(\frac{2\pi}{a}-p\right)$$

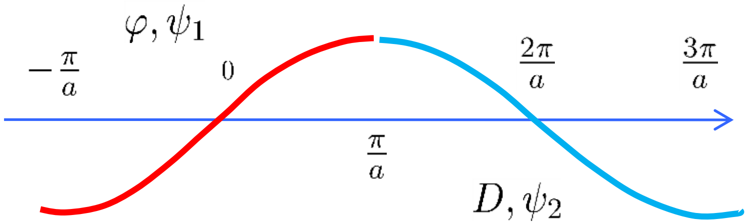
$$\delta_2\Phi(p) = \cos\frac{ap}{4}\alpha\Psi\left(\frac{2\pi}{a}-p\right)$$

$$\delta_2\Psi\left(\frac{2\pi}{a}-p\right) = 4\sin\frac{ap}{4}\alpha\Phi(p)$$

N=2 lattice SUSY algebra

$$\delta_1 = \alpha Q_1, \qquad \delta_2 = \alpha Q_2$$

$$Q_1^2 = Q_2^2 = 2\sin\frac{ap}{2}, \qquad \{Q_1, Q_2\} = 0$$



Lattice super derivative

$$\cos\frac{ap}{4} \;\rightarrow\; \cos\frac{ap}{2}\cos\frac{ap}{4}, \qquad \sin\frac{ap}{4} \;\rightarrow\; -\cos\frac{ap}{2}\sin\frac{ap}{4}$$

$$\begin{cases} D_1\Phi(p) = i\cos\frac{ap}{2}\cos\frac{ap}{4}\Psi(p) \\ D_1\Psi(p) = 4i\cos\frac{ap}{2}\sin\frac{ap}{4}\Phi(p) \end{cases} \qquad \begin{cases} D_2\Phi(p) = \cos\frac{ap}{2}\cos\frac{ap}{4}\Psi(\frac{2\pi}{a}-p) \\ D_2\Psi(\frac{2\pi}{a}-p) = 4\cos\frac{ap}{2}\sin\frac{ap}{4}\Phi(p) \end{cases}$$

Chiral lattice SUSY algebra (D=1,N=2)

$$Q_\pm = \frac{1}{2}(Q_1 \pm iQ_2) \qquad\qquad D_\pm = \frac{1}{2}(D_1 \pm iD_2)$$

$$\{Q_+, Q_-\} = 2\sin\frac{ap}{2}, \quad Q_+^2 = Q_-^2 = 0$$

$$\{D_+, D_-\} = -2\underline{\cos^2\frac{ap}{2}}\sin\frac{ap}{2}, \quad D_+^2 = D_-^2 = 0$$

$$\boxed{\{Q_\pm, D_\pm\} = \{Q_\pm, D_\mp\} \;=\; 0}$$

No influence to the cont. limit

Chiral conditions

truncation of
species doub. d.o.f.

$$D_-\Phi(p) = i\cos\tfrac{ap}{2}\cos\tfrac{ap}{4}\tfrac{1}{2}\left\{\Psi(p) - \Psi\left(\tfrac{2\pi}{a}-p\right)\right\} = 0$$

$$D_-\Psi(p) = i\sin ap\ \ \tfrac{1}{2}\left\{\frac{\Phi(p)}{\cos\frac{ap}{4}} - \frac{\Phi\left(\frac{2\pi}{a}-p\right)}{\sin\frac{ap}{4}}\right\} = 0$$

rescaled field !
meaning ?

$$\phi(p) \equiv \frac{\Phi(p)}{\cos\frac{ap}{4}}$$

$$\phi(p) - \phi\left(\tfrac{2\pi}{a}-p\right) = 0$$

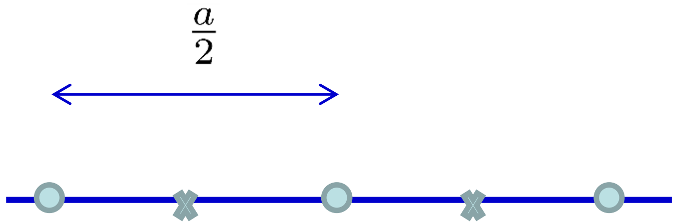
$$2\phi^{(s)}_{(a)}(p) = \phi(p) \pm \phi\left(\tfrac{2\pi}{a}-p\right)$$

$$2\Psi^{(s)}_{(a)}(p) = \Psi(p) \pm \Psi\left(\tfrac{2\pi}{a}-p\right)$$

		Q_+	Q_-
chiral	$\phi^{(s)}(p)$	$i\Psi^{(s)}(p)$	0
	$\Psi^{(s)}(p)$	0	$-2i\sin\frac{ap}{2}\phi^{(s)}(p)$
anti-chiral	$\phi^{(a)}(p)$	0	$i\Psi^{(a)}(p)$
	$\Psi^{(a)}(p)$	$-2i\sin\frac{ap}{2}\phi^{(a)}(p)$	0

⟸ $D_+\Phi = D_+\Psi = 0$

The meaning of $\phi(p) \equiv \frac{\Phi(p)}{\cos\frac{ap}{4}}$



$$\Phi(p) = \sum_x e^{ipx}\Phi(x)$$

$$(x = n\tfrac{a}{2})$$

$$\Phi(p+\tfrac{4\pi}{a}) = \sum_x e^{ipx} e^{i\frac{4\pi}{a}n\frac{a}{2}}\Phi(x) = \sum_x e^{ipx}\Phi(x) = \Phi(p)$$

$$(x = n\tfrac{a}{2}+\tfrac{a}{4})$$

$$\Phi(p+\tfrac{4\pi}{a}) = \sum_x e^{ipx} e^{i\frac{4\pi}{a}(n\frac{a}{2}+\frac{a}{4})}\Phi(x) = -\sum_x e^{ipx}\Phi(x) = -\Phi(p)$$

$$\phi(p+\tfrac{4\pi}{a}) = \frac{\Phi(p+\frac{4\pi}{a})}{\cos\frac{a}{4}(p+\frac{4\pi}{a})} = -\frac{\Phi(p)}{\cos\frac{ap}{4}} = -\phi(p) \qquad (\Phi(p+\tfrac{4\pi}{a}) = \Phi(p))$$

$$\Longrightarrow \quad x = n\tfrac{a}{2}+\tfrac{a}{4}$$

$\phi(p) \equiv \frac{\Phi(p)}{\cos\frac{ap}{4}}$ is shifted $\frac{a}{4}$ from $\Phi(p)$ in the coordinate

