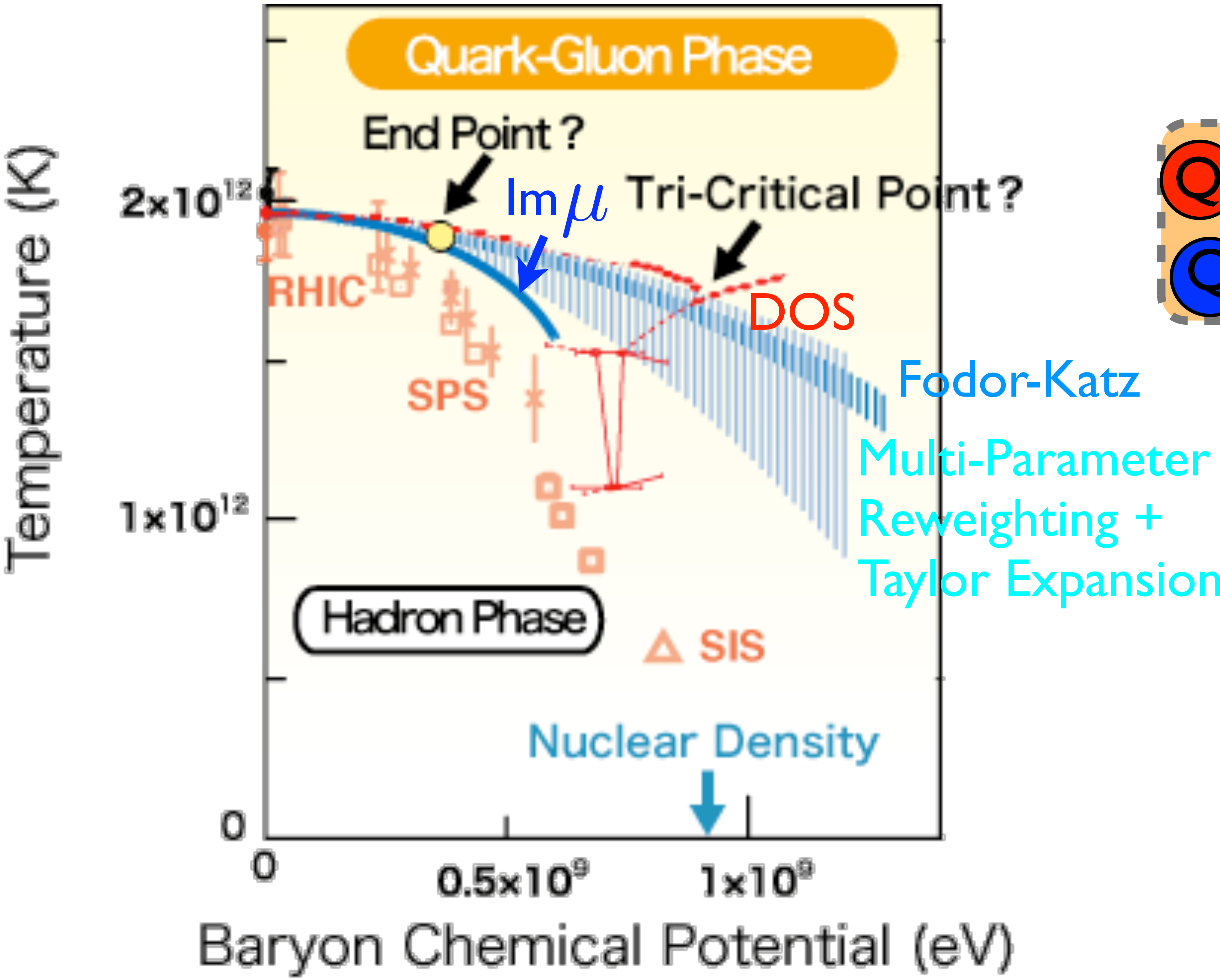
有限密度格子QCD

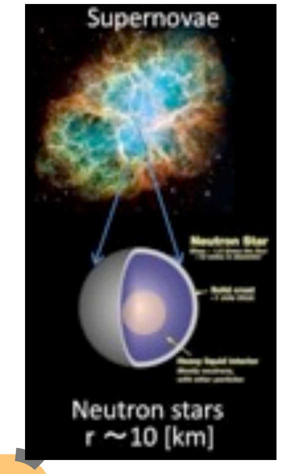
Lattice QCD at Finite Density

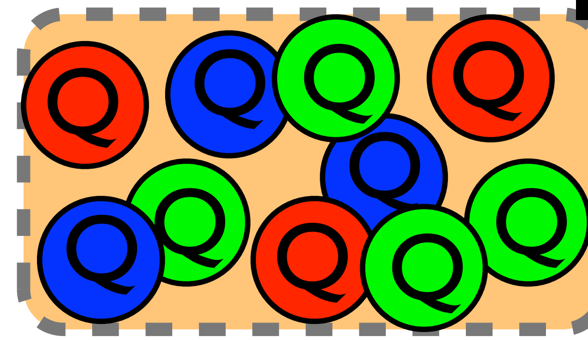
ゼロからの格子ＱＣＤ入門

-- 有限バリオン密度系の研究を目指して --

素核宇宙融合 レクチャーシリーズ」第９回


Quark-Gluon Phase
End Point ?
Im μ
Tri-Critical Point ?
DOS
RHIC
SPS
Fodor-Katz
Multi-Parameter
Reweighting +
Taylor Expansion
Hadron Phase
SIS
Nuclear Density
Temperature (K)
2x10^12
1x10^12
0
0
0.5x10^9
1x10^9
Baryon Chemical Potential (eV)


Supernovae
Neutron stars
r ~10 [km]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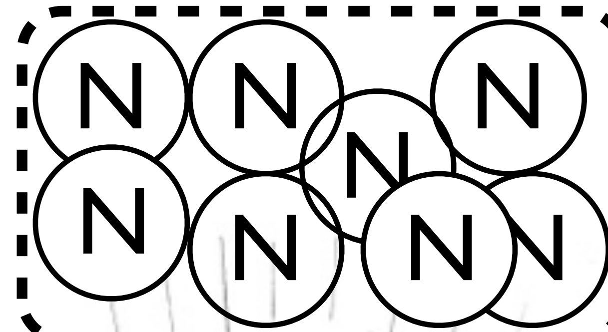
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Lattice QCD

- 1st Principles Calculation
- Sound Base for Hadron/Quark Systems at finite Temperature and Density

Origin of the Sign-Problem in Lattice QCD Simulation at finite Density

QCD at finite density

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi\, e^{-\beta S_G - \bar{\psi}\Delta\psi}$$

$$= \int \mathcal{D}U \prod_f \det \Delta(m_f)\, e^{-\beta S_G}$$

$$\Delta(\mu) = D_\nu \gamma_\nu + m + \mu\gamma_0$$

$$\boxed{\Delta(\mu)^\dagger = -D_\nu\gamma_\nu + m + \mu^*\gamma_0 = \gamma_5 \Delta(-\mu^*)\gamma_5}$$

$$(\det \Delta(\mu))^* = \det \Delta(\mu)^\dagger = \det \Delta(-\mu^*)$$

$(\det\Delta(\mu))^* = \det\Delta(\mu)^\dagger = \det\Delta(-\mu^*)$

For $\mu = 0$

$$(\det\Delta(0))^* = \det\Delta(0)$$

$\det\Delta \Rightarrow$ *Real*

For $\mu \neq 0$ (in general)

$\det\Delta \Rightarrow$ *Complex*

$$Z = \int \mathcal{D}U \boxed{\prod_f \det\Delta(m_f, \mu_f)} e^{-\beta S_G}$$

Complex $\Rightarrow$ Sign Problem

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U \, O \, \det \Delta \, e^{-\beta S_G}$$

In Monte Carlo simulation, configurations are generated according to the Probability:

$$\det \Delta \, e^{-\beta S_G} / Z$$

$\det \Delta$: *Complex*

Monte Carlo Simulations very difficult !

$$\langle O \rangle = \frac{\int DU O \det \Delta e^{-S_G}}{\int DU \det \Delta e^{-S_G}}$$

$$\det \Delta = |\det \Delta| e^{i\theta}$$

$$\langle O \rangle = \frac{\int DU O |\det \Delta| e^{i\theta} e^{-S_G}}{\int DU |\det \Delta| e^{-S_G}} \times \frac{\int DU |\det \Delta| e^{-S_G}}{\int DU |\det \Delta| e^{i\theta} e^{-S_G}}$$

$$= \frac{\left\langle O e^{i\theta} \right\rangle_{|\det|}}{\left\langle e^{i\theta} \right\rangle_{|\det|}}$$

π 凝縮の恐怖

Phase Quench = Finite-Isospin

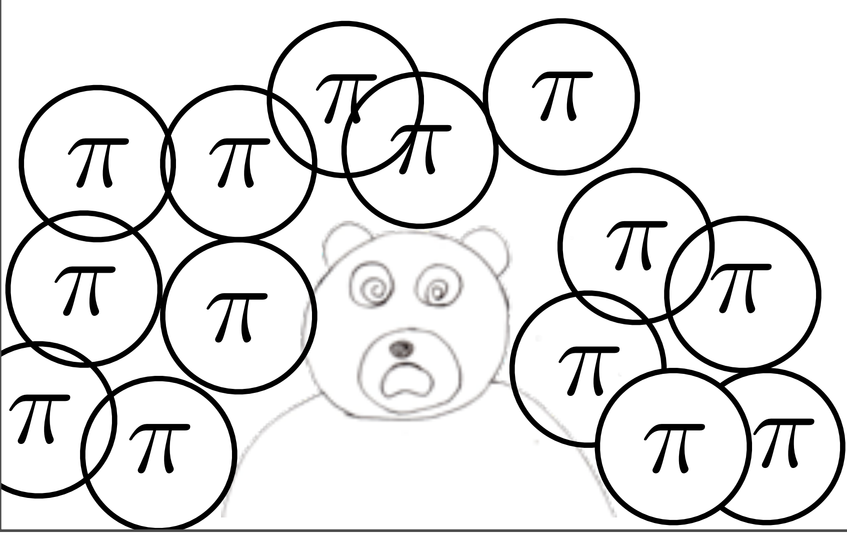
$$\int |\det \Delta(\mu)|^2 e^{-S_G} = \int \det \Delta(\mu) \det \Delta(\mu)^* e^{-S_G}$$

