「離散的手法による場と時空のダイナミクス」＠KEK, Sept. 29, 2013

情報幾何とAdS/CFT対応の関係

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Some historical roots to interdisciplinary reaserch

1975: Search for quantum-classical correspondence
Suzuki-Trotter decomposition (M. Suzuki)
Quantum Monte-Carlo simulation

1980 – 1990: Development of strongly correlated electron physics
Quantum Hall effect, GMR manganite
High-Tc superconductivity (Bednorz-Muller, 1986)
Conformal field theory (1984)

1990-2000: New theoretical & numerical approaches
DMRG (S. White, 1993)
AdS/CFT correspondence (J. Maldacena, 1997)

2000-2010: DMRG meets with Quantum Information
2009-present: close communication among various research fields

Demands for Numerical Techniques

Familiar techniques
(1) Exact Diagonalization: only for small clusters
(2) DMRG: quasi-exact for large systems, but only in 1D
(3) Quantum Monte Carlo: powerful, but negative sign appears

New concept coming from quantum-information community (2004)
→ 'Quantum Entanglement'
Field-theory side: universal scaling of entanglement entropy

DMRG: Density Matrix Renormalization Group
→ But Variational Method rather than RG
→ EE is crucial for variational reformulation of DMRG !

DMRG-based variational approach is still promising for 2D cases, if we could take account of proper entanglement structure.

Strongly Correlated Electron Systems

Competition among itinerant and localized characters of electrons
→ high-Tc, Manganite, Heavy Fermions, Organic, and many

Hubbard model

$$H = -\sum_{i,j,\sigma} t_{ij} \left(c_{i,\sigma}^{+} c_{j,\sigma} + H.c. \right) + U \sum_{i} n_{i,\uparrow} n_{i,\downarrow}$$

t-J Model：large-U expansion of the Hubbard model（$J=4t^2/U$）

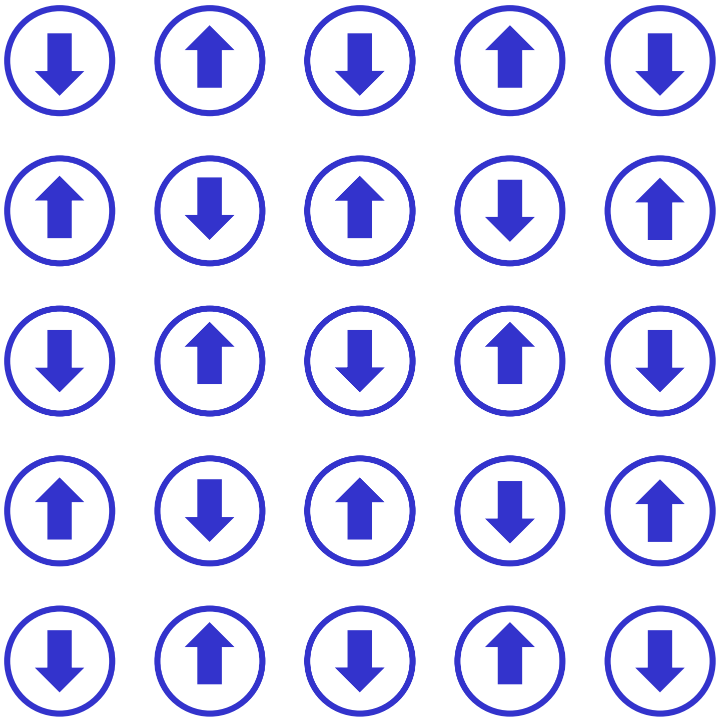
$$H = -\sum_{i,j,\sigma} t_{ij} \left(\tilde{c}_{i,\sigma}^{+} \tilde{c}_{j,\sigma} + H.c. \right) + J \sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j$$

V-t model

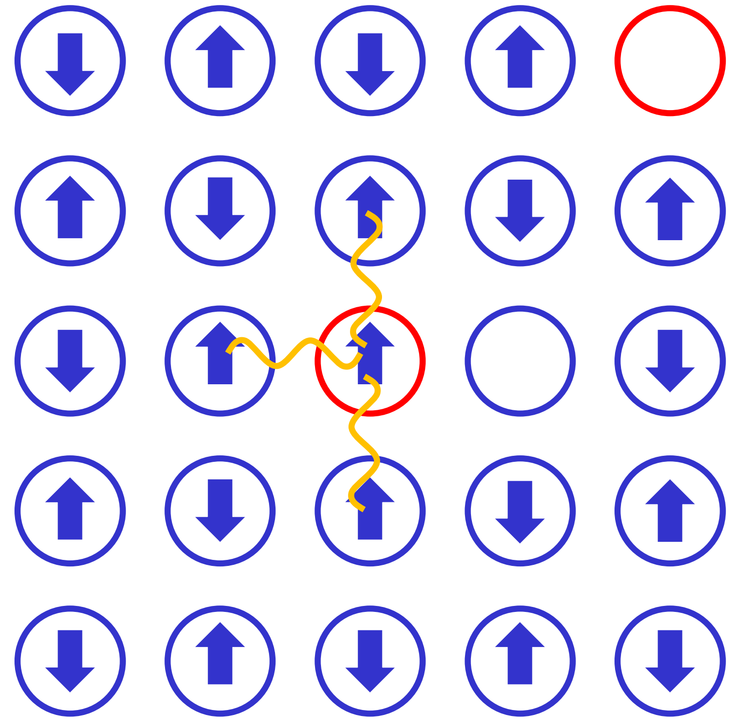
$$H = -\sum_{i,j} t_{ij} \left(c_i^{+} c_j + H.c. \right) + V \sum_{i} n_i n_{i+1}$$

t–J model for high-T_c superconductivity

$$H = -\sum_{i,j,\sigma} t_{ij}\left(\tilde{c}^{+}_{i,\sigma}\tilde{c}_{j,\sigma} + H.c.\right) + J\sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j$$



Half Filling (Mott Insulator)



Doping (Superconducting)

$$\Delta E_{exchange} \approx +3J$$

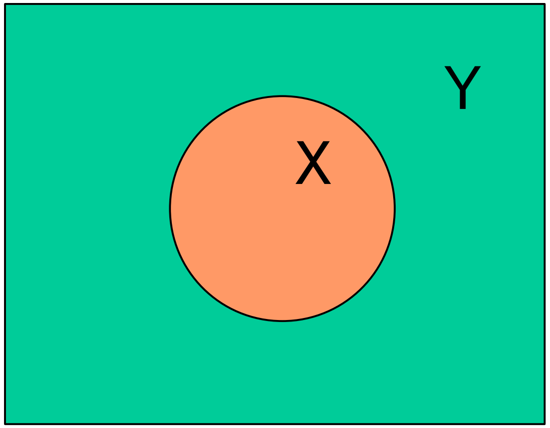
$$\Delta E_{hopping} \approx -t$$

High resolution ARPES & Neutron

Entanglement entropy

Total system= X+Y: “Superblock”, “Universe”

$$|\psi\rangle = \sum_{x,y} \psi(x,y) |x\rangle \otimes |y\rangle \qquad \begin{array}{l} x \in X \\ y \in Y \end{array}$$



Density matrix for subsystems X and Y

$$\rho_X = \mathrm{Tr}_Y |\psi\rangle\langle\psi|$$
$$\rho_Y = \mathrm{Tr}_X |\psi\rangle\langle\psi|$$

Entanglement entropy

$$S_X = -\mathrm{Tr}_X \left(\rho_X \log \rho_X\right)$$
$$S_Y = -\mathrm{Tr}_Y \left(\rho_Y \log \rho_Y\right)$$

Entanglement entropy → Log. of correlation function
→ We can directly pick up exponents in critical cases

Singular Value Decomposition

Singular Value Decomposition (SVD) of Ψ

$$\psi(x,y)=\sum_l U_l(x)\sqrt{\Lambda_l}\,V_l(y)$$

Λ_l : singular value (non-negative, uniquely determined)

$U_l(x), V_l(y)$: (unitary matrices, various choices)

$$\rho_X(x,x')=\sum_y \psi(x,y)\psi^*(x',y)=\sum_l U_l(x)\,\Lambda_l\,U_l^*(x')$$

$$\rho_Y(y,y')=\sum_x \psi(x,y)\psi^*(x,y')=\sum_l V_l(y)\,\Lambda_l\,V_l^*(y')$$

Von Neumann Entropy → Area-law scaling

$$S_X=-\sum_l \lambda_l \log\lambda_l = S_Y \qquad \lambda_l=\Lambda_l/\sum_l \Lambda_l$$

Area-law Scaling for Entanglement Entropy and its Violation

* CFT Analysis, Numerical Study, ･･･

Gapped 1D, general d>1 → Area Law Scaling

$$S = \alpha L^{d-1} + \cdots$$

1D Critical, Systems with Fermi Surface: Log. Violation of Area Law

$$S = \frac{1}{3} C L^{d-1} \log L + \cdots$$

C : Effective # of Excitation Modes (Central Charge in 1D)

Topological Entanglement Entropy (d=2)

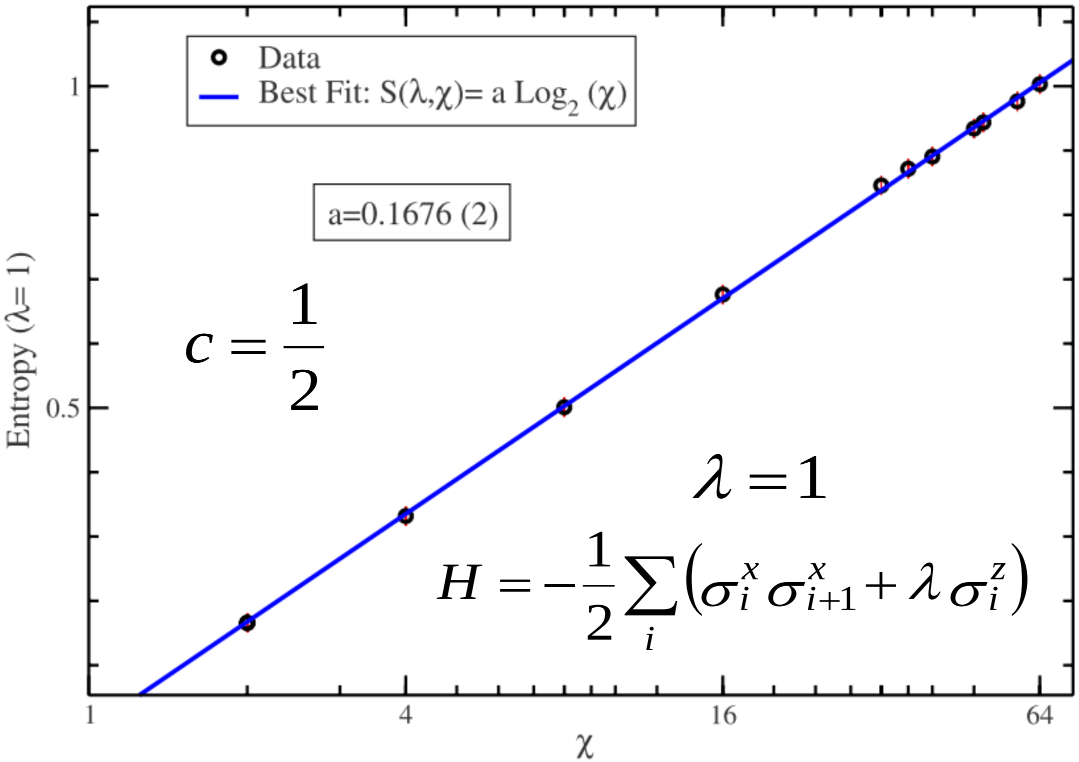
$$S = \alpha L - \log \sqrt{D}$$

* MPS (d=1) → very good for gapped cases

* We need χ =O(N) for critical cases.