Exact Lattice SUSY action for N=2 D=1

Super charge exact form ⟹ exact lattice SUSY invariant

$$\{Q_+, Q_-\} = 2\sin\tfrac{ap}{2}, \quad Q_+^2 = Q_-^2 = 0$$

$$S = \int d\hat{p}_1 \cdots d\hat{p}_n \delta(\hat{p}_1 + \cdots + \hat{p}_n)\left(\prod_{j=2}^n \cos\tfrac{ap_j}{2}\right) Q_+ Q_- \{\phi^{(a)}(p_1)\phi^{(s)}(p_2)\cdots\phi^{(s)}(p_n)\}$$

$$= \int d\hat{p}_1 \cdots d\hat{p}_n \delta(\hat{p}_1 + \cdots + \hat{p}_n) \frac{2}{\sin\frac{ap_1}{4}}\left(\prod_{j=2}^n \frac{\cos\frac{ap_j}{2}}{\cos\frac{ap_j}{4}}\right)$$

$$\{2\sin^2\tfrac{ap_1}{4}\,\Phi(p_1)\Phi(p_2)\cdots\Phi(p_n) + \tfrac{n-1}{4}\sin\tfrac{a(p_1-p_2)}{4}\Psi(p_1)\Psi(p_2)\Phi(p_3)\cdots\Phi(p_n)\}$$

↑ lattice momentum conservation

$$\hat{p}_i = 2\sin\tfrac{ap}{2} \qquad d\hat{p}_i = a\,dp\cos\tfrac{ap_i}{2} \qquad \text{integration range } [-\tfrac{\pi}{a}, \tfrac{3\pi}{a}]$$

$n = 4$

$$2\sin^2\tfrac{ap_1}{4}\;\Phi(p_1)\Phi(p_2)\Phi(p_3)\Phi(p_4) + \tfrac{3}{4}\sin\tfrac{a(p_1-p_2)}{4}\Psi(p_1)\Psi(p_2)\Phi(p_3)\Phi(p_4)$$

$$\sim \varphi^4 + \boxed{\varphi^3 D} + \varphi^2 D^2 + \varphi D^3 + D^4 + \boxed{\psi_1\psi_2\varphi^2} + \psi_1\psi_2\varphi D + \psi_1\psi_2 D^2$$

In the continuum only these terms appear !

$$[\psi_1] = [\psi_2] = 0, \quad [\varphi] = L^{\frac{1}{2}}, \quad [D] = L^{-\frac{1}{2}}$$

New $*$ product and Leibniz rule (coordinate rep.)

New star $*$ product

$$\hat{p} = 2\sin\frac{ap}{2}$$

$$(F * G)(p) = \int d\hat{p}_1 d\hat{p}_2 F(p_1) G(p_2) \delta(\hat{p} - \hat{p}_1 - \hat{p}_2)$$

$$(F * G)(x) = F(x) * G(x) = \int d\hat{p}\ e^{-ipx}\ (F * G)(p)$$

$$(x = n\tfrac{a}{2}, y = m\tfrac{a}{2}, z = l\tfrac{a}{2})$$

$$= \frac{2}{a}\int_{-\infty}^{\infty} d\tau J_{n\pm1}(\tau) \sum_{m,l} J_{m\pm1}(\tau) J_{l\pm1}(\tau) F(y) G(z)$$

$$J_n(\tau) = \frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha} e^{i(n\theta - \tau\sin\theta)} d\theta$$

Leibniz rule in lattice momentum space

$$\hat{p}\ (F * G)(p) = \int\ d\hat{p}_1\ d\hat{p}_2 \left[\hat{p}_1\ F(p_1)\ G(p_2) + F(p_1)\ \hat{p}_2\ G(p_2)\right] \delta(\hat{p} - \hat{p}_1 - \hat{p}_2)$$

Leibniz rule on $*$ product (coordinate rep.)

$$i\hat{\partial}(F(x) * G(x)) = (i\hat{\partial}F(x)) * G(x) + F(x) * (i\hat{\partial}G(x))$$

N=2 Wess-Zumino model in two dimensions

$$\gamma^\mu = (\sigma^3, \sigma^1)$$

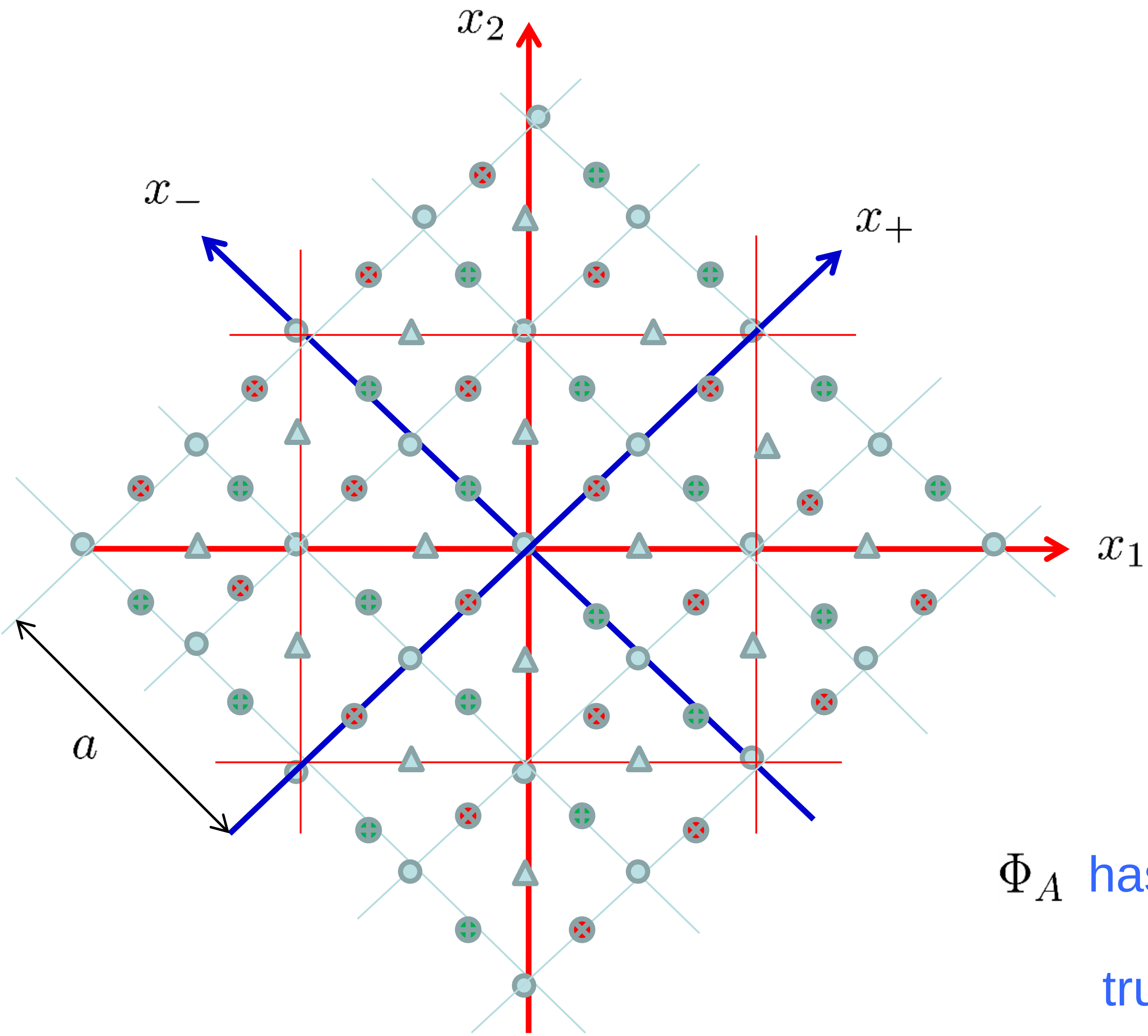
N=D=2 algebra: $\{Q_{\alpha i}, Q_{\beta j}\} \ = \ 2\delta_{ij}(\gamma^\mu)_{\alpha\beta}P_\mu$