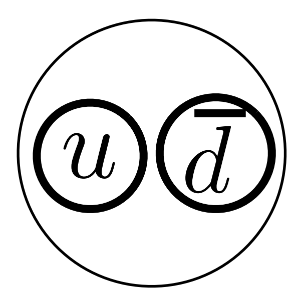
$$= \int \det \Delta(\mu) \det \Delta(-\mu) e^{-S_G}$$

$$= \int \det \Delta(\mu_u) \det \Delta(\mu_d) e^{-S_G}$$

$$\mu_u = \mu, \quad \mu_d = -\mu$$

$\mu > \dfrac{m_\pi}{2}$ ならば





π^+を外場 μ がいくらでも作ってしまう

QCD In-Equality

Do you have enough Time ?

$$\mathrm{Tr}AB^\dagger \leq \sqrt{\mathrm{Tr}AA^\dagger}\sqrt{\mathrm{Tr}BB^\dagger}$$

(Cauchy-Schwarz)

Meson $\bar{\psi}\Gamma\psi$

Propagators

$$\mathrm{Tr}\Delta^{-1}(x,0)\Gamma\boxed{\Delta^{-1}(0,x)}\Gamma$$

$$= \mathrm{Tr}\Delta^{-1}(x,0)\Gamma\boxed{\gamma_5\Delta^{-1}(x,0)^\dagger\gamma_5}\Gamma$$

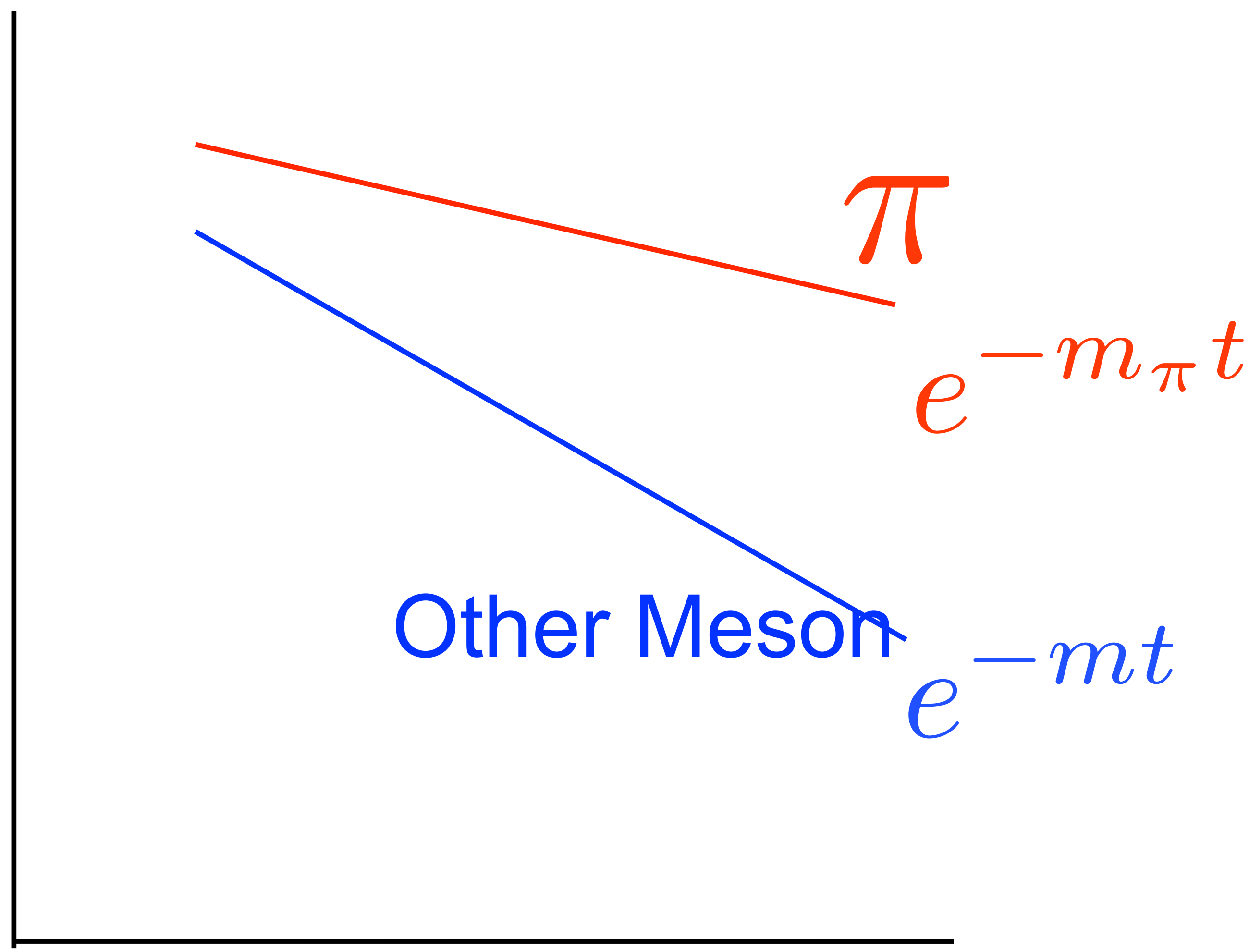
$$\leq \sqrt{\Delta^{-1}(x,0)\Delta^{-1}(x,0)^\dagger}$$

$$\sqrt{\Gamma\gamma_5\Delta^{-1}(x,0)^\dagger\gamma_5\Gamma\left(\Gamma\gamma_5\Delta^{-1}(x,0)^\dagger\gamma_5\Gamma\right)^\dagger}$$

$$= \mathrm{Tr}\Delta^{-1}(x,0)\Delta^{-1}(x,0)^\dagger$$

πのPropagators

$$\left(\text{Any Meson Propagators} \right) \leq \left(\text{pi Meson Propagators} \right)$$



$$\boxed{m_\pi \leq m}$$

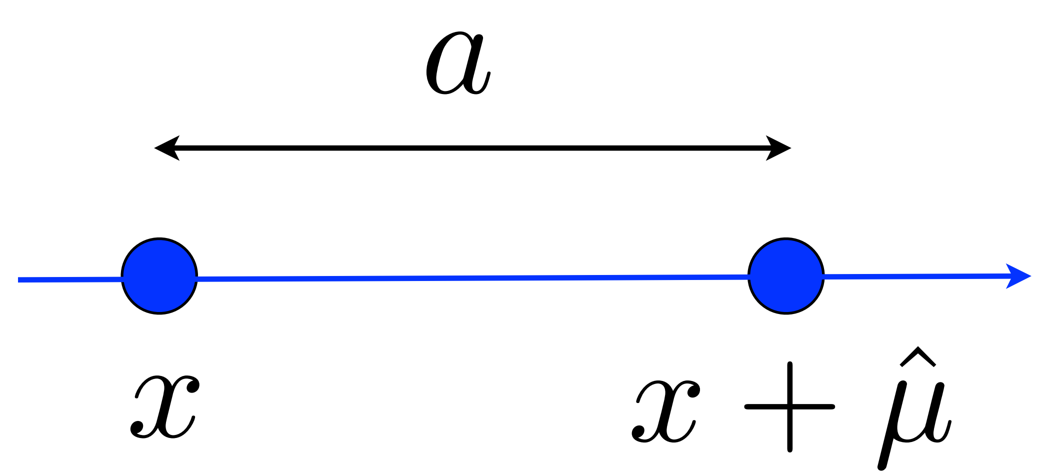
π is lightest.

This does not hold for $\mu \neq 0$

格子の上では

$$\bar{\psi}(x)\partial_\mu \psi(x)$$

$$\bar{\psi}(x)\psi(x+\hat{\mu})$$



$$\bar{\psi}(x)e^{i\int_x^{x+\hat{\mu}} A_\mu dx_\mu}\psi(x+\hat{\mu})$$

$$= \bar{\psi}(x)U_\mu(x)\psi(x+\hat{\mu})$$

$$U_\mu(x) = e^{iA_\mu(x)a}$$

化学ポテンシャルはどこに入る

$$\Delta(x,x') = \delta_{x,x'} - \kappa \sum_{i=1}^{3} \left\{ (1-\gamma_i) U_i(x) \delta_{x',x+\hat{i}} \right.$$
$$\left. +(1+\gamma_i) U_i^\dagger(x') \delta_{x',x-\hat{i}} \right\}$$
$$-\kappa \left\{ e^{+\mu} (1-\gamma_4) U_4(x) \delta_{x',x+\hat{4}} \right.$$
$$\left. + e^{-\mu} (1+\gamma_4) U_4^\dagger(x') \delta_{x',x-\hat{4}} \right\}$$

$$\Delta(x,x') = m\delta_{x,x'} + \frac{1}{2} \sum_{i=1}^{3} \eta_i(x) \left\{ U_i(x) \delta_{x',x+\hat{i}} \right.$$
$$\left. - U_i^\dagger(x') \delta_{x',x-\hat{i}} \right\}$$
$$+ \frac{1}{2} \eta_4(x) \left\{ e^{+\mu} U_4(x) \delta_{x',x+\hat{4}} - e^{-\mu} U_4^\dagger(x') \delta_{x',x-\hat{4}} \right\}$$