$$S_{MPS} = \frac{1}{\sqrt{12/c} + 1} \log \chi$$

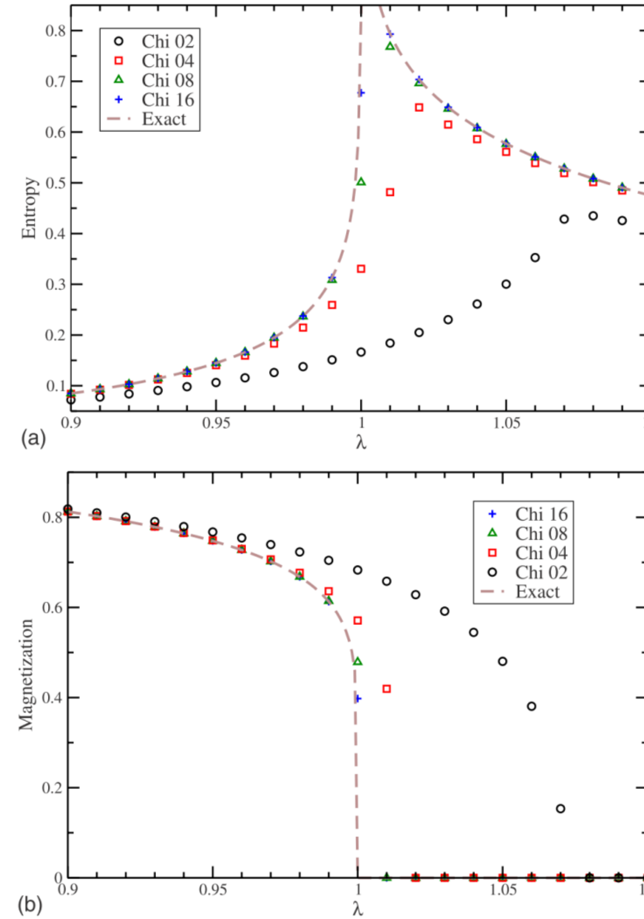
Finite-χ scaling for quantum Ising model in transverse field



$$c = \frac{1}{2}$$

$$\lambda = 1$$

$$H = -\frac{1}{2}\sum_i \left(\sigma_i^x \sigma_{i+1}^x + \lambda \sigma_i^z\right)$$



$$S = \frac{c\kappa}{6}\log\chi = \frac{c}{6}\log\xi$$

$$\kappa = \frac{6}{c\left(\sqrt{12/c}+1\right)}$$

L. Tagliacozzo et al., PRB78, 024410 (2008)

F. Pollmann et al., PRL102, 255701 (2009)

Ching-Yu Huang and Feng-Li Lin, PRA81, 032304 (2010)

H. Matsueda, arXiv:1106.5624

What is quantum entanglement ?: the simplest example

S=1/2 Heisenberg Antiferromagnet (2 sites)

$$H = J\,\vec{S}_1 \cdot \vec{S}_2 = \frac{J}{2}\left(S_1^+ S_2^- + S_1^- S_2^+\right) + J\,S_1^z S_2^z$$

Basis set: $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

$$H = \begin{pmatrix} \frac{J}{4} & 0 & 0 & 0 \\ 0 & -\frac{J}{4} & \frac{J}{2} & 0 \\ 0 & \frac{J}{2} & -\frac{J}{4} & 0 \\ 0 & 0 & 0 & \frac{J}{4} \end{pmatrix}$$

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$E_0 = -\frac{3}{4}J$$

$$\vec{S} = \frac{1}{2}\vec{\sigma}$$

General variational function

A, B, C, D: Non-local

$$|\psi\rangle = A|\uparrow\uparrow\rangle + B|\uparrow\downarrow\rangle + C|\downarrow\uparrow\rangle + D|\downarrow\downarrow\rangle$$

→ Minimize $E = \langle\psi|H|\psi\rangle / \langle\psi|\psi\rangle$

Local Approximation

$a^{\uparrow}$, $a^{\downarrow}$, $c^{\uparrow}$, $c^{\downarrow}$: local

$$|\psi\rangle = \sum_{s_1=\uparrow,\downarrow} a^{s_1}|s_1\rangle \otimes \sum_{s_2=\uparrow,\downarrow} c^{s_2}|s_2\rangle$$

$$= \left(a^{\uparrow}|\uparrow\rangle + a^{\downarrow}|\downarrow\rangle\right) \otimes \left(c^{\uparrow}|\uparrow\rangle + c^{\downarrow}|\downarrow\rangle\right)$$

$$= a^{\uparrow}c^{\uparrow}|\uparrow\uparrow\rangle + a^{\uparrow}c^{\downarrow}|\uparrow\downarrow\rangle + a^{\downarrow}c^{\uparrow}|\downarrow\uparrow\rangle + a^{\downarrow}c^{\downarrow}|\downarrow\downarrow\rangle$$

The local approx. cannot describe the singlet.

$$|\psi\rangle = \sum_{s_1=\uparrow,\downarrow} a^{s_1}|s_1\rangle \otimes \sum_{s_2=\uparrow,\downarrow} c^{s_2}|s_2\rangle$$

$$= a^{\uparrow}c^{\uparrow}|\uparrow\uparrow\rangle + a^{\uparrow}c^{\downarrow}|\uparrow\downarrow\rangle + a^{\downarrow}c^{\uparrow}|\downarrow\uparrow\rangle + a^{\downarrow}c^{\downarrow}|\downarrow\downarrow\rangle$$

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$a^{\uparrow}c^{\uparrow} = 0$$

$$a^{\downarrow}c^{\downarrow} = 0$$

$$a^{\uparrow}c^{\downarrow} = 1/\sqrt{2}$$

$$a^{\downarrow}c^{\uparrow} = -1/\sqrt{2}$$

No solution !

$$|\psi\rangle = |1\rangle \otimes |2\rangle$$ Product state

$$\rho_1 = Tr_2|\psi\rangle\langle\psi| = |1\rangle\langle 1| \qquad S_1 = -Tr_1\rho_1\log\rho_1 = 0$$

Local Approx. → No Entanglement → How to recover it ?

Vector product state

$$|\psi\rangle = \sum_{s_1,s_2} a^{s_1} c^{s_2} |s_1 s_2\rangle \Longrightarrow \sum_{s_1,s_2} A^{s_1} C^{s_2} |s_1 s_2\rangle$$

$$A^{s_1} = \left(a_1^{s_1}, a_2^{s_1}\right)$$

$$C^{s_2} = \begin{pmatrix} c_1^{s_2} \\ c_1^{s_2} \end{pmatrix}$$

This looks local decomposition, but A and C get entangled !

Introduction of <u>additional index</u> that represents entanglement

$$|\psi\rangle = \sum_{\alpha=1}^{\chi=2} \left\{ \sum_{s_1=\uparrow,\downarrow} a_\alpha^{s_1} |s_1\rangle \otimes \sum_{s_2=\uparrow,\downarrow} c_\alpha^{s_2} |s_2\rangle \right\}$$

$$= \left(a_1^\uparrow c_1^\uparrow + a_2^\uparrow c_2^\uparrow\right)|\uparrow\uparrow\rangle + \left(a_1^\uparrow c_1^\downarrow + a_2^\uparrow c_2^\downarrow\right)|\uparrow\downarrow\rangle$$

$$+ \left(a_1^\downarrow c_1^\uparrow + a_2^\downarrow c_2^\uparrow\right)|\downarrow\uparrow\rangle + \left(a_1^\downarrow c_1^\downarrow + a_2^\downarrow c_2^\downarrow\right)|\downarrow\downarrow\rangle$$

$$a_1^\uparrow = c_2^\uparrow = a_2^\downarrow = c_1^\downarrow = 0$$

$$a_2^\uparrow c_2^\downarrow = 1/\sqrt{2}$$

$$a_1^\downarrow c_1^\uparrow = -1/\sqrt{2}$$

$$|\psi\rangle = |0\rangle$$

Exact for χ =2 !

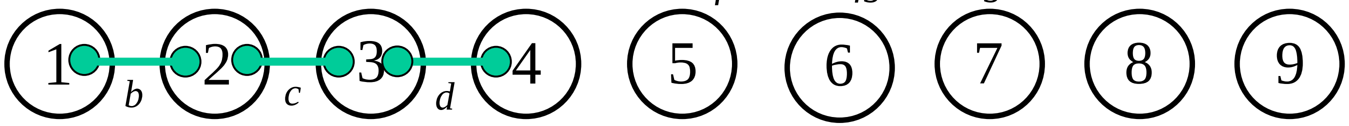
$$A^\uparrow = (x, y), A^\downarrow = (z, w), C^\uparrow = \begin{pmatrix} \frac{y}{xw-yz} \\ \frac{x}{yz-xw} \end{pmatrix}, C^\downarrow = \begin{pmatrix} \frac{w}{xw-yz} \\ \frac{z}{yz-xw} \end{pmatrix}$$

Matrix Product State (MPS): DMRG optimizes MPS

MPS for open boundary cases (vectors on two edges)

$$|\psi\rangle = \sum_{\{s_1, s_2, \cdots, s_n\}} \langle s_1 | A_2^{s_2} A_3^{s_3} \cdots A_{n-1}^{s_{n-1}} | s_n \rangle | s_1 s_2 \cdots s_n \rangle$$

$\langle s_1| = A_b^{s_1}$ $\quad A_{bc}^{s_2}$ $\quad A_{cd}^{s_3}$ $\quad A_{de}^{s_4}$ $\quad A_{ef}^{s_5}$ $\quad A_{fg}^{s_6}$ $\quad A_{gh}^{s_7}$ $\quad A_{hi}^{s_8}$ $\quad |s_9\rangle = A_i^{s_9}$