Light cone coordinate

$$\{Q_+^{(+)}, Q_+^{(-)}\} = P_+, \qquad \{Q_-^{(+)}, Q_-^{(-)}\} = P_-,$$

$$(Q_+^{(+)})^2 = (Q_+^{(-)})^2 = 0 \qquad (Q_-^{(+)})^2 = (Q_-^{(-)})^2 = 0$$

2-dim. = (1 dim.) x (1 dim.)



$\Phi_A \quad x = (x_+, x_-)$

Φ ● $\left(n\frac{a}{2}, m\frac{a}{2}\right)$

Ψ_1 ● $\left(n\frac{a}{2} + \frac{a}{4}, m\frac{a}{2}\right)$

Ψ_2 ● $\left(n\frac{a}{2}, m\frac{a}{2} + \frac{a}{4}\right)$

F ▲ $\left(n\frac{a}{2} + \frac{a}{4}, m\frac{a}{2} + \frac{a}{4}\right)$

$n, m \in Z$

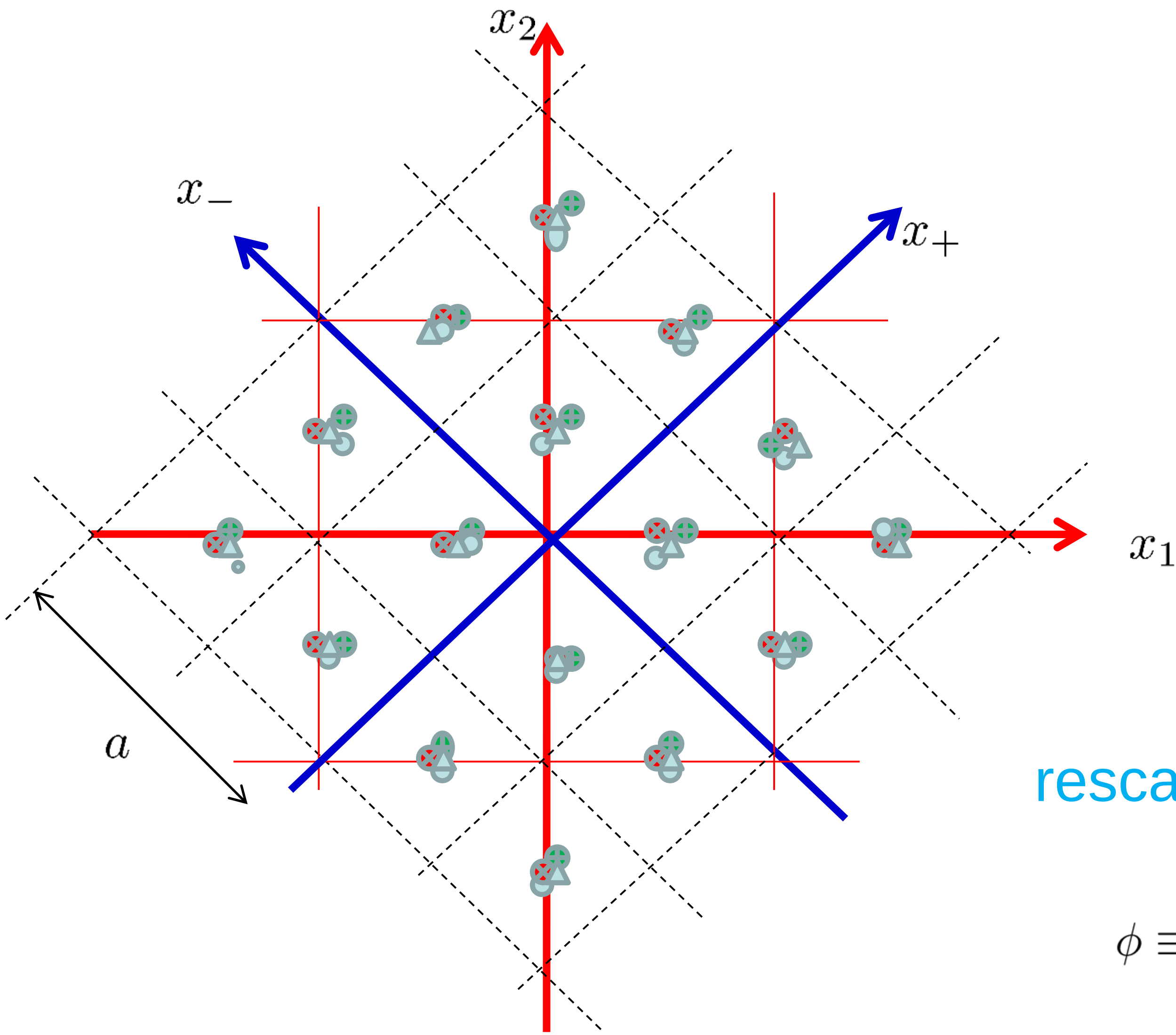
Φ_A has 4 species doublers

⬇

truncation needed

⬇

chiral conditions



$$(n\tfrac{a}{2}+\tfrac{a}{4}, m\tfrac{a}{2}+\tfrac{a}{4})$$

$$n, m \in Z$$

$$x = (x_+, x_-)$$

rescaling of fields

$$\phi \equiv \frac{\Phi(p_+,p_-)}{\cos\frac{ap_+}{4}\cos\frac{ap_-}{4}}$$

$$\psi_1 \equiv \frac{\Psi_1(p_+,p_-)}{\cos\frac{ap_-}{4}}, \quad \psi_2 \equiv \frac{\Psi_2(p_+,p_-)}{\cos\frac{ap_+}{4}},$$

D=N=2 lattice SUSY transformation

Chiral

	$Q_+^{(+)}$	$Q_+^{(-)}$	$Q_-^{(+)}$	$Q_-^{(-)}$
$\phi(p)$	$i\psi_1(p)$	0	$i\psi_2(p)$	0
$\psi_1(p)$	0	$-2i\sin\frac{ap_+}{2}\phi(p)$	$-F(p)$	0
$\psi_2(p)$	$F(p)$	0	0	$-2i\sin\frac{ap_-}{2}\phi(p)$
$F(p)$	0	$2\sin\frac{ap_+}{2}\psi_2(p)$	0	$-2\sin\frac{ap_-}{2}\psi_1(p)$

Anti-chiral

	$Q_+^{(+)}$	$Q_+^{(-)}$	$Q_-^{(+)}$	$Q_-^{(-)}$
$\bar{\phi}(p)$	0	$i\bar{\psi}_1(p)$	0	$i\bar{\psi}_2(p)$
$\bar{\psi}_1(p)$	$-2i\sin\frac{ap_+}{2}\bar{\phi}(p)$	0	0	$-\bar{F}(p)$
$\bar{\psi}_2(p)$	0	$\bar{F}(p)$	$-2i\sin\frac{ap_-}{2}\bar{\phi}(p)$	0
$\bar{F}(p)$	$2\sin\frac{ap_+}{2}\bar{\psi}_2(p)$	0	$-2\sin\frac{ap_-}{2}\bar{\psi}_1(p)$	0

Wess-Zumino action in two dimensions

Super charge exact form ⟹ exact lattice SUSY inv.