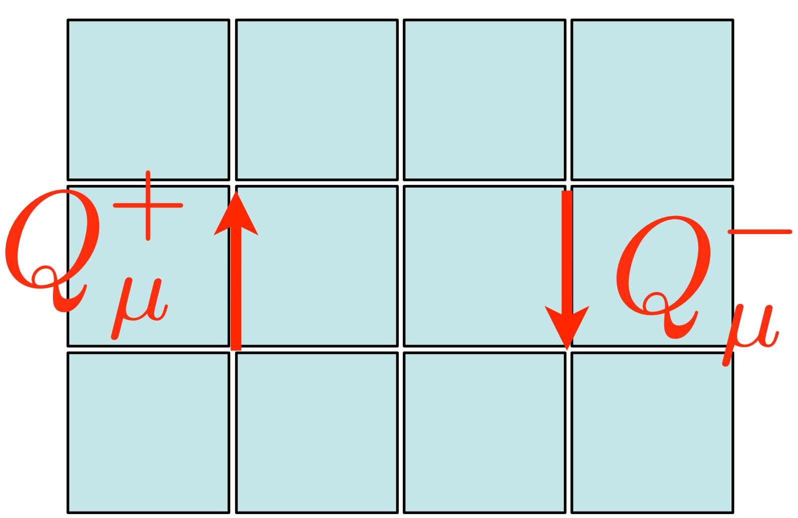
符号問題のオリジン

Wilson Fermions $\quad \Delta = I - \kappa Q$

KS(Staggered) Fermions $\quad \Delta = m - Q'$

$$= m\left(I - \frac{1}{m}Q\right)$$

$$Q = \sum_{i=1}^{3}\left(Q_i^+ + Q_i^-\right) + \left(e^{+\mu}Q_4^+ + e^{-\mu}Q_4^-\right)$$



$$Q_\mu^+ = **U_\mu(x)\delta_{x',x+\hat{\mu}}$$

$$Q_\mu^- = **U_\mu^\dagger(x')\delta_{x',x-\hat{\mu}}$$

$$\det\Delta = e^{\mathrm{Tr}\log\Delta} = e^{\mathrm{Tr}\log(I-\kappa Q)}$$

$$= e^{-\sum_n \frac{1}{n}\kappa^n \mathrm{Tr} Q^n}$$

Hopping Parameter 展開

or

Large Mass 展開

消えないのは閉ループになる項

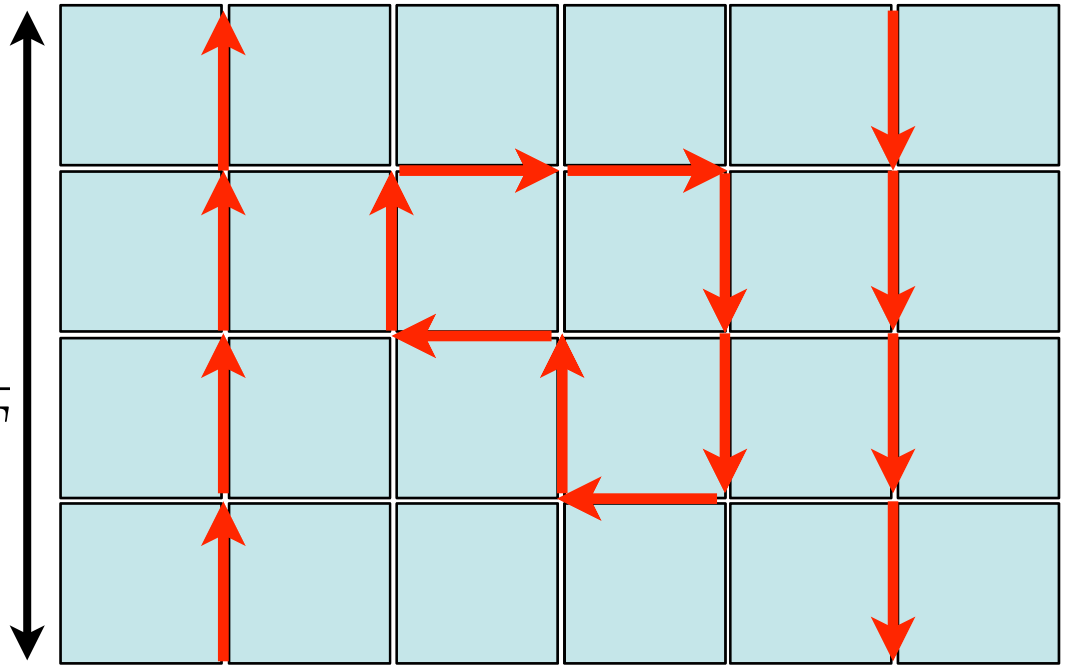
μ **依存性を持つ最低次は**

$$\kappa^{N_t} e^{\mu N_t} \mathrm{Tr}(Q^+ \cdots Q^+)$$
$$= **\,\kappa^{N_t} e^{\mu/T} \mathrm{Tr} L$$

$$\kappa^{N_t} e^{-\mu N_t} \mathrm{Tr}(Q^- \cdots Q^-)$$
$$= **\,\kappa^{N_t} e^{-\mu/T} \mathrm{Tr} L^\dagger$$

$\mathrm{Tr}L$: Polyakov Loop

$N_t = \frac{1}{T}$



両方の寄与

$$**\,\kappa^{N_t}\left(\cosh\frac{\mu}{T}\Re\mathrm{Tr}L + i\sinh\frac{\mu}{T}\Im\mathrm{Tr}L\right)$$

符号問題が無い場合もあります

え、本当？

考えれば
すぐ分かるよ

最終解決では
ないけど

符号問題が無い場合

1.虚数化学ポテンシャル

- $$(\det\Delta(\mu))^* = \det\Delta(-\mu^*)$$

$$\mu = i\mu_I \quad \Rightarrow \quad (\det\Delta(\mu_I))^* = \det\Delta(\mu_I)$$

2.カラーSU(2)

$$U_\mu^* = \sigma_2 U_\mu \sigma_2$$

$$\det\Delta(U,\gamma_\mu)^* = \det\Delta(U^*,\gamma_\mu^*) = \det\sigma_2\Delta(U,\gamma_\mu^*)\sigma_2$$

$$= \det\Delta(U,\gamma_\mu)$$

3.アイソベクトル型（finite iso-spin)

$$\mu_d = -\mu_u$$

$$\det\Delta(\mu_u)\det\Delta(\mu_d) = \det\Delta(\mu_u)\det\Delta(-\mu_u)$$

$$= \det\Delta(\mu_u)\det\Delta(\mu_u)^* = \left|\det\Delta(\mu_u)\right|^2$$

(Phase Quench)

Multi-Parameter Reweighting の誤解と真実？

有限密度QCDは

Fodor-Katz
が解決したって

聞いたけど

Towards large density QCD
What is a real Obstacle ?

- Multi-parameter Re-weighting by Fodor-Katz

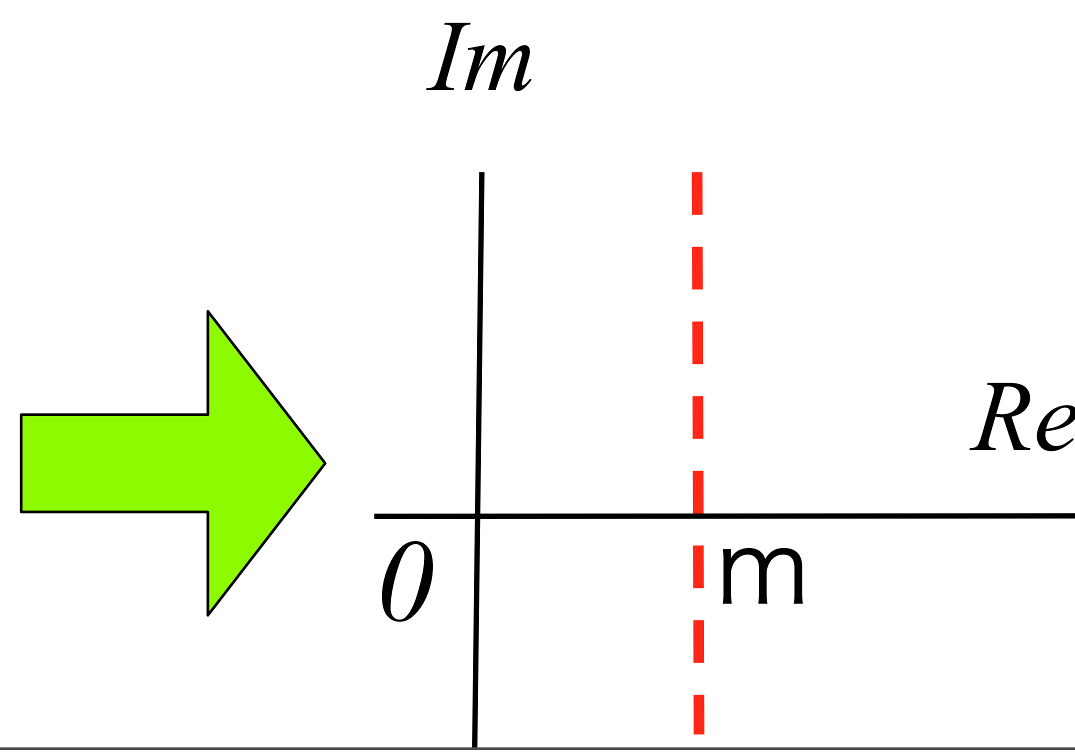
$$\langle O\rangle = \frac{1}{Z}\int \mathcal{D}U\, O\, \det\Delta(\mu)\, e^{-\beta S_G}$$

$$= \frac{1}{Z}\int \mathcal{D}U\, O\, \underbrace{\det\Delta(0)\, e^{-\beta_0 S_G}}_{\text{測度}}\, \underbrace{\frac{\det\Delta(\mu)}{\det\Delta(0)}\, e^{(\beta_0-\beta)S_G}}_{\text{Reweighting Factor}}$$