$A_j^{s_j}$ χ ×χ matrix, χ : unphysical, artificial degree

$s_j = \uparrow, \downarrow$: physical degree

Matrix = projection of unphysical degree on physical one

↓

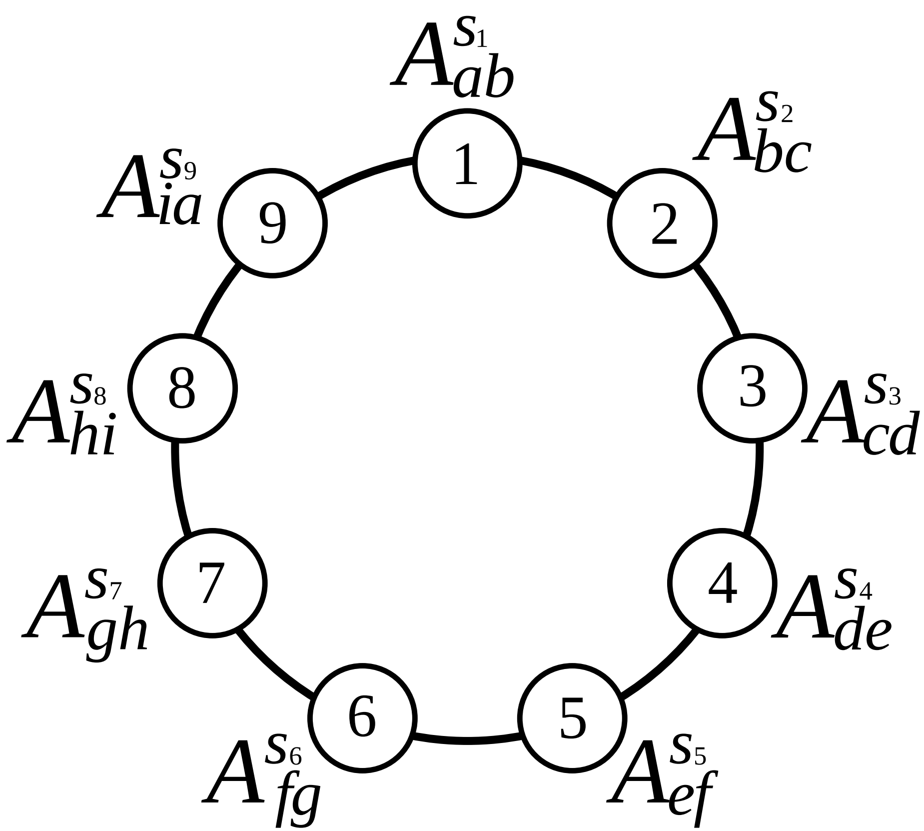
What is unphysical degree of freedom ?

Ψ : product of local matrices → non-local correlation

MPS for periodic n-sites system

$$|\psi\rangle = \sum_{\{s_1, s_2, \cdots, s_n\}} tr\left(A_1^{s_1} A_2^{s_2} \cdots A_n^{s_n}\right)|s_1 s_2 \cdots s_n\rangle$$

Trace → rotational invariance in the n→∞ limit



Numerical optimization of MPS: Generalized Eigenvalue Problem

$$\left|\psi\right\rangle = \sum_{\alpha,\beta} A^{\alpha} B^{\beta} \left|\alpha\beta\right\rangle$$

B → fix
A → variational parameters

$$\left\langle\psi\right|H\left|\psi\right\rangle = \sum_{\alpha,\beta,\gamma,\delta} \left\langle\gamma\delta\right|\left(A^{\gamma} B^{\delta}\right) H \left(A^{\alpha} B^{\beta}\right)\left|\alpha\beta\right\rangle$$

$$= \sum_{\alpha,\beta,\gamma,\delta} \sum_{i,j} \left\langle\gamma\delta\right| A_j^{\gamma} B_j^{\delta} H A_i^{\alpha} B_i^{\beta} \left|\alpha\beta\right\rangle$$

$$A_i^{\alpha} \Rightarrow \vec{A} = \begin{pmatrix} A_1^{\uparrow} \\ A_2^{\uparrow} \\ A_1^{\downarrow} \\ A_2^{\downarrow} \end{pmatrix}$$

$$= \sum_{\alpha,i} \sum_{\gamma,j} A_j^{\gamma} \left(\sum_{\beta,\delta} \left\langle\gamma\delta\right| B_j^{\delta} H B_i^{\beta} \left|\alpha\beta\right\rangle \right) A_i^{\alpha}$$

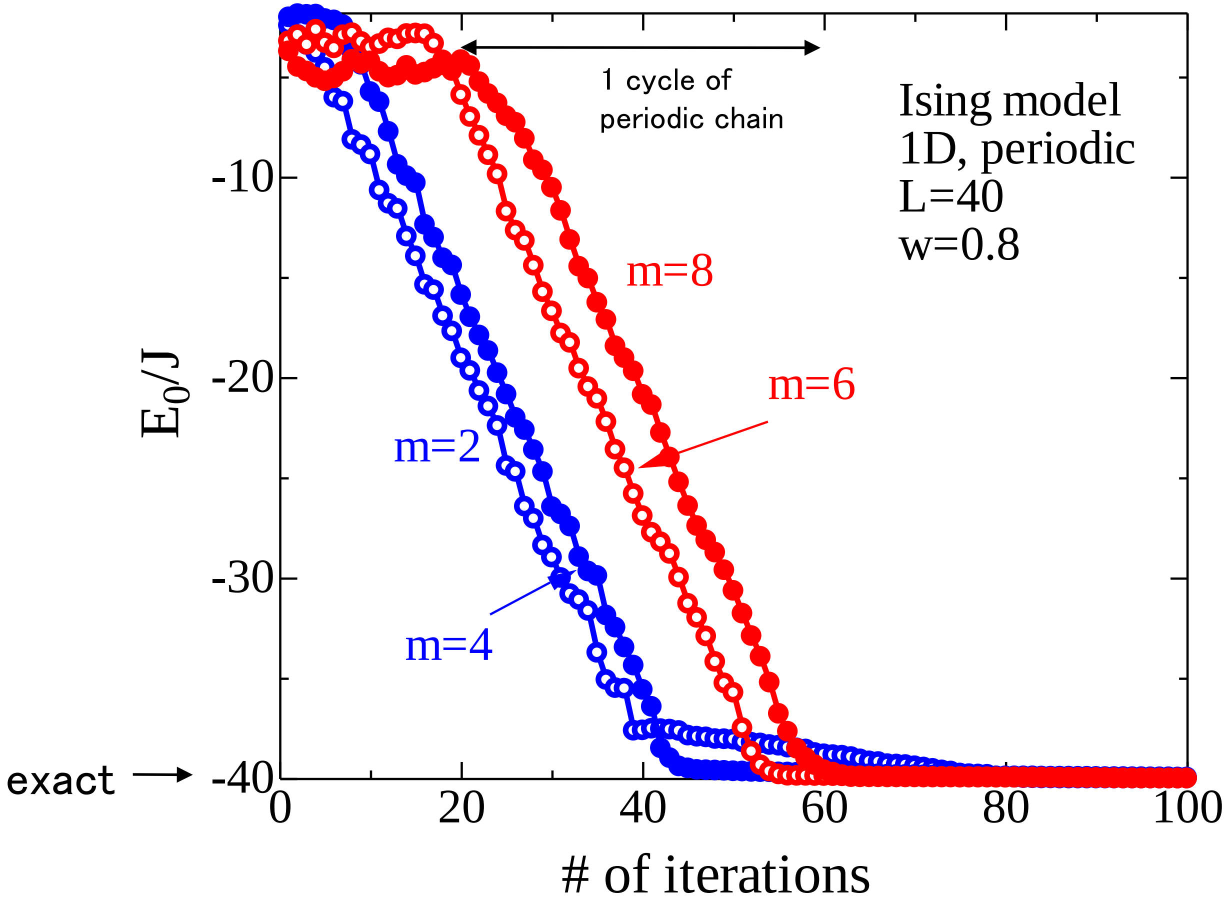
$$= \vec{A}^{+} H_{eff} \vec{A}$$

$$\left\langle\psi\middle|\psi\right\rangle = \vec{A}^{+} N_{eff} \vec{A}$$

$$H_{eff} \vec{A} = \lambda N_{eff} \vec{A}$$

A → fix
B → variational parameters
……

Variational optimization of MPS for Ising chain (critical, c=1/2)