Kinetic term

$$S_K = \int d\hat{p}_+ d\hat{p}_- d\hat{q}_+ d\hat{q}_- \delta(\hat{p}_+ + \hat{q}_+)\delta(\hat{p}_- + \hat{q}_-) Q_+^{(-)} Q_-^{(-)} Q_+^{(+)} Q_-^{(+)} \{\bar{\phi}(p)\phi(q)\}$$

$$= \int d\hat{p}_+ d\hat{p}_- d\hat{q}_+ d\hat{q}_- \delta(\hat{p}_+ + \hat{q}_+)\delta(\hat{p}_- + \hat{q}_-)$$

$$[-4\bar{\phi}(p) \sin \tfrac{aq_+}{2} \sin \tfrac{aq_-}{2} \phi(q) - \bar{F}(p)F(q)$$

$$+2\bar{\psi}_2(p) \sin \tfrac{aq_+}{2} \psi_2(q) + 2\bar{\psi}_1 \sin \tfrac{aq_-}{2} \psi_1(q)]$$

Interaction term

$$S_I = \int d^2\hat{p}_1 \cdots d^2\hat{p}_n \delta^{(2)}(\hat{p}_1 + \cdots + \hat{p}_n) Q_+^{(+)} Q_-^{(+)} \{\phi(p_1)\phi(p_2)\cdots\phi(p_n)\}$$

$$= [\textstyle\sum_{j=1}^n iF(p_j) \prod_{l(\neq j)} \phi(p_l) + \sum_{j,k;j\neq k} \psi_2(p_j)\psi_1(p_k) \prod_{l(\neq j,k)} \phi(p_l)]$$

N=2 Wess-Zumino actions in coordinate

$*$ product actions in two dimensions

Kinetic term

$$S_K = \sum_{x_+,x_-}\{-4\bar{\phi}(x)*\partial_+\partial_-\phi(x) - \bar{F}(x)*F(x) + 2\bar{\psi}_2(x)*\partial_+\psi_2(x) + 2\bar{\psi}_1(x)*\partial_-\psi_1(x)\}$$

Interaction term

$$S_I = \sum_{x_+,x_-}\{iF(x)*(\phi(x))^{n-1} + \psi_2(x)*\psi_1(x)*(\phi(x))^{n-2}\}$$

SUSY algebra with Leibniz rule is satisfied on $*$ product !

1) Is the SUSY realized in the quantum level ?

2) How does the non-local nature influence to the physical quantities ?

Check of the Ward-Takahashi identities

3) Does the translational invariance recover in the continuum limit ?

Numerical check !

Exact lattice SUSY at the quantum level

Asaka, D'Adda, N.K. Kondo (2013)

One loop Ward-Takahashi Identity:

Example:

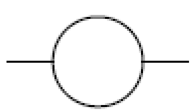
$$< \psi_1(p)\overline{\psi}_1(-p) >_{1-loop} + \hat{p}_+ < \phi(p)\overline{\phi}(-p) >_{1-loop}$$

$$= [< \psi_1(p)\overline{\psi}_1(-p) >_{tree} + \hat{p}_+ < \phi(p)\overline{\phi}(-p) >_{tree}] A^{(N)}_{1-loop}(p) = 0$$

$$< \psi_1(p)\overline{\psi}_1(-p) >_{tree} = \frac{\hat{p}_+}{D(\hat{p})} \qquad < \phi(p)\overline{\phi}(-p) >_{tree} = \frac{-1}{D(\hat{p})}$$

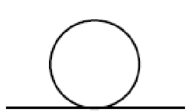
$$D(\hat{p}) = \hat{p}_+\hat{p}_- - m^2 \qquad (\hat{p}_\pm = \frac{2}{a} sin \frac{ap_\pm}{2})$$

Φ^3



$$A^{(3)}_{1-loop} = -2g_3^2 \frac{\hat{p}_+\hat{p}_- + m^2}{D(\hat{p})} \int d^2\hat{k} \frac{1}{D(\hat{k})D(\hat{p}-\hat{k})}$$

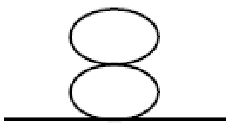
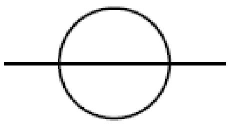
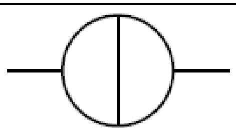
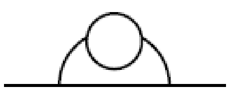
Φ^4



$$A^{(4)}_{1-loop}(p) = 0$$

$$< \psi_1(p)\overline{\psi}_1(-p) >_{1-loop} + \hat{p}_+ < \phi(p)\overline{\phi}(-p) >_{1-loop}$$

$$= [< \psi_1(p)\overline{\psi}_1(-p) >_{tree} + \hat{p}_+ < \phi(p)\overline{\phi}(-p) >_{tree}] X(\hat{p}) = 0$$

Loop diagram	$X(\hat{p})$
	1
	0
	$-6g_4^2 \frac{\hat{p}_+\hat{p}_- + m^2}{D(p)} I_2$
	$16m^2 g_3^4 \frac{2\hat{p}_+\hat{p}_- + m^2}{D(\hat{p})} I_3$
	$8g_3^4 \frac{\hat{p}_+\hat{p}_- + m^2}{D(\hat{p})} I_4$