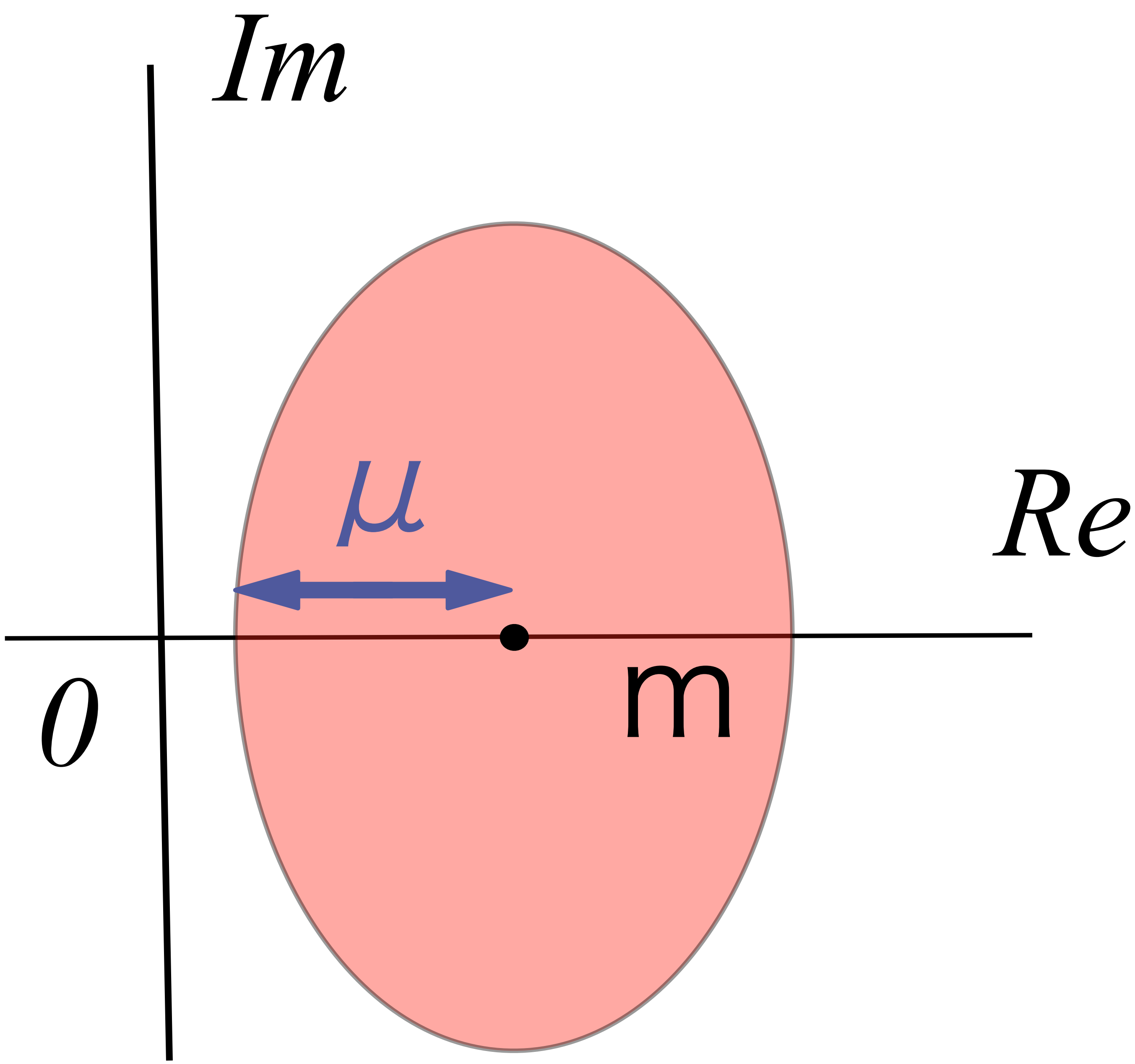
$$\Delta(\mu = 0) = D_\nu\gamma_\nu + m$$

$D_\nu\gamma_\nu$: anti-Hermite

Eigen Value Distribution



When μ increases



$$\frac{|\lambda|_{\max}}{|\lambda|_{\min}} \to \infty$$

Conjugate Gradient to calculate

$$\Delta(\mu)^{-1}$$

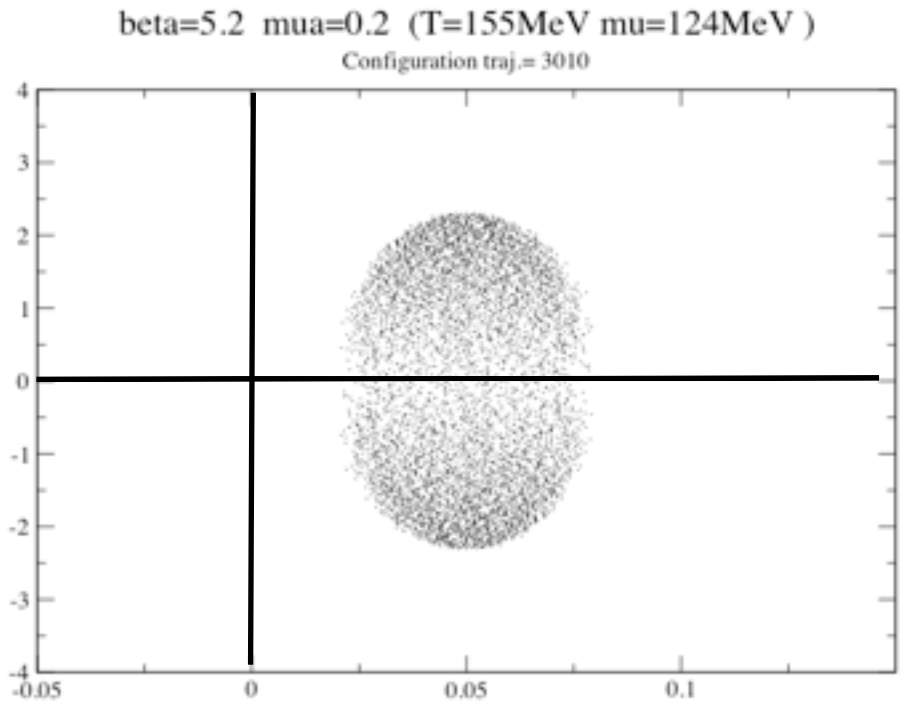
does not converge

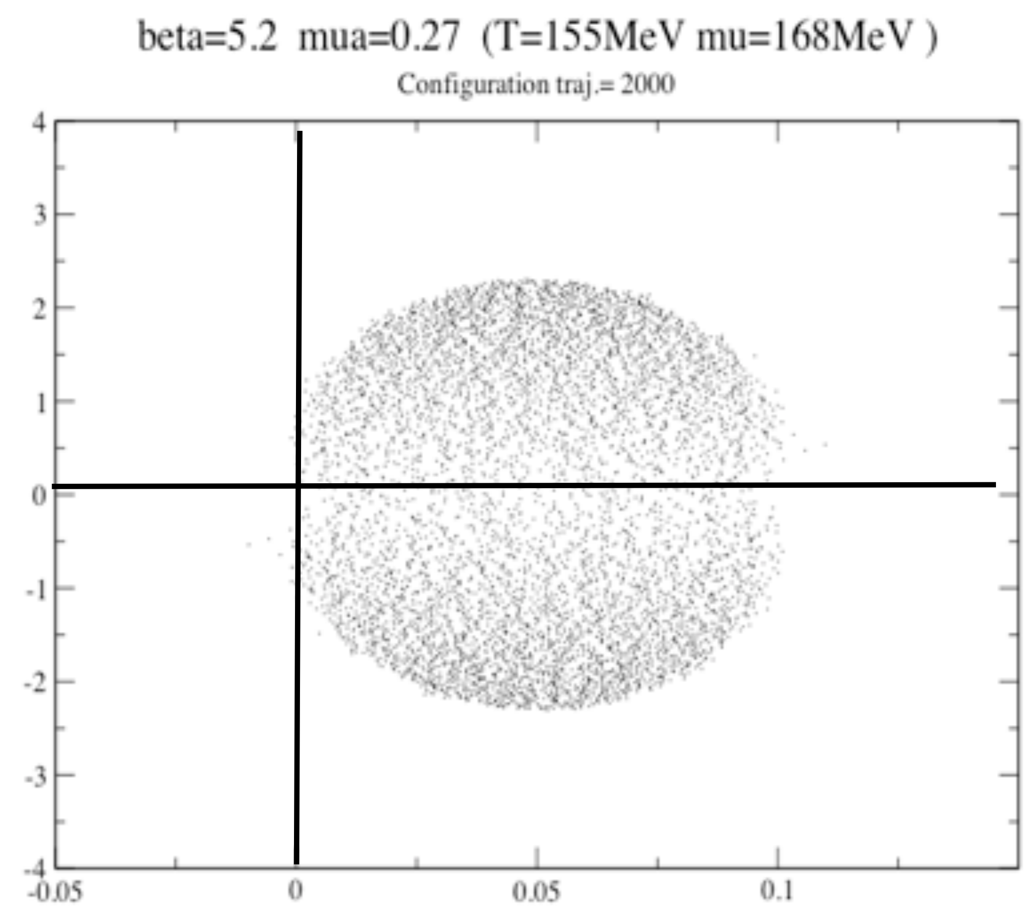
(Imaginary chemical Potential calculation does not have this problem.)

Eigen Value Distribution

- All full QCD update algorithms require $\Delta(\mu)^{-1}$
- Fodor-Katz algorithm does not calculate $\Delta(\mu)^{-1}$ but evaluate

$$\frac{\det\Delta(\mu)}{\det\Delta(0)}$$


beta=5.2 mua=0.2 (T=155MeV mu=124MeV)
Configuration traj.= 3010
4
3
2
1
0
-1
-2
-3
-4
-0.05
0
0.05
0.1

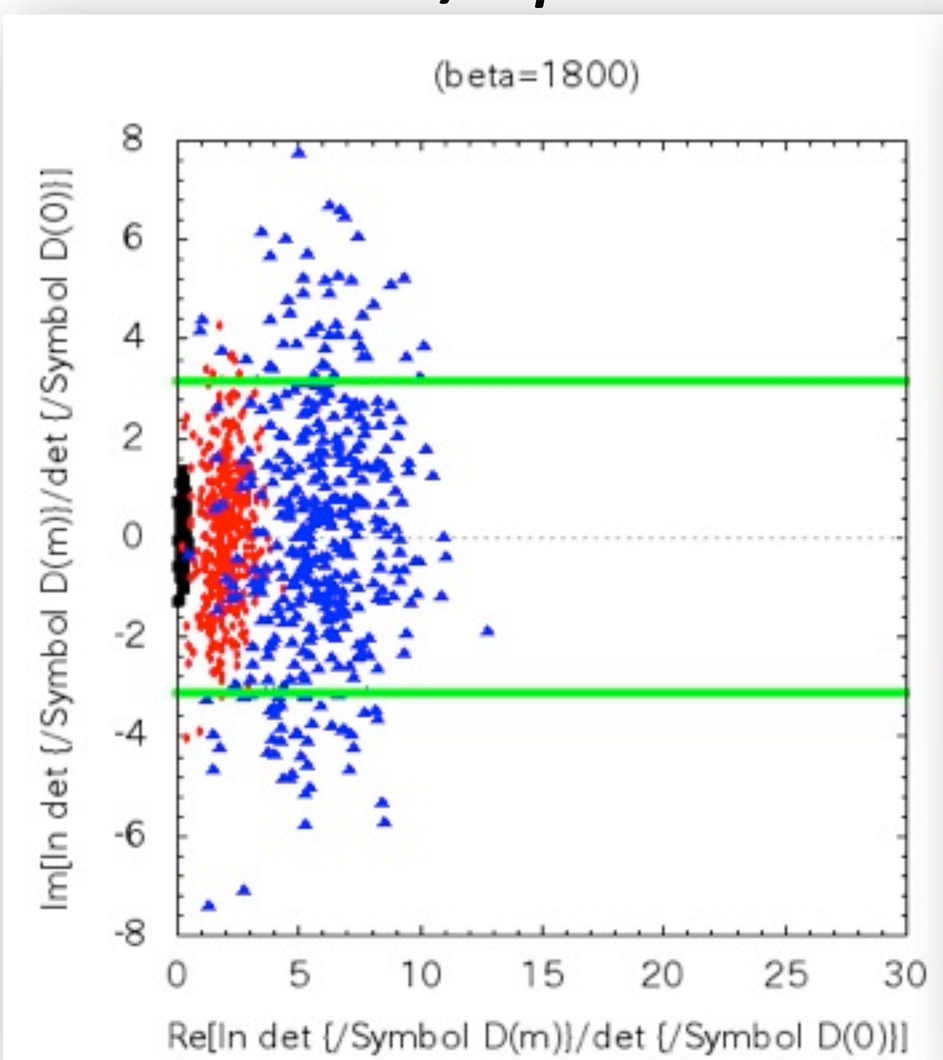

beta=5.2 mua=0.27 (T=155MeV mu=168MeV)
Configuration traj.= 2000
4
3
2
1
0
-1
-2
-3
-4
-0.05
0
0.05
0.1