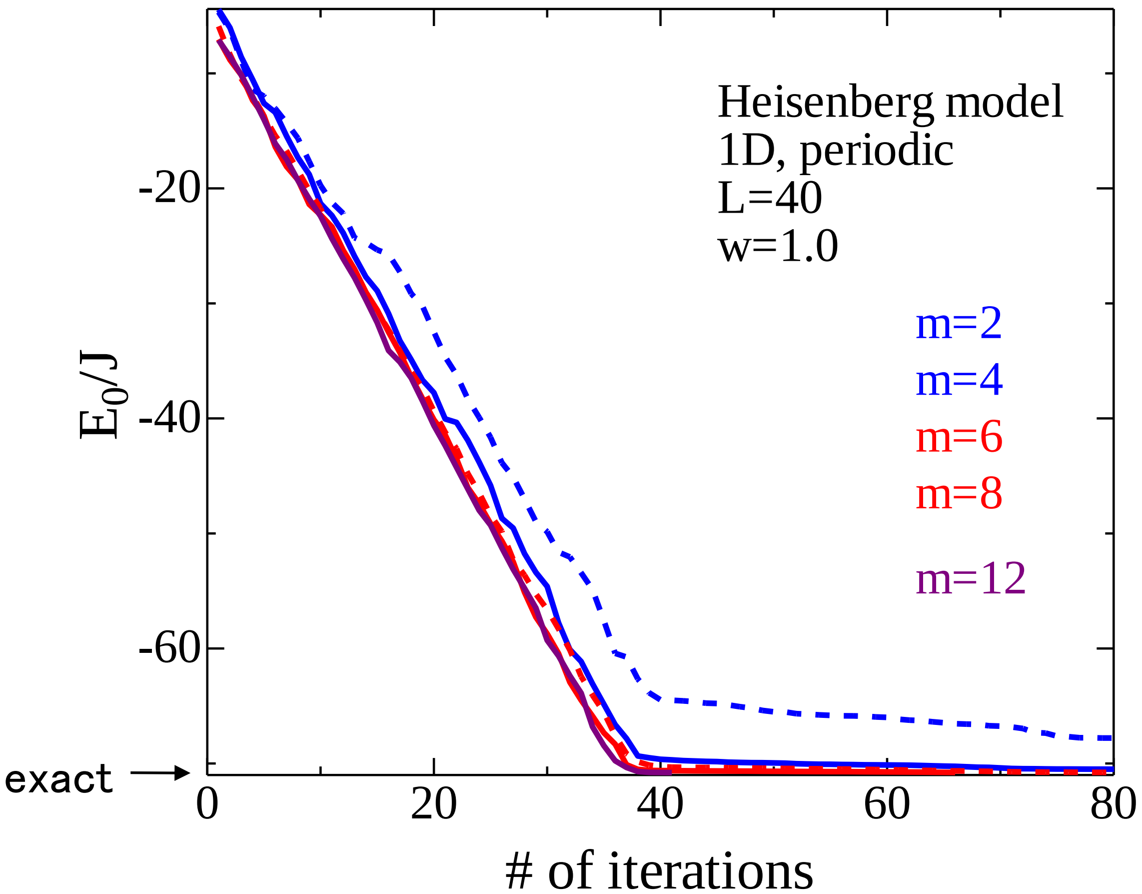


$$w\,A_{new} + (1-w)\,A_{old} \Rightarrow A_{new}$$

Ising (classical) → no quantum entanglement

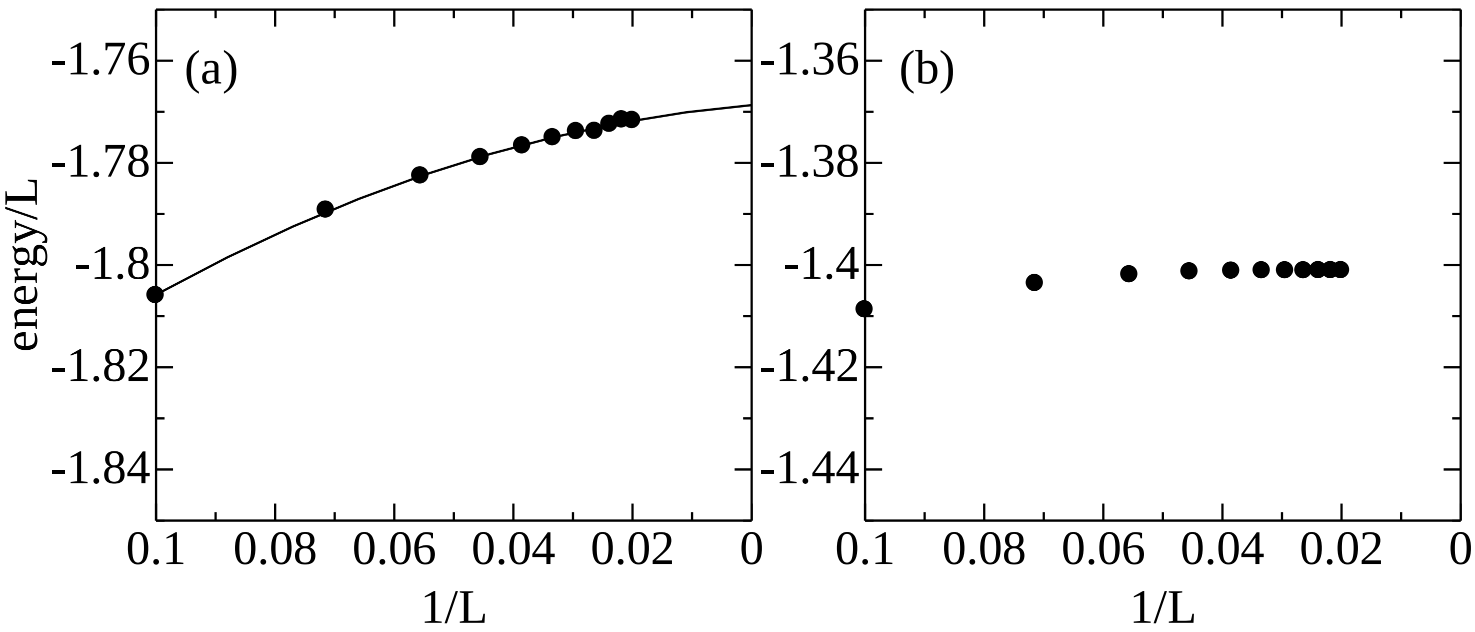
Small m is enough for sufficient numerical accuracy for G.S.

Variational optimization of MPS for S=1/2 Heisenberg chain



We need large m for critical cases

Critical system (S=1/2, (a)) and Haldane-Gap system (S=1, (b))



S=1/2: close to the exact Bethe-Hulthen result

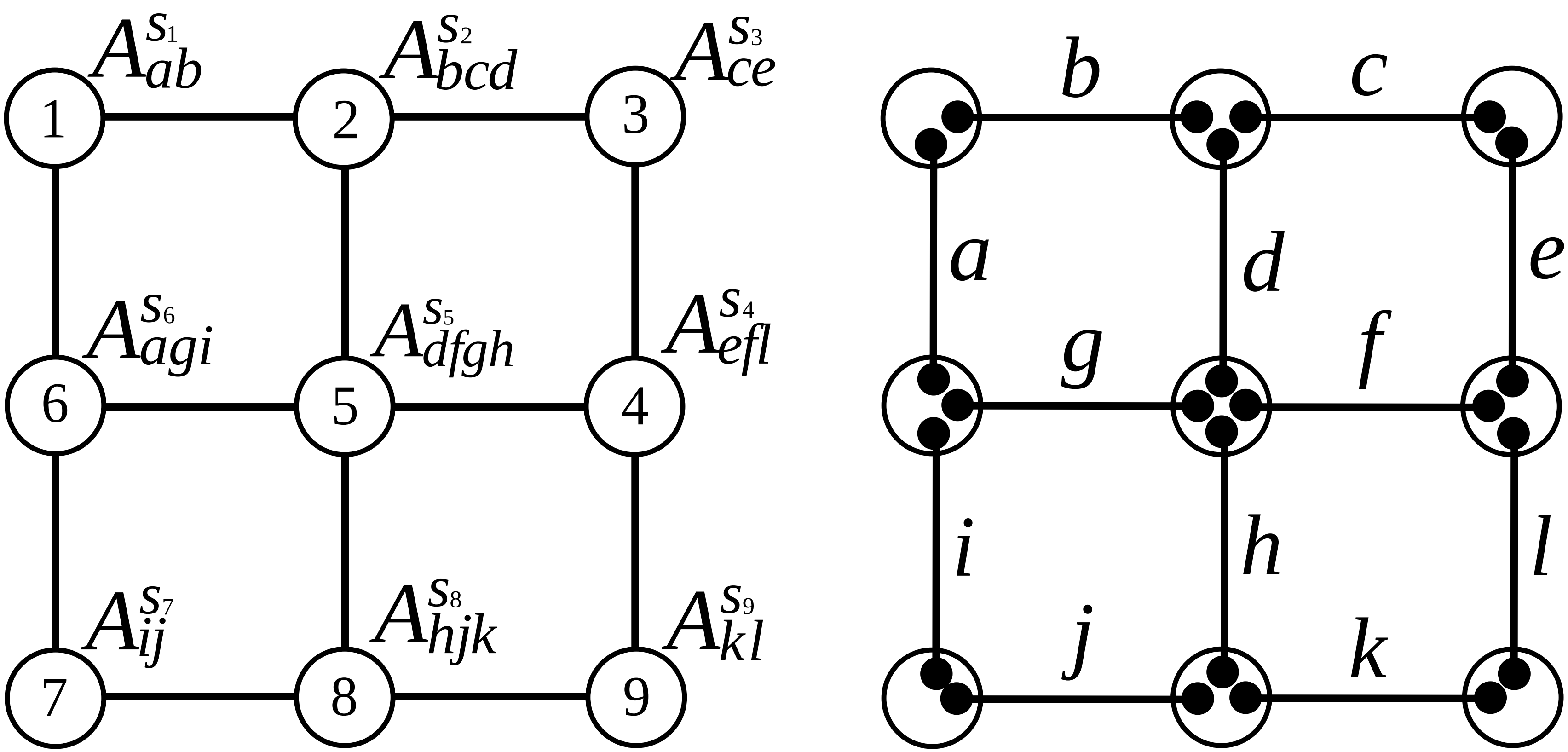
$$4E/LJ = 1 - 4\ln 2 = -1.772$$

S=1: excitation gap → exponential decay of correlation

Tensor Product State (TPS)

Tensor Network State (TNS)

Projected Entangled Pair State (PEPS)

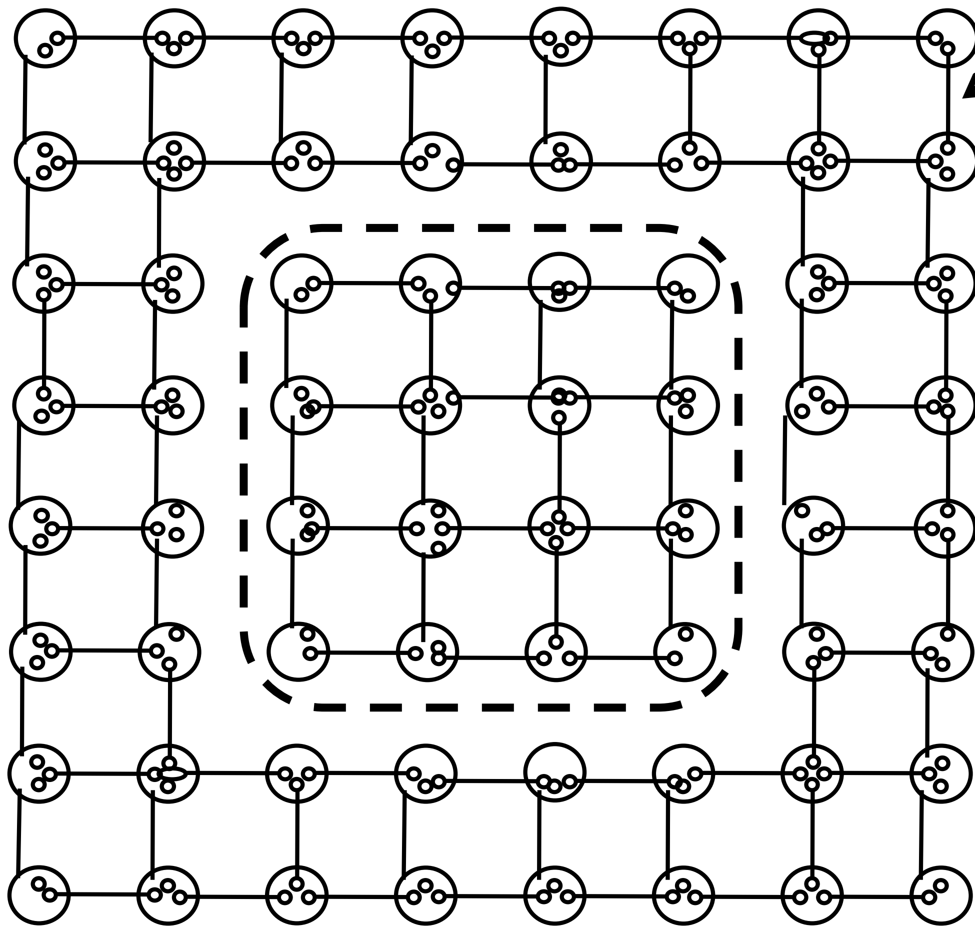


(a) (b)

$$|\psi\rangle = \sum_{\{s_j\}} \sum_{a,b,\ldots,l} A^{s_1}_{ab} A^{s_2}_{bcd} A^{s_3}_{ce} A^{s_4}_{efl} A^{s_5}_{dfgh} A^{s_6}_{agi} A^{s_7}_{ij} A^{s_8}_{hjk} A^{s_9}_{kl} |s_1 s_2 \cdots s_9\rangle$$

Entanglement structure of TPS

Maximally entangled bond



$$|\psi\rangle = \frac{1}{\sqrt{\chi}} \sum_{\alpha=1}^{\chi} |\alpha\alpha\rangle$$

Entanglement entropy between system and environment

$$S = N_{bond} \log \chi$$

Proper value of χ

Gapped cases (Area Law):

$$S = N_{bond} \log \chi \sim \alpha L^{d-1}$$

$$N_{bond} \sim L^{d-1}$$

χ is a constant O(ξ)