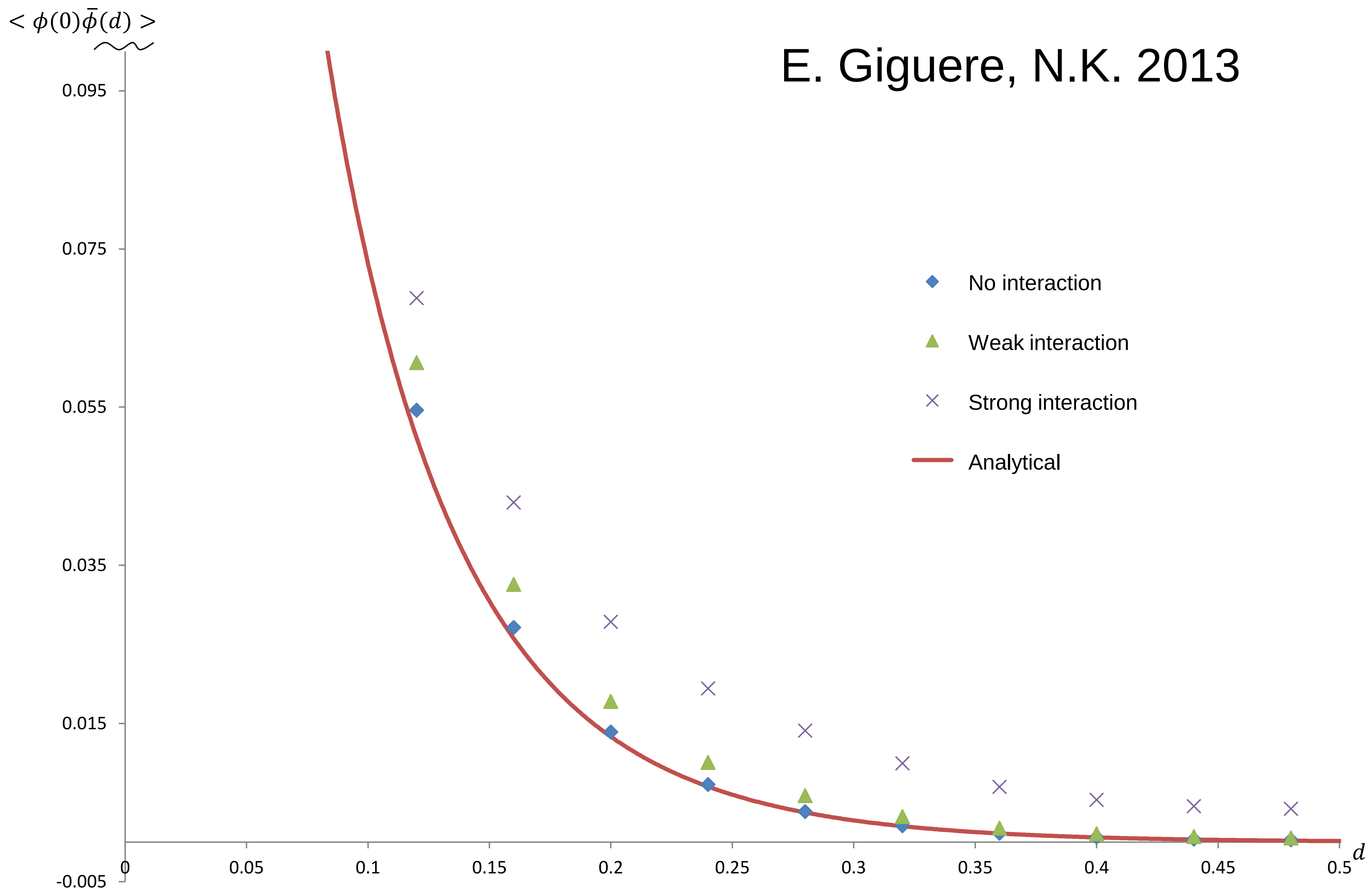
$$I_2 = \int \frac{d\hat{k}_1^2}{(2\pi)^2} \frac{d\hat{k}_2^2}{(2\pi)^2} \frac{1}{D(\hat{k}_1)D(\hat{k}_2)D(\hat{p} - \hat{k}_1 - \hat{k}_2)},$$

$$I_3 = \int \frac{d\hat{k}_1^2}{(2\pi)^2} \frac{d\hat{k}_2^2}{(2\pi)^2} \frac{1}{D(\hat{k}_1)D(\hat{k}_2)D(\hat{k}_1 + \hat{p})D(\hat{k}_2 + \hat{p})D(\hat{k}_1 - \hat{k}_2)},$$

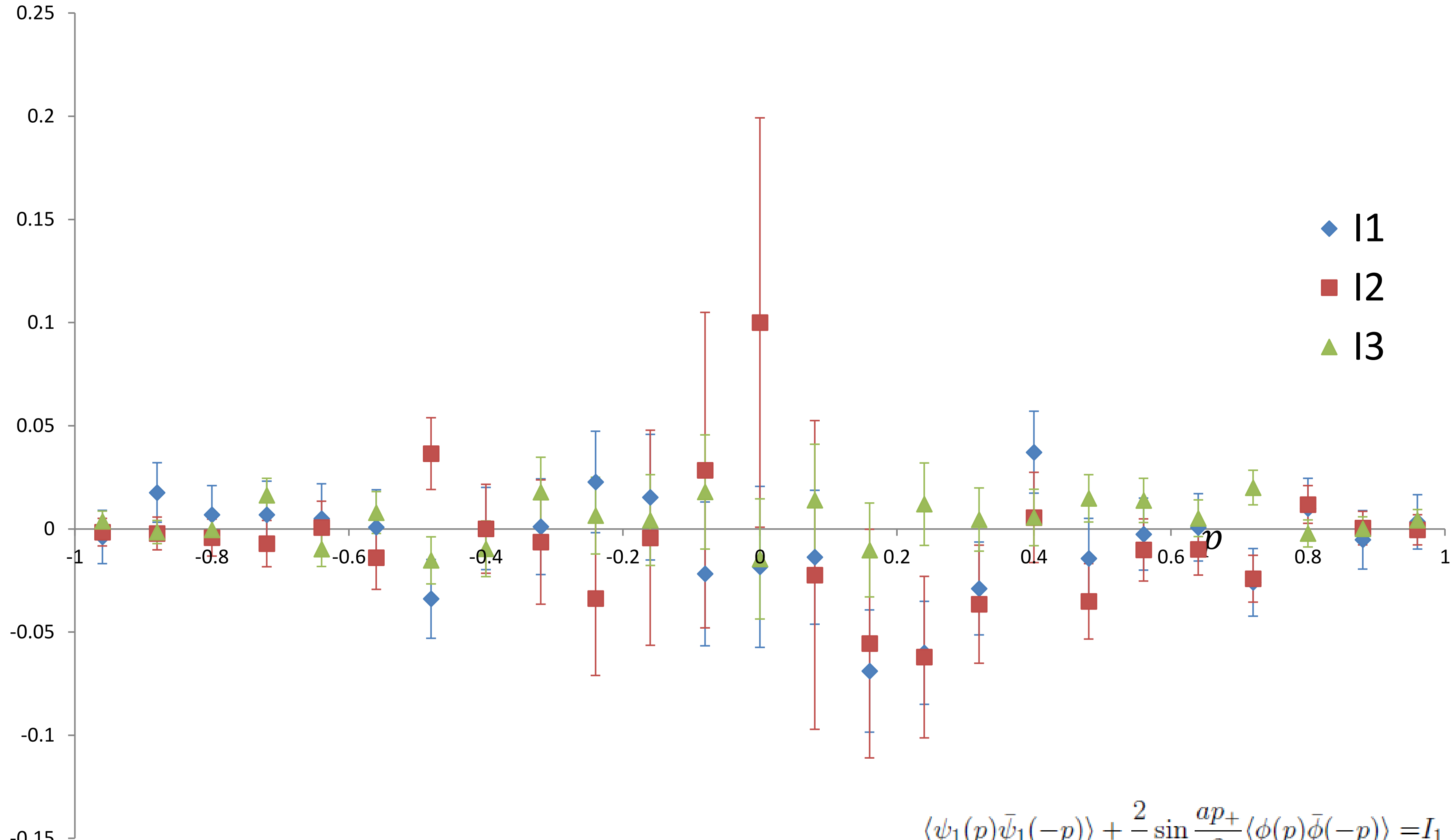
$$I_4 = \int \frac{d^2\hat{k}_1 d^2\hat{k}_2}{(2\pi)^2(2\pi)^2} \frac{\hat{k}_1^2 + m^2}{D(\hat{k}_1)^2 D(\hat{k}_2)} \int \frac{d^2\hat{k}}{(2\pi)^2} \frac{1}{D(\hat{k})D(\hat{k}_1 - \hat{k})}$$