Reweighting factor

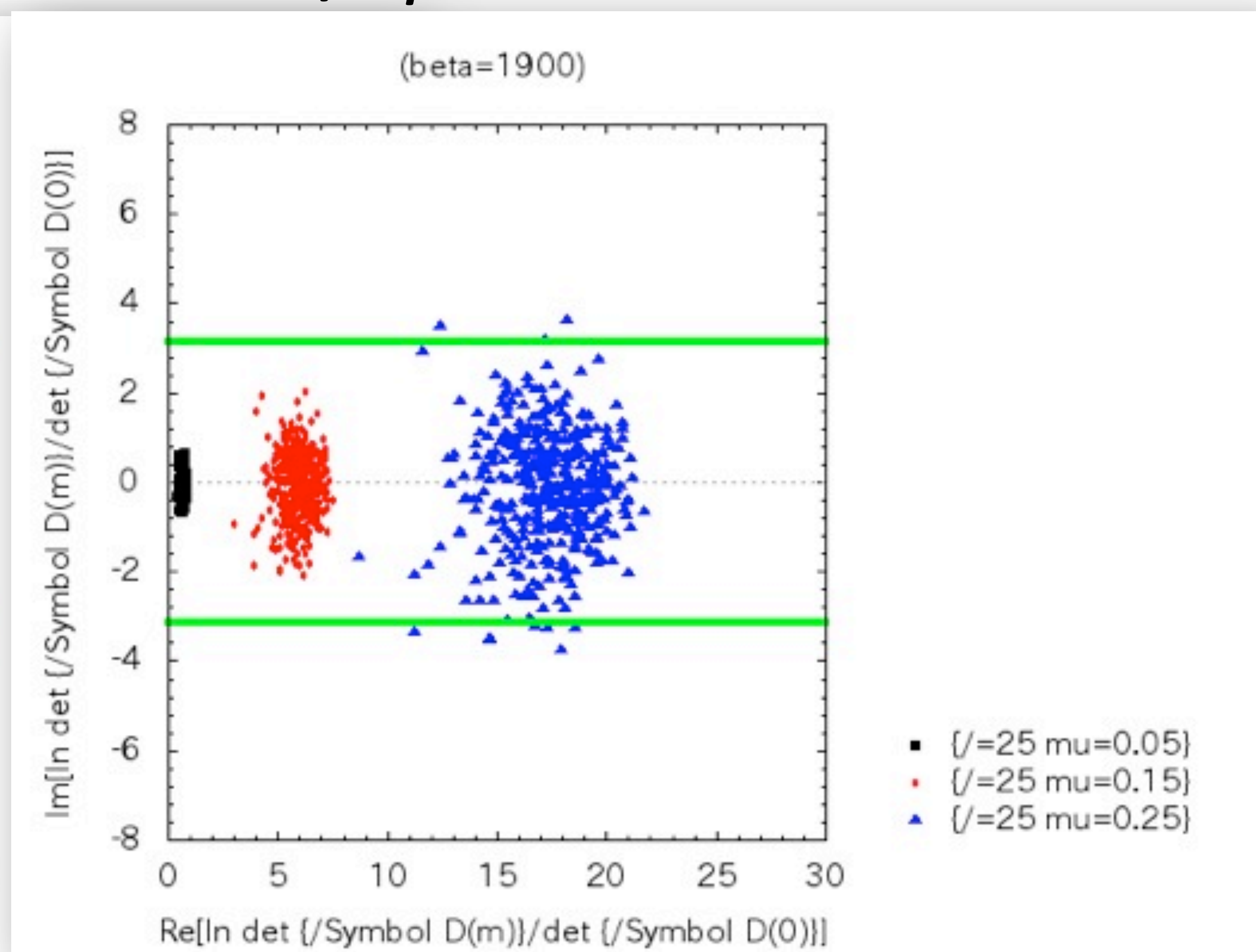
$$R(\mu,0) = \frac{[\det D(\mu)]^{Nf}}{[\det D(0)]^{Nf}}$$

Re [ln R] v.s. Im[ln R]

T/Tpc=0.95



T/Tpc=1.1



テーラー展開法

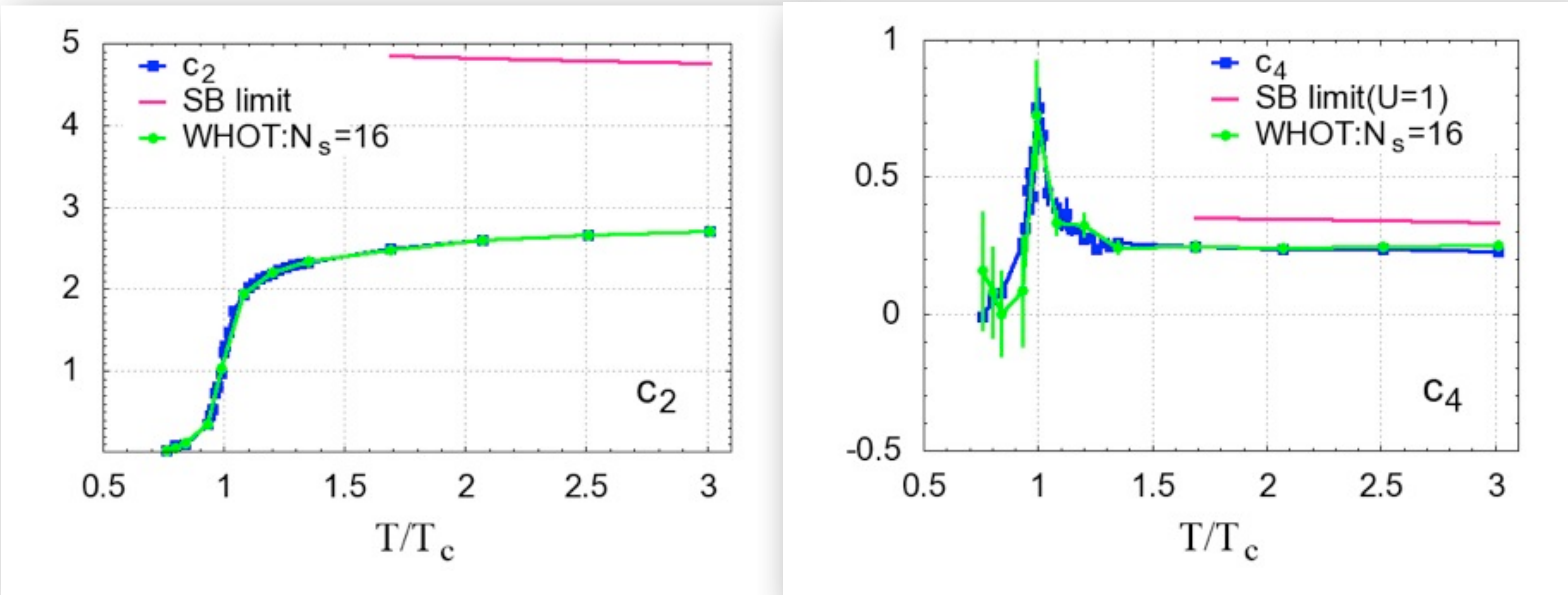
- アイデアはシンプル
  - 有限のμで計算できないのなら、μ=0でテーラー展開

$$f(\mu) = f(0) + \mu f'(0) + \frac{1}{2}\mu^2 f''(0) + \cdots$$

  - QCD-TARO: Phys. Rev. D65 (2002) 054501 $\Rightarrow$ Screening Mass
  - Swansea-Bielefeld: Phys. Rev. D66, 074507 (2002) $\Rightarrow$ EoS

$$\frac{p(\mu)}{T^4} = c_0 + c_2\left(\frac{\mu}{T}\right)^2 + c_4\left(\frac{\mu}{T}\right)^4 + \cdots$$

Taylor coefficients(c2 & c4)



Nagata, Nakamura, JHEP 1204 (2012) 092

"WHOT" : WHOT QCD,
Ejiri et al., PR**D82**, 014508 (2010).