Critical cases (Log. violation of the area law):

$$S = N_{bond} \log \chi \sim \frac{1}{3} C L^{d-1} \log L$$

$$\chi \sim L^{C/3}$$

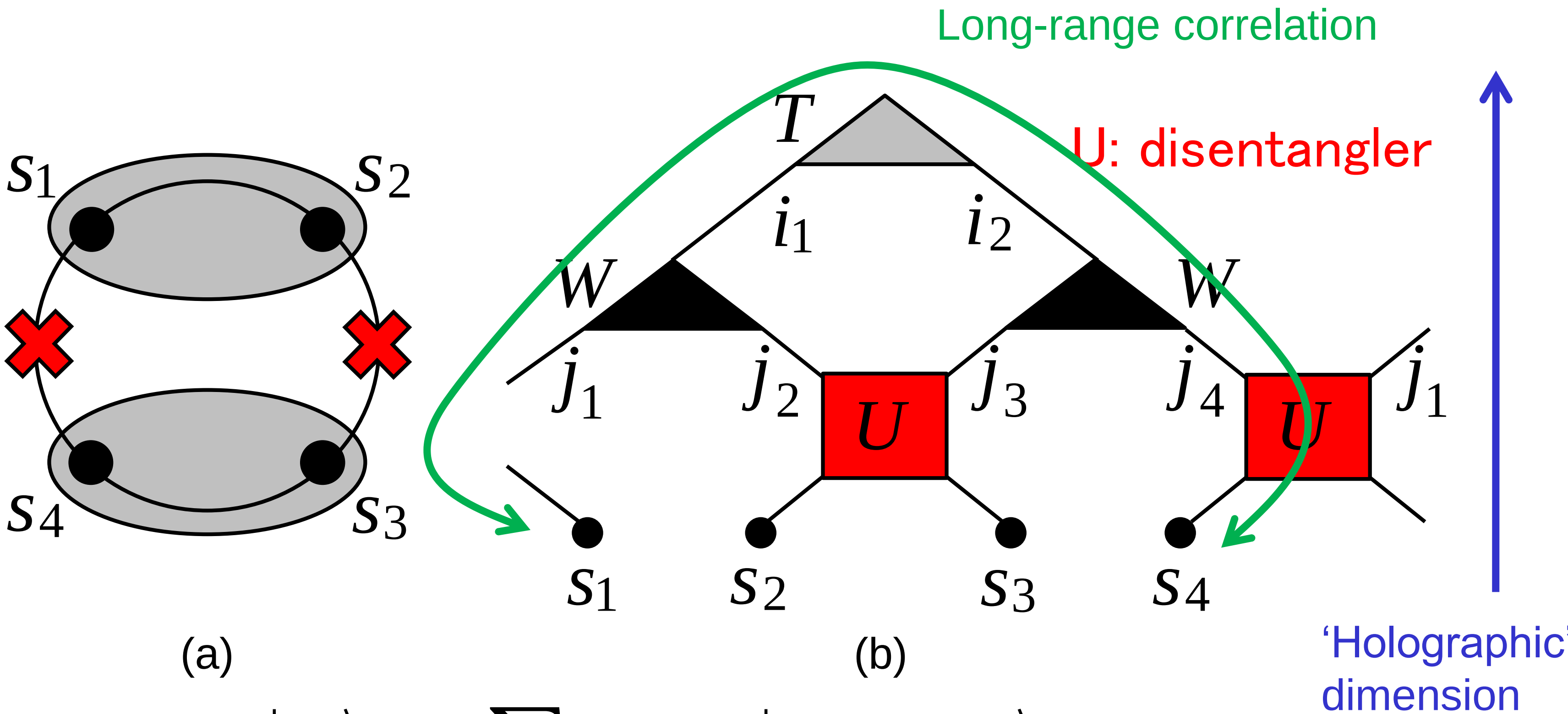
We need a large χ value.

How can we treat log. term of EE ?

Hierarchical tensor network
→ 'renormalization', 'coarse graining'

Hierarchical Tensor Network

Multiscale Entanglement Renormalization Ansatz (MERA)

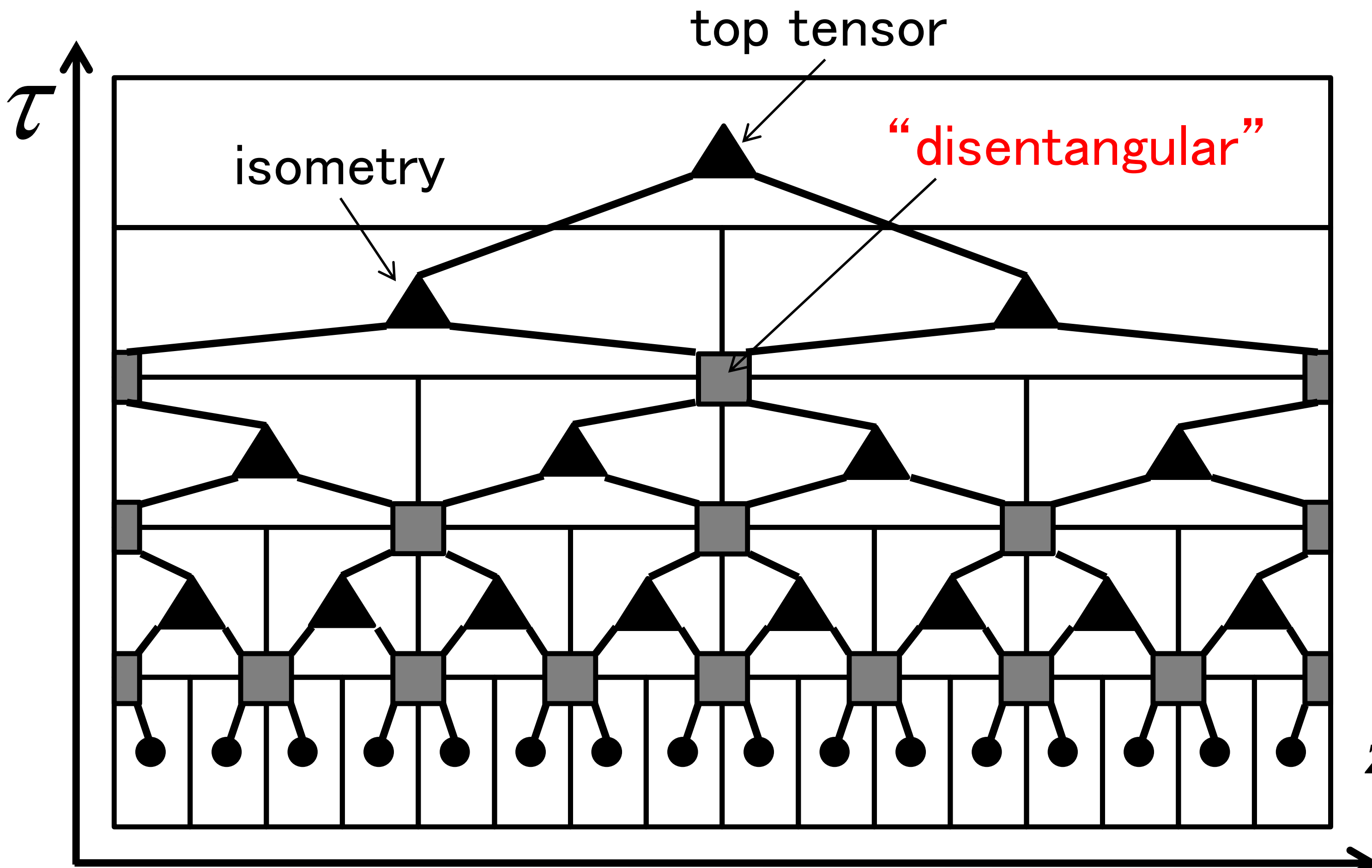


(a) (b)

$$\left|\psi\right\rangle = \sum_{s_1, s_2, s_3, s_4,} T_{s_1 s_2 s_3 s_4} \left| s_1 s_2 s_3 s_4 \right\rangle$$

$$\left|\psi\right\rangle = \sum_{i_1, i_2} \sum_{j_1, \ldots, j_4} \sum_{s_1, \ldots, s_4} T_{i_1 i_2} W^{i_1}_{j_1 j_2} W^{i_2}_{j_3 j_4} U^{j_2 j_3}_{s_2 s_3} U^{j_4 j_1}_{s_4 s_1} \left| s_1 s_2 s_3 s_4 \right\rangle$$

(Multiscale Entanglement Renormalization Ansatz, MERA)



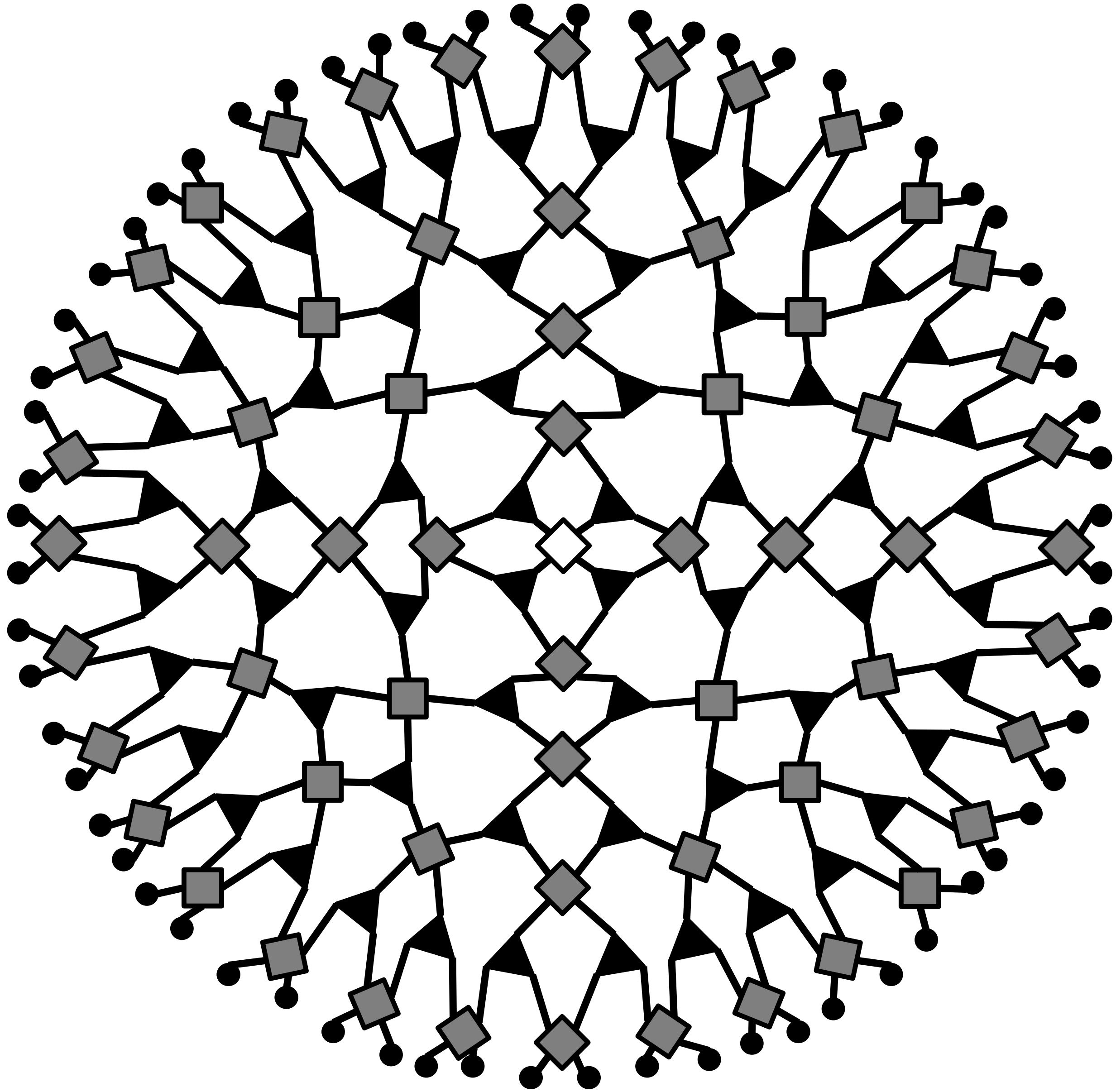
$$z = 2^{\tau/l}\, l$$

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + dx^2\right) = \left(d\tau \log 2\right)^2 + \left(2^{-\tau/l} dx\right)^2$$