loop W.I. $\sim$ tree W.I.

2 points functions for 2d Wess-Zumino



Ward-Takashi Identity for 2d Wess-Zumino



$$\langle \psi_1(p)\bar{\psi}_1(-p)\rangle + \frac{2}{a}\sin\frac{ap_+}{2}\langle \phi(p)\bar{\phi}(-p)\rangle = I_1$$

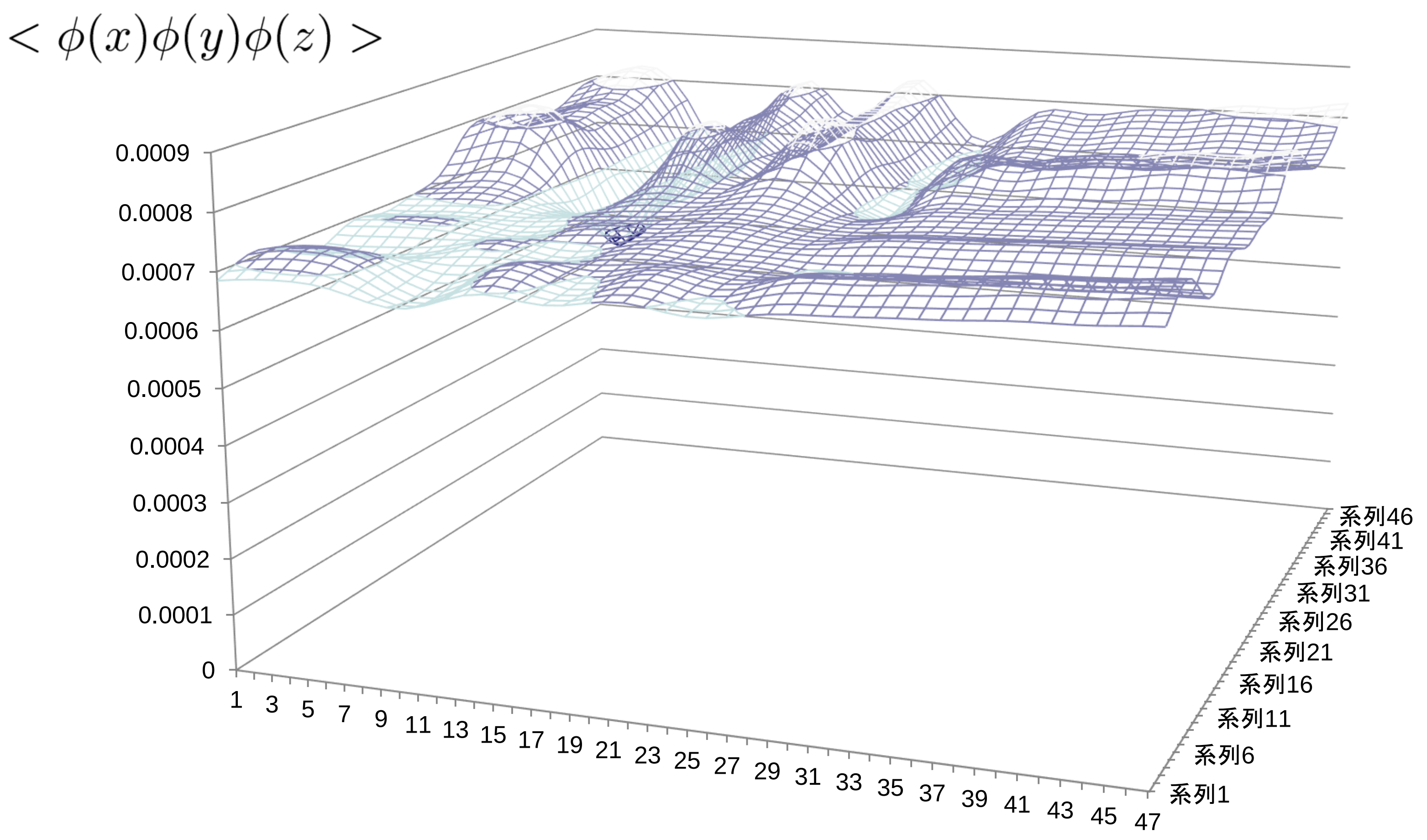
$$\langle \psi_1(p)\bar{\psi}_1(-p)\rangle \frac{2}{a}\sin\frac{ap_-}{2} - \langle F(p)\bar{F}(-p)\rangle = I_2$$

$$i\langle \psi_1(p)\psi_2(-p)\rangle - \frac{2}{a}\sin\frac{aq_+}{2}\langle \phi(p)F(-p)\rangle = I_3$$

Investigations of Ward-Takahashi identities of Wess-Zumino models in one and two dimensions by cut-off model has also been studies by