For staggered, see e.g.
Allton et al., PRD71, 054508 ('05)

clover-Wilson + RG-gauge(Nf=2)

Volume : 8^3x4

quark mass : mps/mV ~ 0.8

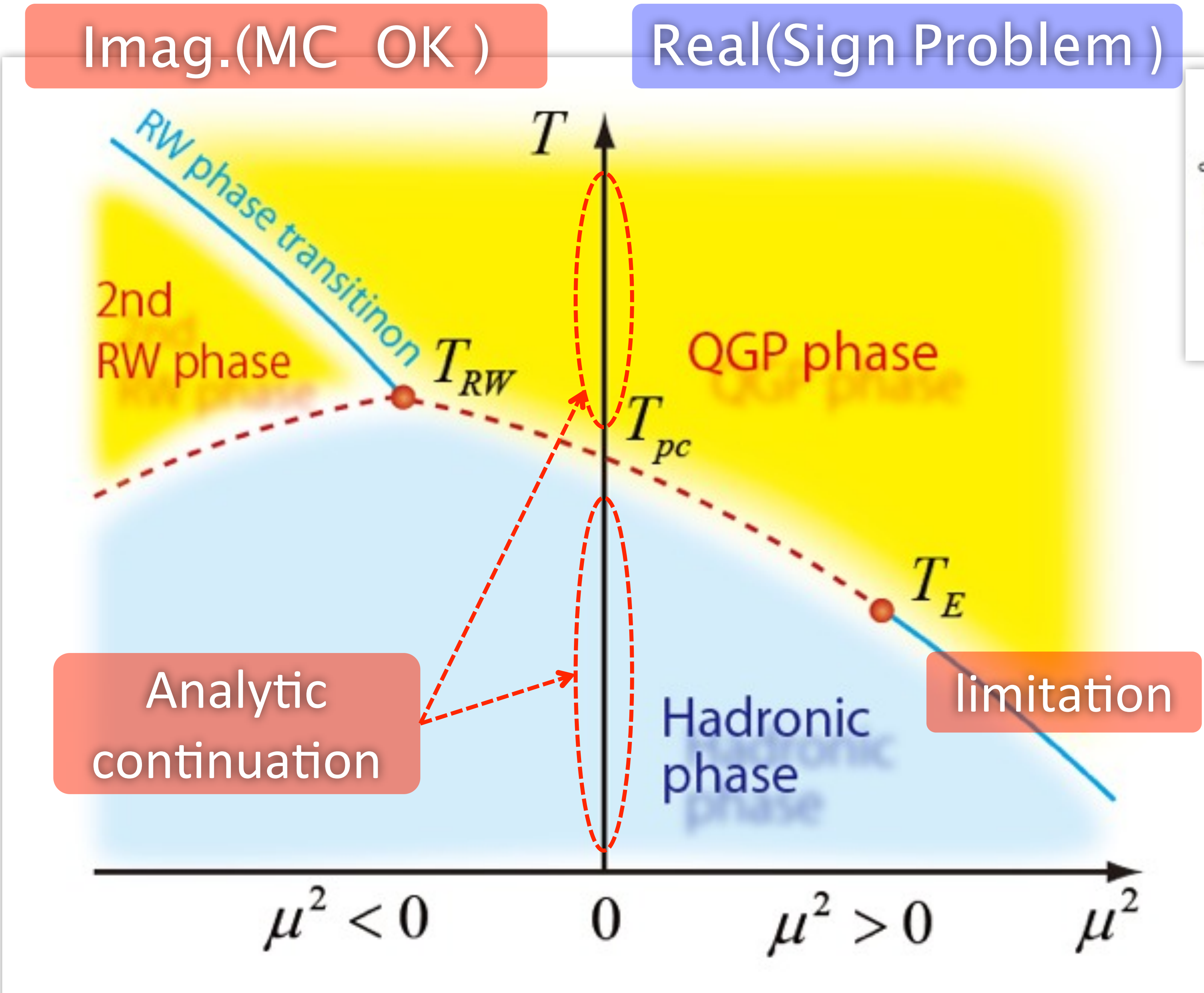
Configurations : HMC at mu=0

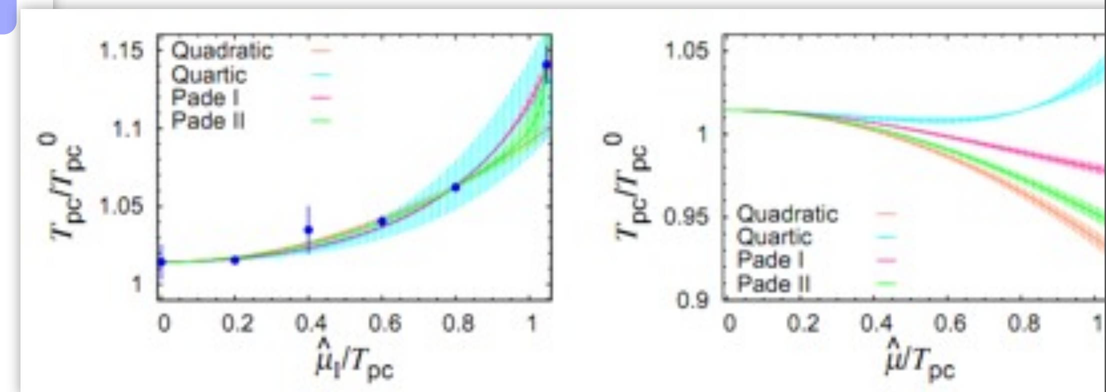
11 K steps including (therm. as 3-5K)

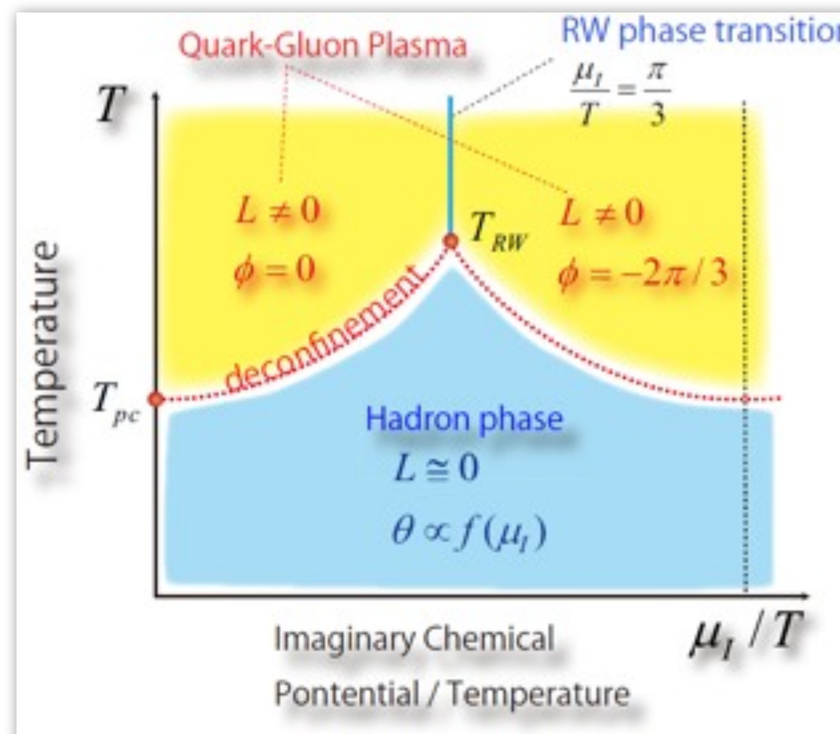
Eigen values : 400 configs.

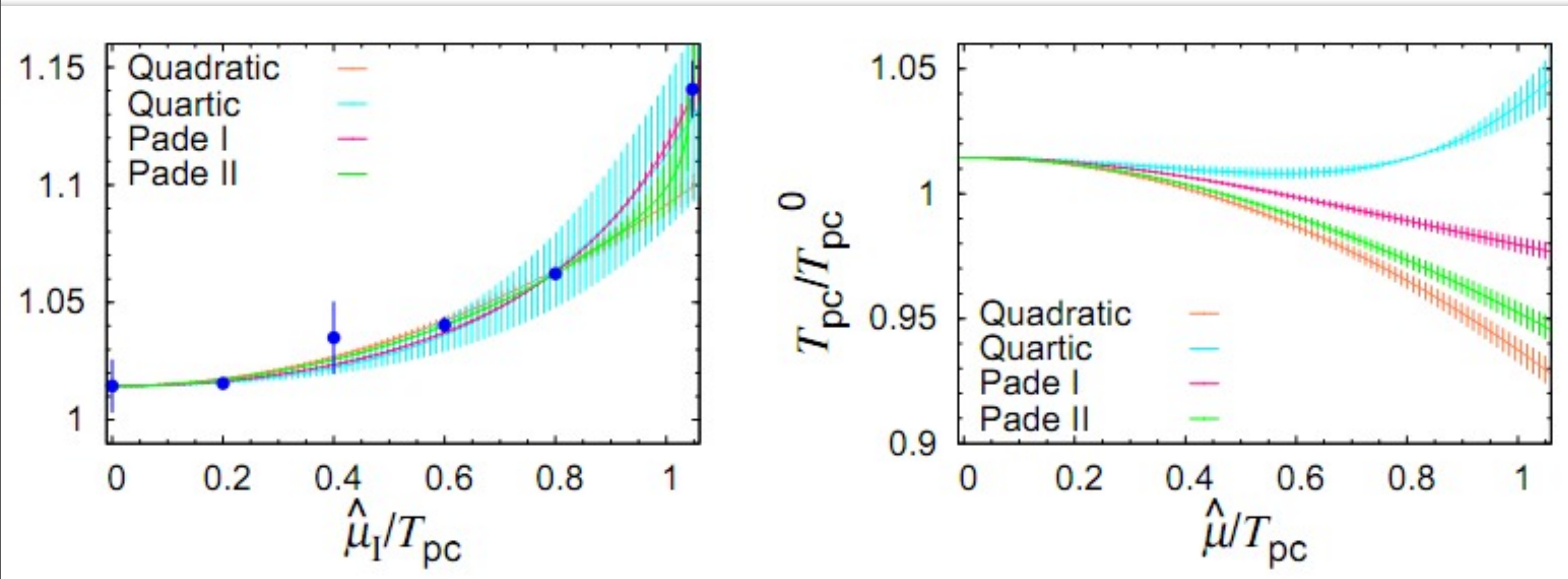
Imaginary chemical potential

- *MC simulations in mu^2<0 region and analytic continuation*








Quadratic
Quartic
Pade I
Pade II
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
$\hat{\mu}_I/T_{pc}$
T_{pc}/T_{pc}^0
1.05
1
0.95
0.9
Quadratic
Quartic
Pade I
Pade II
$\hat{\mu}/T_{pc}$

他の攻め方は？

統計力学で
何か教わっ
たっけ？

Canonical Approach

- Miller and Redlich
  - Phys. Rev. D35 (1987) 2524
- A.Hasenfratz and Toussaint
  - Nucl.Phys.B371 (1992) 539
- Engels, Kaczmarek, Karsch and Laermann
  - Nucl.Phys. B558 (1999) 307 (hep-lat/9903030)
  - hep-lat/9905022
- Forcrand and Kratochvila
  - Nucl. Phys. B (P.S.) 153 (2006) 62 (hep-lat/0602024)
- A.Li, Meng, Alexandru, K-F. Liu
  - PoS LAT2008:032 and 178 (arXiv:0810.2349, arXiv:0811.2112),
  - Phys.Rev. D82, 054502, 2010 (arXiv:1005.4158)