MPS → decomposed into many tensors with different function

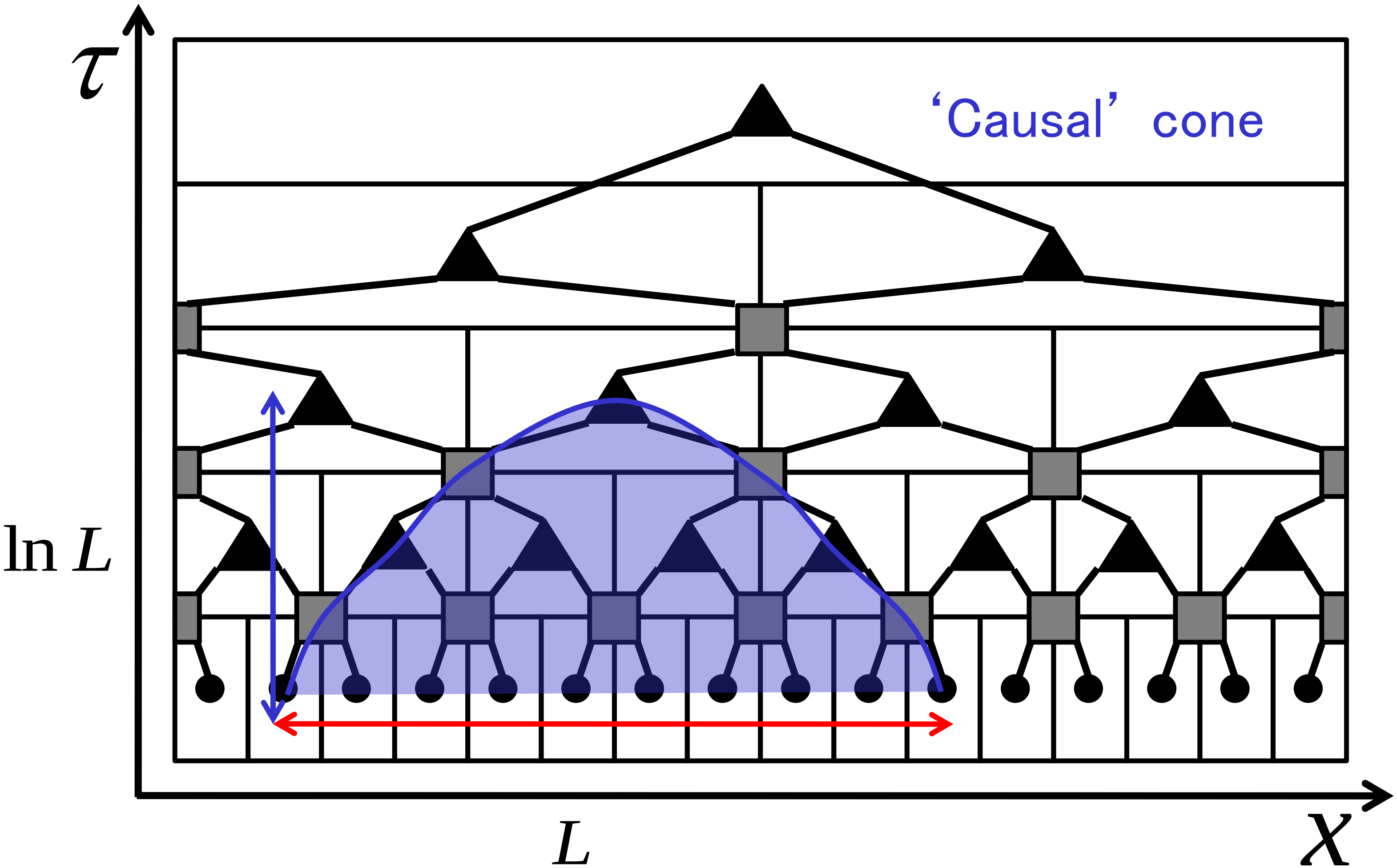
Basis change (disentangler) before renormalization

Poincare Disk Model for MERA Network



How to evaluate entanglement entropy in holographic space ?

Close connection to 'Ryu–Takayanagi formula'
developed in superstring theory



S = minimal surface area in holographic space

Binary decomposition

Spatially 1D cases:

$$\underbrace{\overset{\downarrow}{2}+2+\cdots+2}_{\text{No. of boundary points: } \ln L}=2\ln L$$

Spatially 2D cases:

$$4L\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)=4L\left(2-\frac{1}{2^n}\right)\to 8L$$

Summary before going into more advanced topics

Quantum entanglement

Entanglement-entropy scaling (area law, log. violation)

Area law → PEPS (MPS, TPS)

Log. Violation → hierarchical tensor network, MERA

Entanglement structure of MERA
→ Consistent with Ryu-Takayanagi formula

Thermofield dynamics (TFD) for finite-T wavefunction

Purpose: finite-T MERA and its relation with AdS/CFT

Finite-T → thermal avarage

TFD form → 'thermal vacuum'

Identity state (maximally entangled) $\left|I\right\rangle = \sum_n \left|n\right\rangle \otimes \left|\tilde{n}\right\rangle$

General representation theorem

$$\left|I\right\rangle = \sum_n \left|n\right\rangle \otimes \left|\tilde{n}\right\rangle = \sum_\alpha \left|\alpha\right\rangle \otimes \left|\tilde{\alpha}\right\rangle$$

$$\left|O(\beta)\right\rangle = \rho^{1/2}\left|I\right\rangle$$

$$\left\langle O(\beta)\right|A\left|O(\beta)\right\rangle = \sum_{m,n}\left\langle m\tilde{m}\right|\rho^{1/2} A \rho^{1/2}\left|n\tilde{n}\right\rangle$$

$$= \sum_{m,n}\left\langle m\right|\rho^{1/2} A \rho^{1/2}\left|n\right\rangle \delta_{\tilde{m}\tilde{n}} = tr(\rho A)$$

Thermal state in TFD

$$\left|\psi(\beta)\right\rangle = \sum_{\{m_j\}} \sum_{\{\tilde{n}_j\}} c^{\{m_j\}\{\tilde{n}_j\}} \left| \{m_j\}\{\tilde{n}_j\} \right\rangle$$