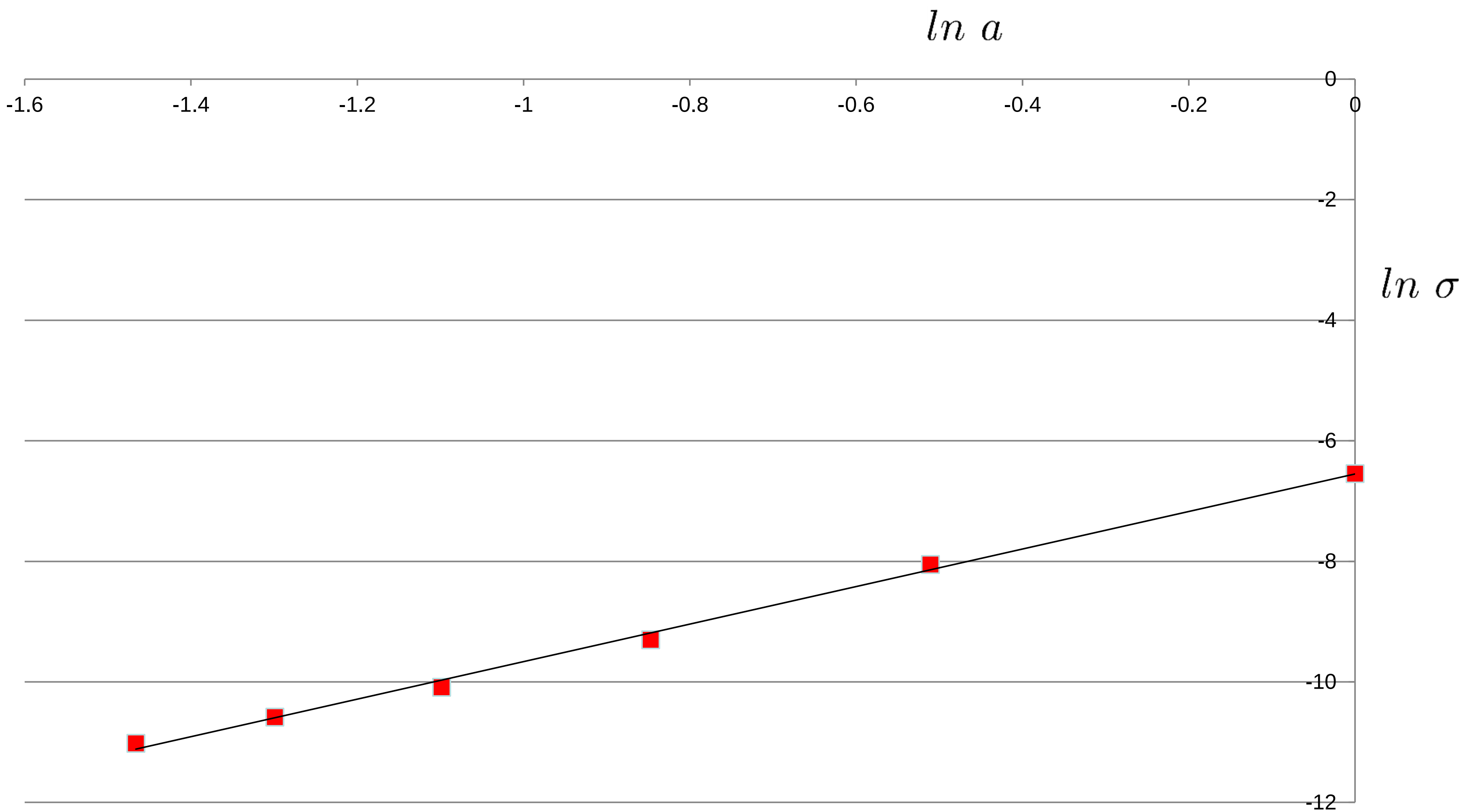
Kado and Suzuki, Kamada and Suzuki..

Translational variance of value of three point function with fixed lattice distance



Recovery of translational invariance

(E. Giguere, N.K. 2013)



Can we generalize this formulation to super Yang-Mills ?

- Breakdown of associativity:

$$(\phi_1 \star (\phi_2 \star \phi_3))(p) \neq ((\phi_1 \star \phi_2) \star \phi_3)(p)$$

$$(\phi_1 \star (\phi_2 \star \phi_3))(p) = \int dp_1 dq \phi_1(p_1)\delta(\hat{p} - \hat{p}_1 - \hat{q})$$

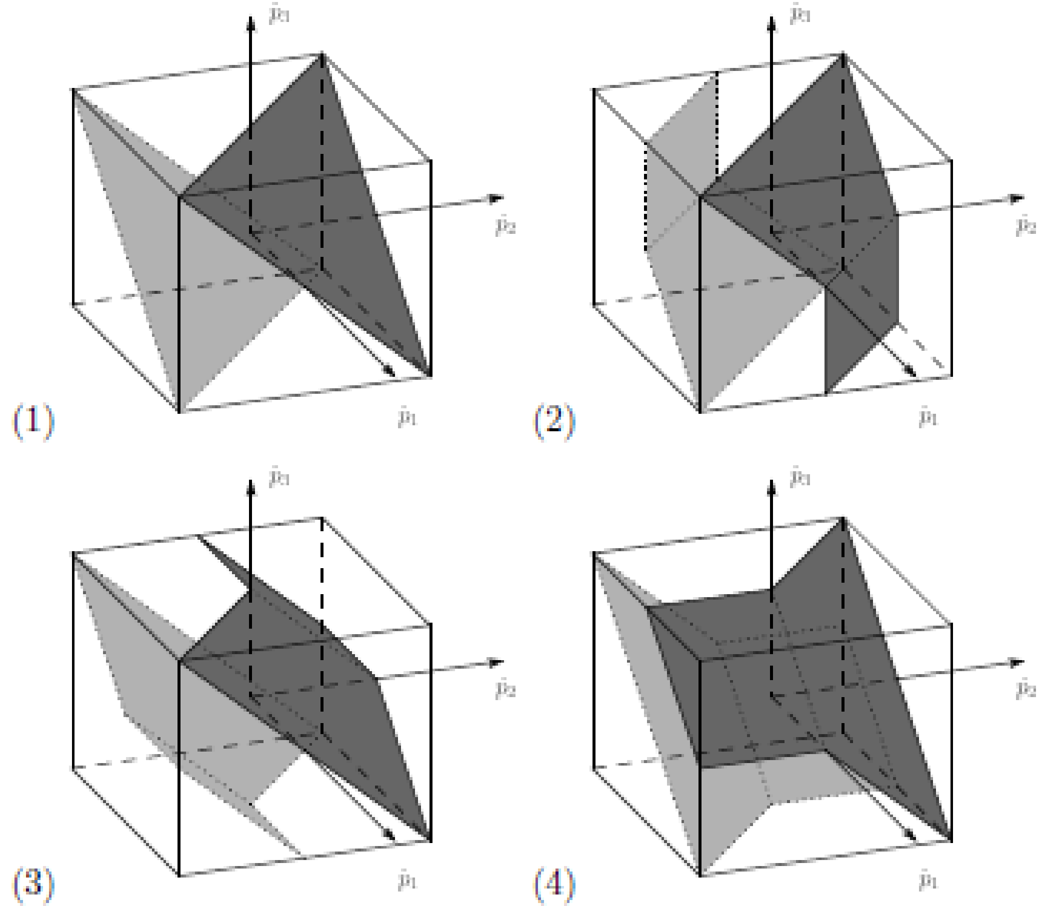
$$\times (\int dp_2 dp_3 \phi_2(p_2)\phi_3(p_3)\delta(\hat{q} - \hat{p}_2 - \hat{p}_3))$$

(2) $|\hat{p}_i| < \frac{2}{a}, \quad |\hat{p}_1 + \hat{p}_2| < \frac{2}{a}, \quad |\hat{p}_1 + \hat{p}_2 + \hat{p}_3| < \frac{2}{a}$

(3) $|\hat{p}_i| < \frac{2}{a}, \quad |\hat{p}_2 + \hat{p}_3| < \frac{2}{a}, \quad |\hat{p}_1 + \hat{p}_2 + \hat{p}_3| < \frac{2}{a}$

$|\hat{p}_i| < \frac{2}{a}$ and $|\sum_i \hat{p}_i| < \frac{2}{a}$

$|\hat{p}_{12}| = |\hat{p}_1 + \hat{p}_2| < \frac{2}{a}$



$|\hat{p}_{23}| = |\hat{p}_2 + \hat{p}_3| < \frac{2}{a}$

$|\hat{p}_{13}| = |\hat{p}_1 + \hat{p}_3| < \frac{2}{a}$

- Non-locality does not seem to cause problem.
- Translational invariance is recovered in the continuum limit.
- Even though associativity is broken, well established product is enough to define Wess-Zumino models since SUSY transformation is linear.
- Immediate extension to gauge theory is not possible if associativity is broken.
- Gauge invariance will be broken by the breakdown of associativity since gauge transformation is non-linear.