Fugacity Expansion of $\det \Delta(\mu)$

$$\det \Delta(\mu) = C_0 \xi^{-Nr/2} \det(Q + \xi),$$

$$\det \Delta(\mu) = C_0 \xi^{-Nr/2} \prod_{n=1}^{Nr} (\lambda_n + \xi)$$

$$\begin{pmatrix} \lambda_1 + \xi & & & \\ & \lambda_2 + \xi & & \\ & & \ddots & \\ & & & \lambda_{N_r} + \xi \end{pmatrix}$$

- *Fugacity expansion*

$$\det \Delta(\mu) = C_0 \sum_{-2N_c N_s^3}^{2N_c N_s^3} c_n \xi^n$$

For details, see
Nagata-Nakamura Phys.Rev.D82:094027,2010

Calculation of Zn

XQCD-J

$$Z_{GC}(\mu, T) = \int \mathcal{D}U \, [\det\Delta(\mu)]^{N_f} e^{-\beta S_G}.$$

$$\det\Delta(\mu) = C_0 \sum c_n \xi^n$$

$$Z_{GC}(\mu) = \int DU \left[\frac{C_0 \sum c_n \xi^n}{\det\Delta(0)}\right]^{N_f} (\det\Delta(0))^{N_f} e^{-S_G}$$

$$Z_{GC}(\mu) = \sum Z_n \xi^n$$

Simulation Setup

clover-Wilson + RG-gauge(Nf=2)

Volume : 8^3x4

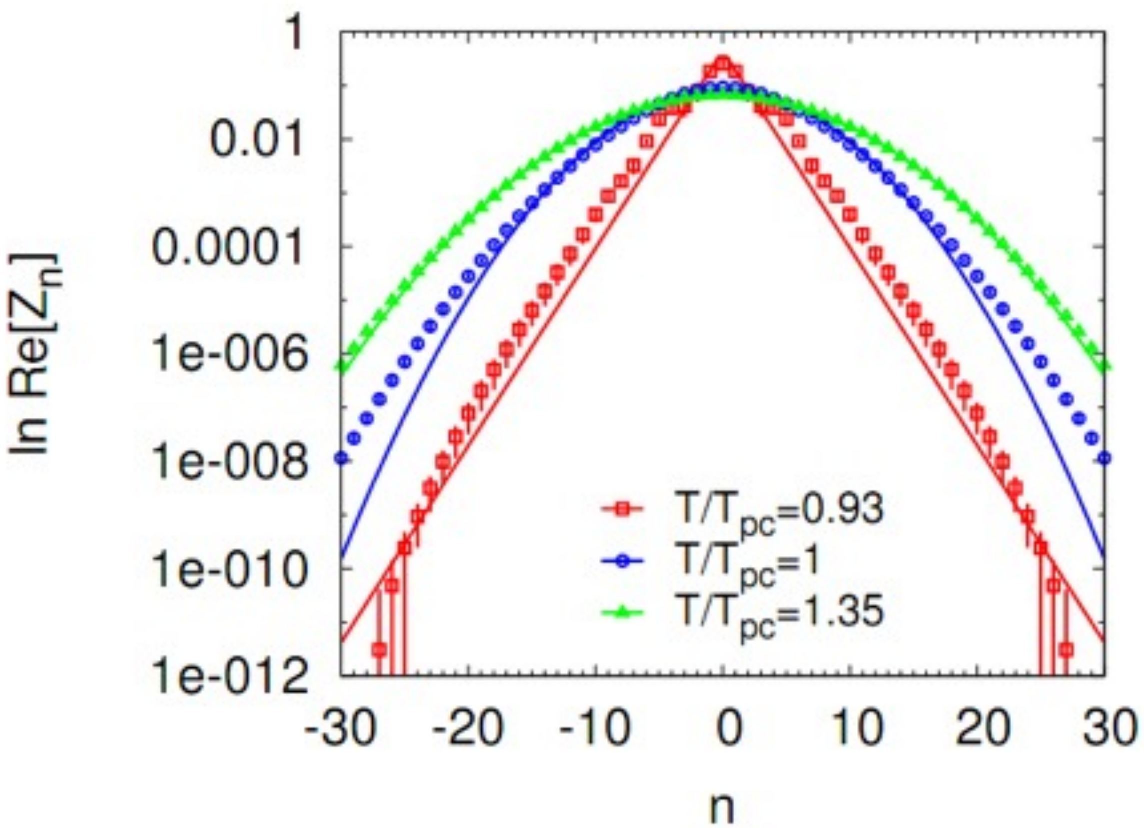
quark mass : mps/mV ~ 0.8

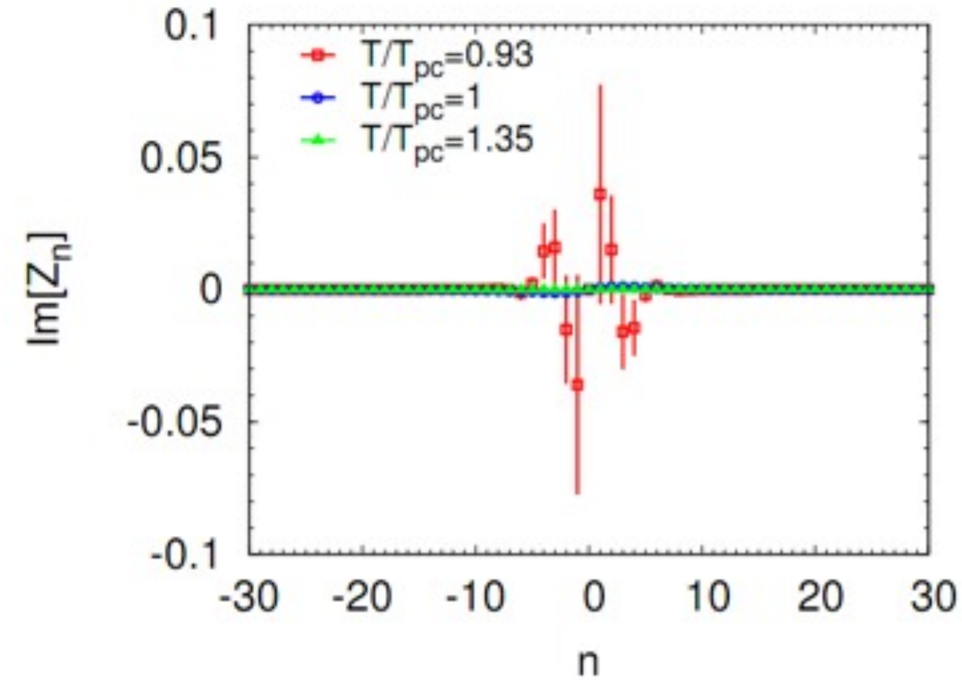
Configurations : HMC at mu=0

Eigen values : 400 configs.

N. E. C. O.

Canonical partition function Z(n)





- *n-dep of the distribution depends on the temperature*
  - *low T : exp(- a |n|)*
  - *high T : Gaussian*
  - *broadening : increase of effective d.o.f. at high T*
  - *large fluctuation of Im[Zn] at low T : severity of sign problem*

$$Z_C(n) \sim \begin{cases} e^{-a_{\rm L}|n|}, & (T < T_{\rm pc}), \\ e^{-a_{\rm H}n^2}, & (T > T_{\rm pc}). \end{cases}$$

N. E. C. O.

Density of State

- Gocksch
  - Phys. Rev. Lett. 61 (1988) 2054.
- Anagnostopoulos and Nishimura
  - Phys.Rev. D66 (2002) 106008 (hep-th/0108041)
- Fodor, Katz and Schmidt
  - JHEP0703:121,2007 (hep-lat/0701022)

現在（近似やモデルを使わずに）もっとも大きな密度まで到達

DOS Simplest Case

$$\int \mathcal{D}U e^{-S(U)}$$

$$\delta(E - S(U))$$

$$\rho(E) = \int \mathcal{D}U \delta(E - S(U)) e^{-S(U)}$$

(Density of State)

$$\langle O \rangle = \frac{1}{Z_\rho} \int dE \rho(E) \langle O \rangle_E$$

$$\langle O \rangle_E \equiv \frac{1}{\rho(E)} \int \mathcal{D}U \delta(E - S) e^{-S} O(U)$$

$$Z_\rho \equiv \int dE \rho(E)$$

DOS 一般形

Muroya, Nonaka, Takaishi, AN, PTP,10, (2003) 615
(hep-lat/0306031) Sec.5.5

$$\rho(E) = \int dU \delta(E - x(U)) g(U)$$

任意！（＊）

$$\langle O \rangle = \frac{1}{Z_\rho} \int dE \rho(E) \langle O \det \Delta e^{-\beta S_g} / g(U) \rangle_E$$