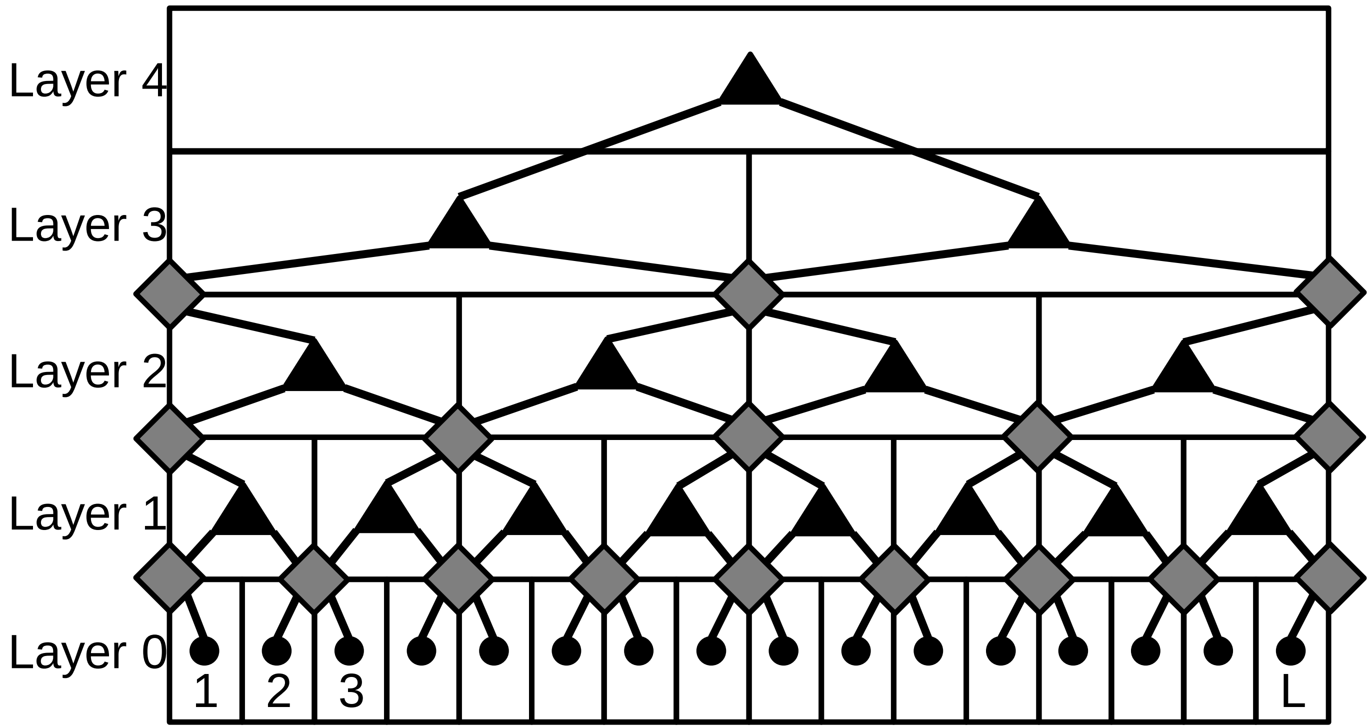
Singular value decomposition

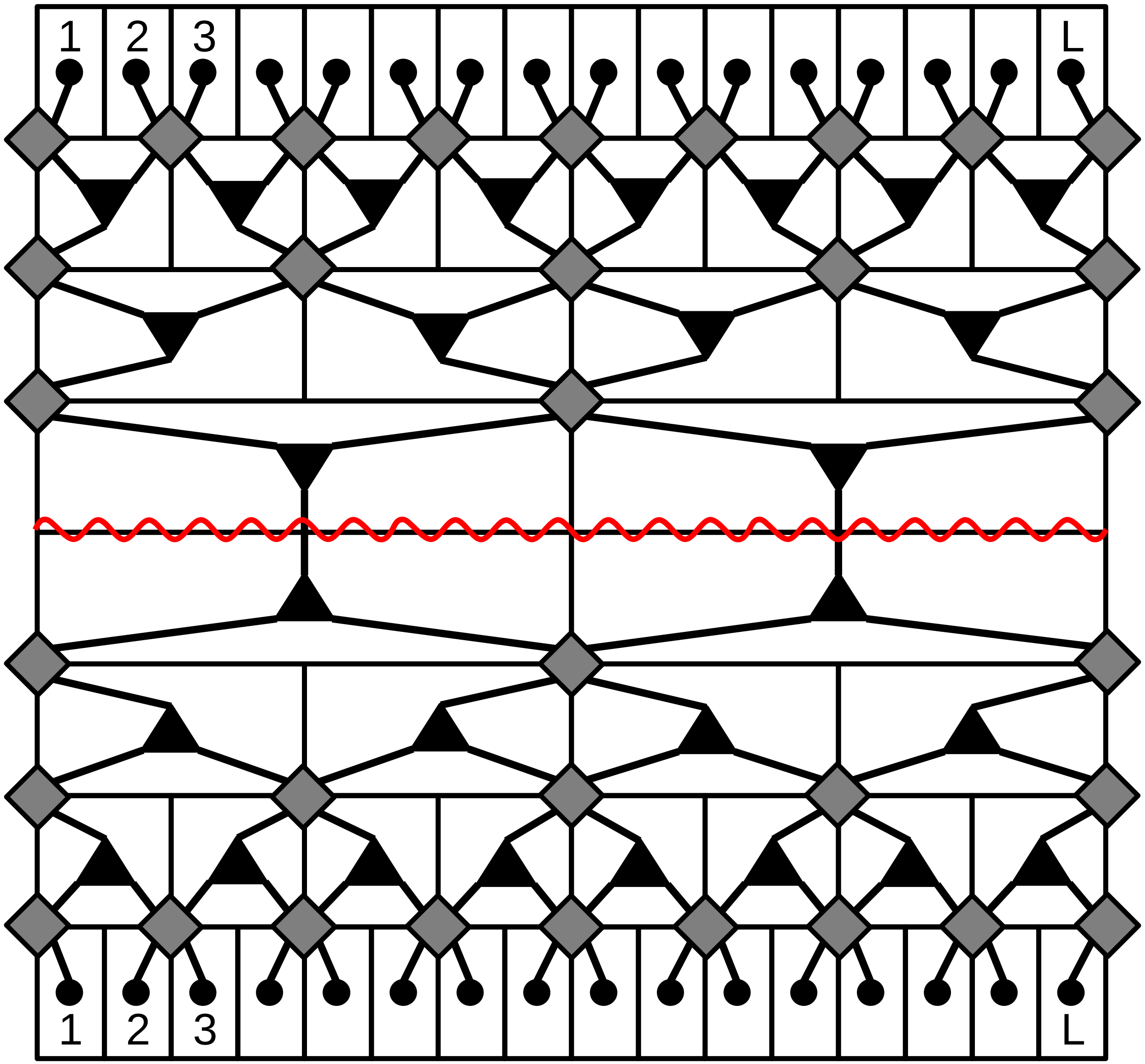
$$c^{\{m_j\}\{\tilde{n}_j\}} = \sum_{\alpha=1}^{\chi} A_\alpha^{\{m_j\}} A_\alpha^{\{\tilde{n}_j\}}$$

α ：event horizon → black hole entropy
= maximally entanglement entropy

Imagine Penrose diagrams …

T=0

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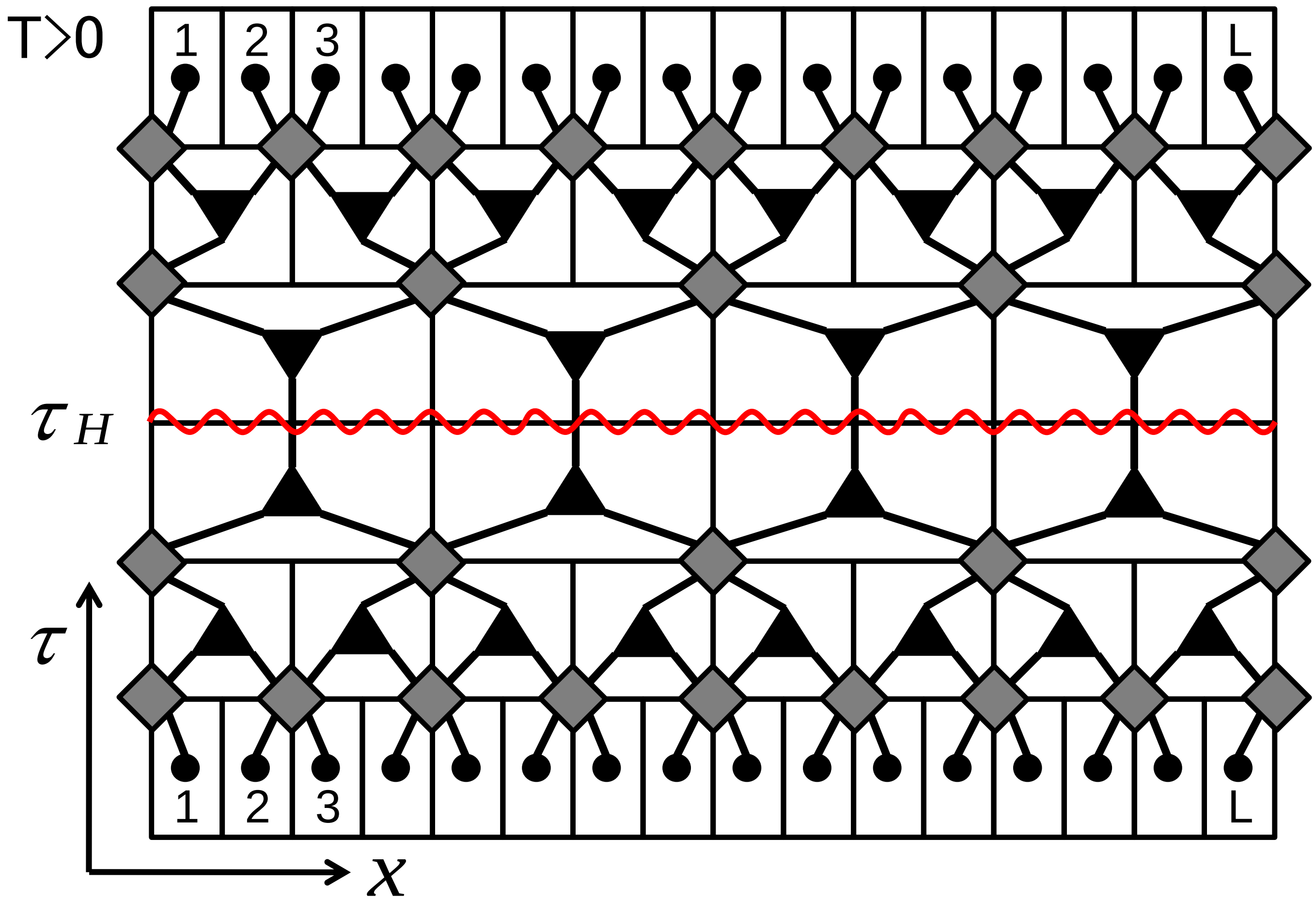
Finite-T MERA Network and AdS Black Hole



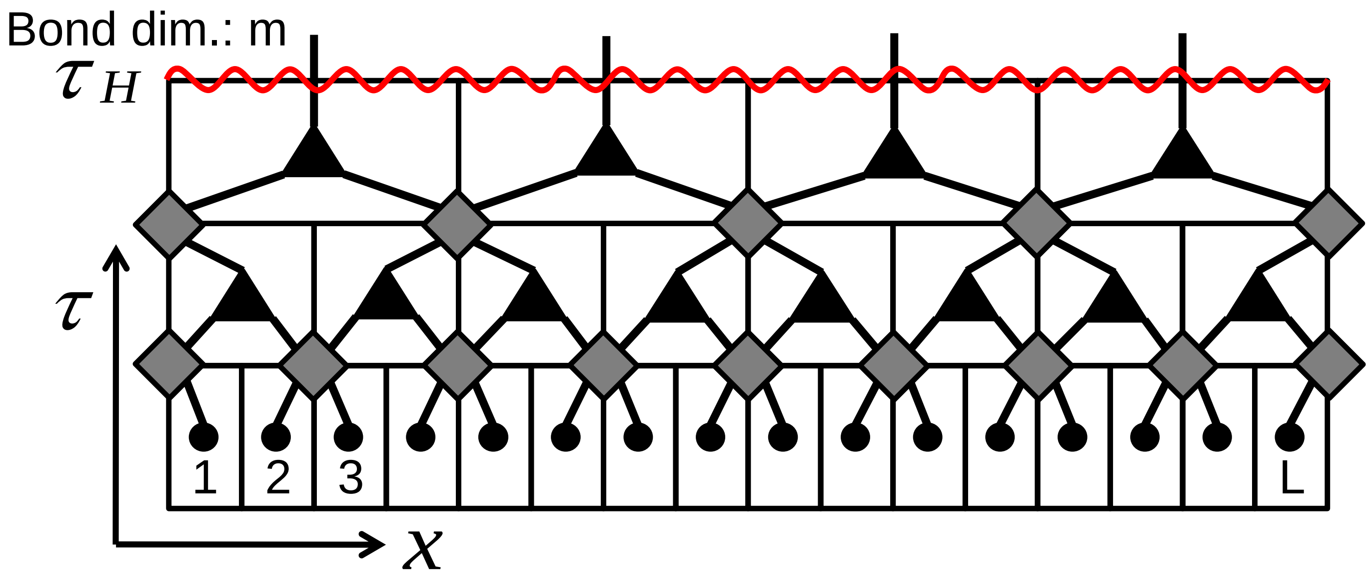
Vertical axis = energy scale, temperature scale

Wave function approach at finite-T → thermofield dynamics

→ Connection between original and tilde spaces

Temperature of MERA Network

Truncation of upper MERA layers = AdS black hole



$$z = 2^{\tau}$$

Area of interface: $\dfrac{L}{2^{\tau_H}} = A$ Total dim. at interface: $\chi = m^A$

Beckenstein–Hawking entropy & Calabrese–Cardy formula:

$$S_{BH} = A \ln m = \frac{L}{z_H} \ln m$$

$$S_{CFT} = \frac{c}{3} \ln\left(\frac{\beta}{\pi \varepsilon} \sinh\left(\frac{\pi L}{\beta} \right) \right)$$

$$k_B T = \left(\frac{3}{c\pi} \ln m \right) \frac{1}{z_H} \propto \frac{1}{z_H}$$

Summary

Formulation of Finite-T MERA

- finite-T wavefunction → TFD formalism
- relation to AdS black hole → $T \propto 1/z_H$
- relation to Penrose diagram / Kruskal extention

Information-geometrical Analysis of Quantum-Classical Correspondence

arXiv:1208.5103

Equivalence formulae between physically different systems

Bulk Geometry ⇔ Algebraic properties of the system on the edge

Statistical physics: Suzuki-Trotter transformation

d-dimensional "quantum" system

⇔ (d+1)-dimensional "classical" system

Statistical physics: Multiscale Entanglement Renormalization Ansatz (MERA)

String theory: Anti-de Sitter space / Conformal Field Theory (AdS/CFT) correspondence

(d+1)-dimensional General relativity on AdS space

⇔ CFT living on the boundary of the space

Condensed matter: Edge Modes in Topological Insulators

(2+1)-dimensional Einstein-Hilbert action

⇔ Chern-Simons action ⇒ Virasoro algebra

Information Geometrical Analysis of AdS/CFT Correspondence

Recent development of study of entanglement entropy
→ Area law, Calabrese-Cardy's formula, holographic entropy

'Relative' entropy = distance of abstract information space
→ We can define the metric of this space.