However we can recover the associativity.

$$\left(-\pi \leq \frac{ap}{2} \leq \pi\right)$$

Non-local

$$\hat{p} = \frac{2}{a} \sin \frac{ap}{2}$$

$$\delta(\hat{p}_1 + \hat{p}_2 + \hat{p}_3)$$

non associative

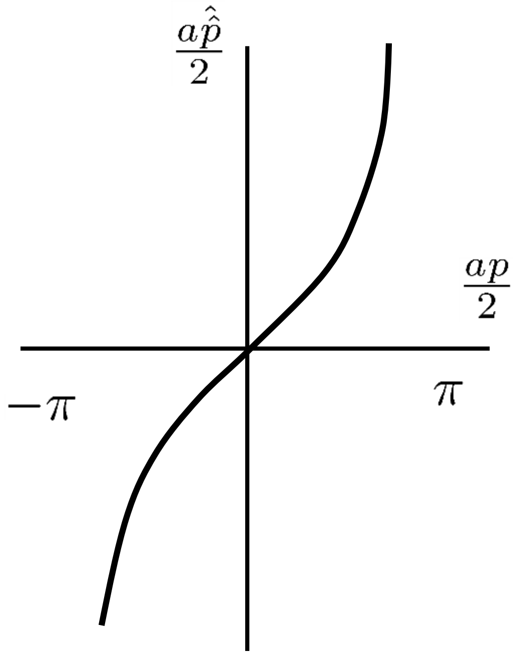
$$-\frac{2}{a} \leq \hat{p}_i \leq \frac{2}{a}$$

$$\frac{a\hat{\hat{p}}}{2} = \frac{1}{2} \ln \frac{1+\sin \frac{ap}{2}}{1-\sin \frac{ap}{2}}$$

$$\delta(\hat{\hat{p}}_1 + \hat{\hat{p}}_2 + \hat{\hat{p}}_3)$$

associative

$$-\infty \leq \hat{\hat{p}}_i \leq \infty$$



Continuum to lattice projection

$$\hat{\hat{p}} = \sin\frac{ap}{2} - \frac{1}{3}\sin\frac{3ap}{2} + \frac{1}{5}\sin\frac{5ap}{2} - \frac{1}{7}\sin\frac{7ap}{2} + \cdots$$

$$\to \delta(x-y-\tfrac{a}{2}) - \delta(x-y+\tfrac{a}{2}) - \tfrac{1}{3}(\delta(x-y-\tfrac{3a}{2}) - \delta(x-y+\tfrac{3a}{2}))$$
$$+ \tfrac{1}{5}(\delta(x-y-\tfrac{5a}{2}) - \delta(x-y+\tfrac{5a}{2})) - \cdots$$

$\hat{\hat{p}}$ has nice derivative structure !

$$p \longleftrightarrow \hat{\hat{p}}$$

$$x = n\tfrac{a}{2} \longleftrightarrow x(\text{continuum})$$

generalized bolocking transformation
of Ginzparg-Wilson type

A proposal for
an alternative solution to chiral fermion problem

1) Introduce half lattice ….. introduction of
double d.o.f. per dimension

2) Truncate fields d.o.f. into half by identification
of species doublers for bosons and fermions:

$$\Phi(p) = \Phi(\frac{2\pi}{a} - p) \longleftrightarrow \Phi(x) = (-1)^{\frac{2x}{a}} \Phi(-x)$$

same chirality for species doublers

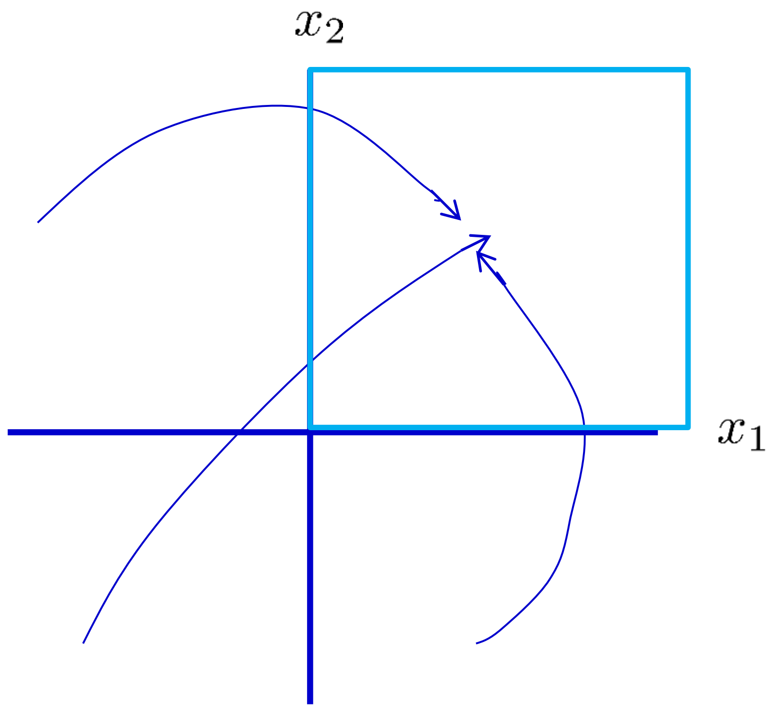
3) Replace the momentum conservation by

$$\hat{\hat{p}} : \quad \delta(\hat{\hat{p}}_1 + \hat{\hat{p}}_2 + \hat{\hat{p}}_3 + \cdots)$$

non-local field theory

$$\Phi(x) = (-1)^{\frac{2x}{a}} \Phi(-x)$$

2 dim.



Because of boundary in the first quadrant region translational invariance is lost but recovered in the limit

Summary for Exact Lattice SUSY

A) Link Approach:
   Hopf algebraic exact SUSY invariance
   Non-commutative super Yang-Mills theory

B) Super doubler approach:
   Exact lattice SUSY on a new star $*$ product
   Non-local field theory

No chiral fermion problem:
   Species doublers are super partners.

Higer dimensions, gauge extension of B)