Relative von Neumann entropy (Kullback-Leibler divergence):

$$V(\lambda, \Lambda) = \sum_{i=1}^{m} \lambda_i \ln\left(\frac{\lambda_i}{\Lambda_i}\right) = \langle \Gamma \rangle - \langle \gamma \rangle$$

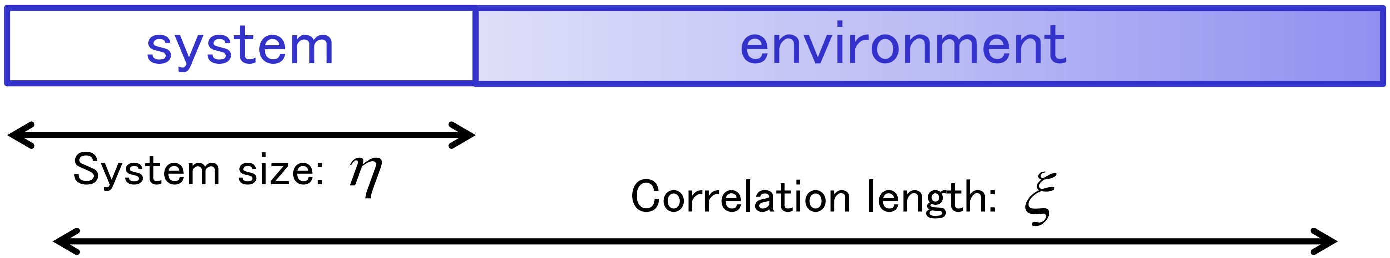
λ, Λ : two probability distributions of quantum critical systems

$$\lambda_i = e^{-\gamma_i} \qquad \Lambda_i = e^{-\Gamma_i}$$

$$\langle \gamma \rangle = \sum_{i=1}^{m} \lambda_i \gamma_i = -\sum_{i=1}^{m} \lambda_i \ln \lambda_i = S$$

λ depends on 'two' length scales even in 1D

Consider a superblock state Ψ in DMRG set up



Near a quantum critical point, $\xi > \eta$

Partial density matrix: $\rho = Tr_{env} |\psi\rangle\langle\psi|$

Entanglement spectrum

$$\lambda_i = e^{-\gamma_i}$$

DMRG → take m states with large eigenvalues

＊ This truncation limits the correlation length.

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m \qquad \sum_{i=1}^{m} \lambda_i = 1 \qquad \xi = \xi(m)$$

Numerical precision of DMRG is determined by m and ξ .

$$\lambda_i = \lambda_i(m, \eta) = \lambda_i(\xi, \eta) = \lambda_i(x^1, x^2)$$

Functional form of λ (ξ ,η)

Calabrese–Cardy's formula for entanglement entropy

$$S = \frac{1}{6} cA \ln \xi + b$$

c : central charge

A : number of boundary points

b : constant

$$S = -\sum_{i=1}^{m} \lambda_i \ln \lambda_i = \sum_{i=1}^{m} \lambda_i \gamma_i = \langle \gamma \rangle \qquad \lambda_i = e^{-\gamma_i}$$

$$C = \frac{1}{6} cA$$

$$\gamma_i(\xi,\eta) = C \ln \xi + a_i^0 + a_i^1 \frac{\eta}{\xi} + a_i^2 \left(\frac{\eta}{\xi}\right)^2 + \cdots$$

$$\langle a^0 \rangle = b$$

$$\langle a^n \rangle = 0, \ \ n = 1,2,\ldots$$

$$\lambda_i(\xi,\eta) = \frac{1}{\xi^C} \exp\left(a_i^0 - a_i^1 \frac{\eta}{\xi} - \cdots \right)$$

Derivation of AdS coordinate from Fisher information

$$\lambda_i = \lambda_i(x^1, x^2)$$
$$\Lambda_i = \lambda_i(x^1 + dx^1, x^2 + dx^2)$$

$$\Rightarrow \quad V(\lambda, \Lambda) = \frac{1}{2} g_{\mu\nu} dx^\mu dx^\nu$$

Fisher information matrix: γ is a kind of 'seed' to create $g_{\mu\nu}$

$$g_{\mu\nu} = \sum_{i=1}^{m} \lambda_i \frac{\partial \gamma_i}{\partial x^\mu} \frac{\partial \gamma_i}{\partial x^\nu} = \left\langle \frac{\partial \gamma}{\partial x^\mu} \frac{\partial \gamma}{\partial x^\nu} \right\rangle \qquad (\xi, \eta) = (x^1, x^2)$$

$$\gamma_i(\xi, \eta) = C \ln \xi + a_i^0 + a_i^1 \frac{\eta}{\xi} + \cdots \qquad C = \frac{1}{6} cA \qquad \langle a^1 \rangle = \langle a^2 \rangle = \cdots = 0$$

$$g_{zz} = \frac{1}{\xi^2} \left\{ C^2 - 2\langle a^1 \rangle C \left(\frac{\eta}{\xi} \right) + \langle a^1 a^1 \rangle \left(\frac{\eta}{\xi} \right)^2 \right\}$$

$$g_{\xi\eta} = \frac{1}{\xi^2} \left\{ \langle a^1 \rangle C - \langle a^1 a^1 \rangle \left(\frac{\eta}{\xi} \right) \right\}$$

$$g_{\eta\eta} = \frac{1}{\xi^2} \langle a^1 a^1 \rangle$$

RG fixed point → Emergent AdS (hyperbolic) space at IR region

$$V(\lambda,\Lambda)=\frac{1}{2}g_{\mu\nu}dx^{\mu}dx^{\nu}=\frac{C^2}{2}\frac{(d\xi)^2+(d\eta)^2}{\xi^2}$$

$$\langle a^1 a^1\rangle = C^2$$

$$\uparrow$$

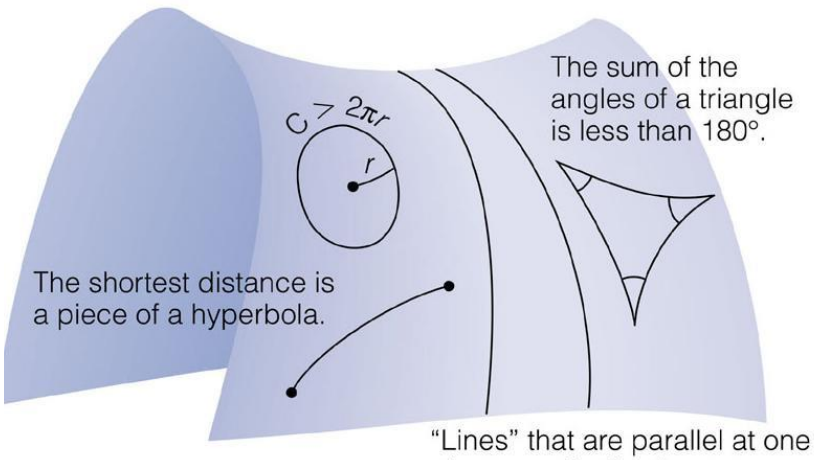
$$w=\eta+i\xi$$

$$\langle a^0\rangle=\frac{1}{6}cA=C$$

C = Curvature radius of AdS

→ Brown-Henneaux symmetry

$$c=\frac{3l}{2G}$$



$$\lambda_i(\xi,\eta)=\frac{1}{\xi^C}\exp\left(a_i^0-a_i^1\frac{\eta}{\xi}-\cdots\right)$$

Comparison with standard CFT

$$\eta << \xi$$

Complex coordinate: $w = \eta + i\xi$

$$\frac{(d\xi)^2 + (d\eta)^2}{\xi^2} \approx \frac{dw d\overline{w}}{|w|^2}$$

$$g_{\mu\nu} dx^\mu dx^\nu = \left\langle \frac{\partial\gamma}{\partial w} \frac{\partial\gamma}{\partial w} \right\rangle dw^2 + \left\langle \frac{\partial\gamma}{\partial \overline{w}} \frac{\partial\gamma}{\partial \overline{w}} \right\rangle d\overline{w}^2 + 2\left\langle \frac{\partial\gamma}{\partial w} \frac{\partial\gamma}{\partial \overline{w}} \right\rangle dw d\overline{w}$$

Laurent expansion:

$$\gamma_i(w,\overline{w}) = g_i + h_i \ln w + \overline{h}_i \ln \overline{w} + \cdots$$

$$\lambda_i(w,\overline{w}) = e^{-g_i} w^{-h_i} \overline{w}^{-\overline{h}_i}$$

$$\lambda_i(Aw, A\overline{w}) = A^{-h_i - \overline{h}_i} \lambda_i(w,\overline{w})$$

Conformal weight

$$\Delta_i = h_i + \overline{h}_i = 2\alpha_i$$

$$h_i = \alpha_i + i\beta_i$$

$$g_{\mu\nu} dx^\mu dx^\nu = 2i\langle\alpha\beta\rangle\left(\frac{dw^2}{w^2} - \frac{d\overline{w}^2}{\overline{w}^2}\right) + 4\langle\alpha^2\rangle \frac{dw d\overline{w}}{|w|^2} \Rightarrow C^2 \frac{dw d\overline{w}}{|w|^2}$$

$$|\alpha_i|^2 = |\beta_i|^2 \qquad \langle\alpha\beta\rangle = 0 \qquad \langle\alpha^2\rangle = \frac{C^2}{4}$$

ν : odd int.

$$\ln w \approx \ln(i\xi) + \frac{\eta}{i\xi} \Rightarrow \gamma_i = \underbrace{\boxed{2\alpha_i}}_{C} \ln\xi + \underbrace{\boxed{(g_i - \beta_i \nu\pi)}}_{a_i^0} + \underbrace{\boxed{2\beta_i}}_{a_i^1} \frac{\eta}{\xi}$$

Summary

<u>Information geometrical interpretation of AdS$_{2(+1)}$/CFT$_{1(+1)}$</u>

・Two length scales & Calabrese-Cardy's formula → $\lambda_i(\xi, \eta)$

$$\lambda_i(\xi,\eta) = \frac{1}{\xi^{\Delta_i}} \exp\left(a_i^0 - a_i^1 \frac{\eta}{\xi} - \cdots \right)$$

・Derivation of metric from Fisher information

▶ asymptotic AdS emerges in IR region.

▶ curvature radius of AdS ⇔ central charge

・Classical side:

The original quantum data are decomposed into a set of different length-scale physics, and they are stored into different layers in AdS. Averaging over microscopic degrees of freedom, the information of criticality only remains, and is converted into the symmetry of the emerged classical space.