$$Z_\rho = \int dE \rho(E) \langle \det \Delta e^{-\beta S_g} / g(U) \rangle_E$$

$$\langle O \rangle_E = \frac{1}{\rho(E)} \int dU \delta(E - x(U)) g(U) O(U)$$

（＊）g(U)として、行列式の近似形を取ることもできるかもしれない。

DOS 位相を選ぶと

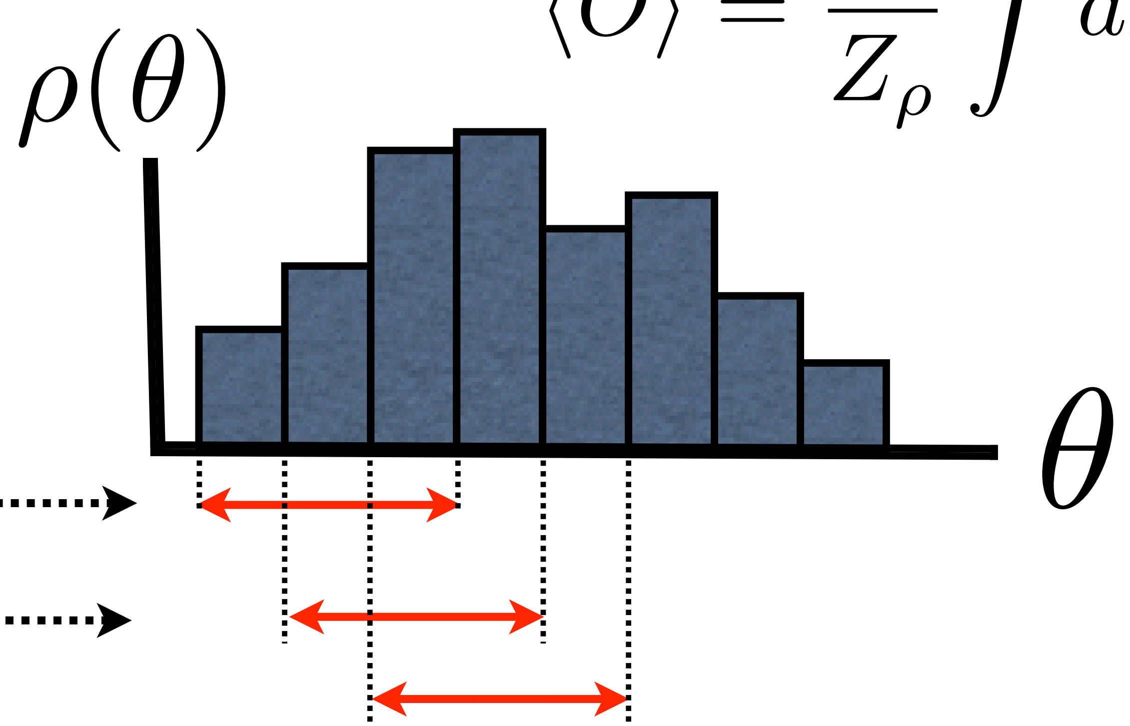
(Gocksch)

$$x(U) = \theta(U)$$ ← $\det \Delta$の位相

$$g(U) = |\det \Delta(\mu)| e^{-\beta S_g}$$

$$\rho(E) = \int dU \delta(E - \theta(U)) |\det \Delta(\mu)| e^{-\beta S_g}$$

$$\langle O \rangle = \frac{1}{Z_\rho} \int dE \rho(E) e^{iE} \langle O \rangle_E$$

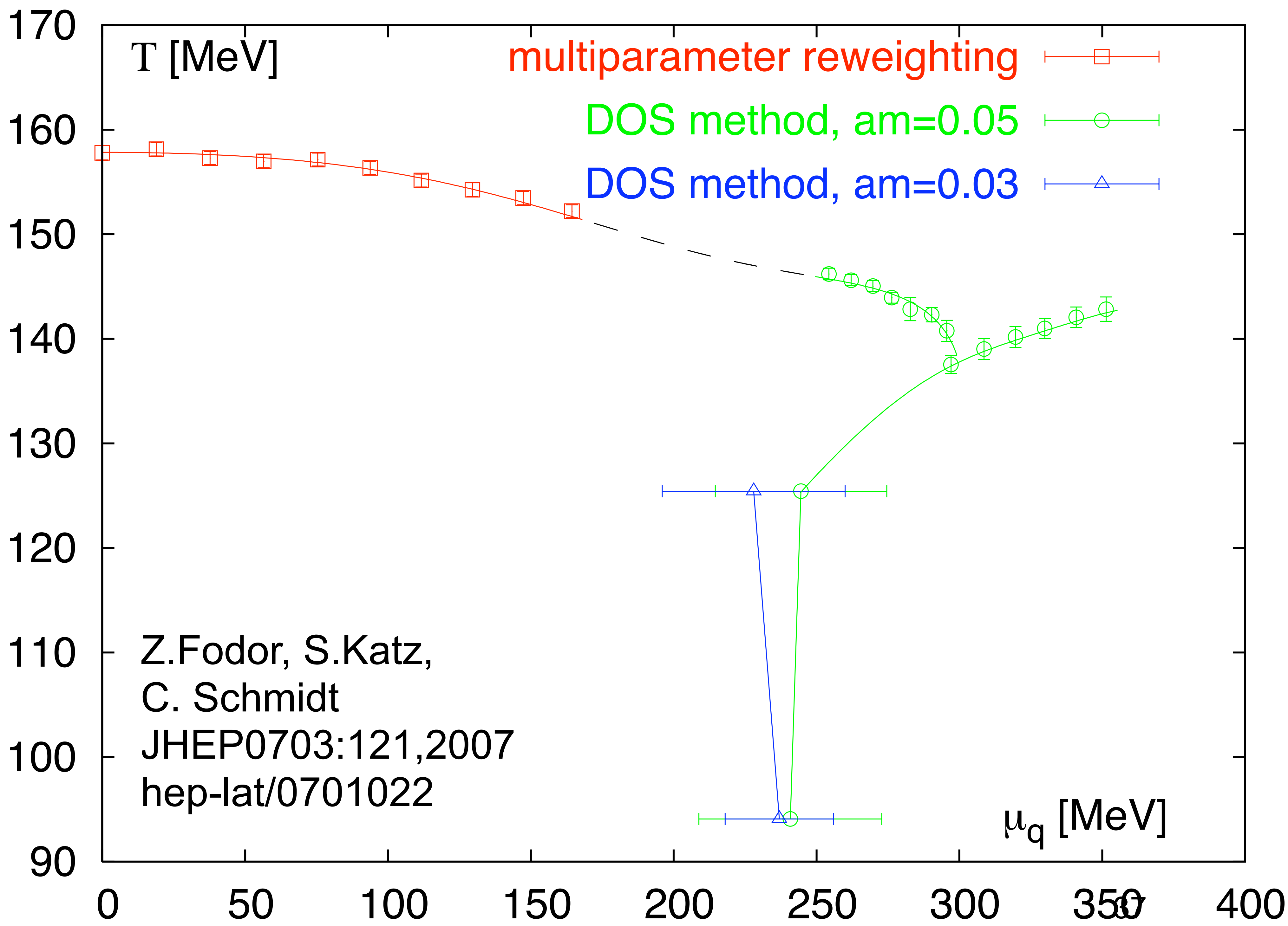


ここに留まってMC

ここに留まってMC

・・・

DOS: Plaq.を選んだケース



Complex Langevin

- Parisi
  - Phys. Lett. 131B (1983),393
- Klauder
  - Phys. Rev. A29 (1984),2036
- Klauder and Petersen
  - J. Stat.Phys., 39 (1985),53
- Karsch and Wyld
  - Phys. Rev. Lett. 21(1985),2242
- Ambjørn and Yang
  - Phys. Lett. B165, (1985)140
- Matsui and A.N
  - Phys. Lett. B194 (1987),262
- Aarts and Stamatescu
  - JHEP 0809:018,2008 (arXiv:0807.1597)

基本的なアイデア

正準量子化 ⟺ 経路積分 ⟺ 確率過程量子化
(Stochastic Quantization

$$P \propto e^{-S}$$

$$\frac{dx(\tau)}{d\tau} = -\frac{dS}{dx} + \eta(\tau)$$

η : 量子ノイズ

τ :仮想時間

どこにも確率は現れない！

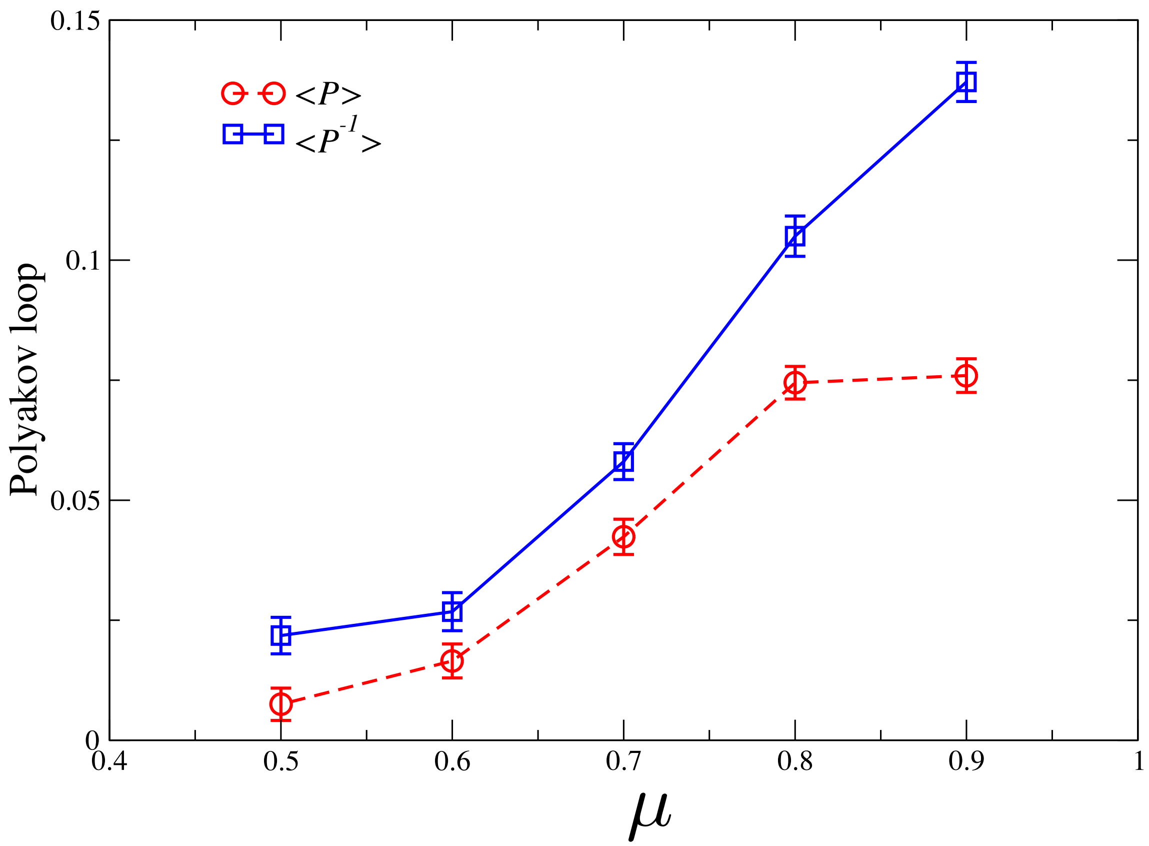
Sが複素数でもこの方程式は解ける

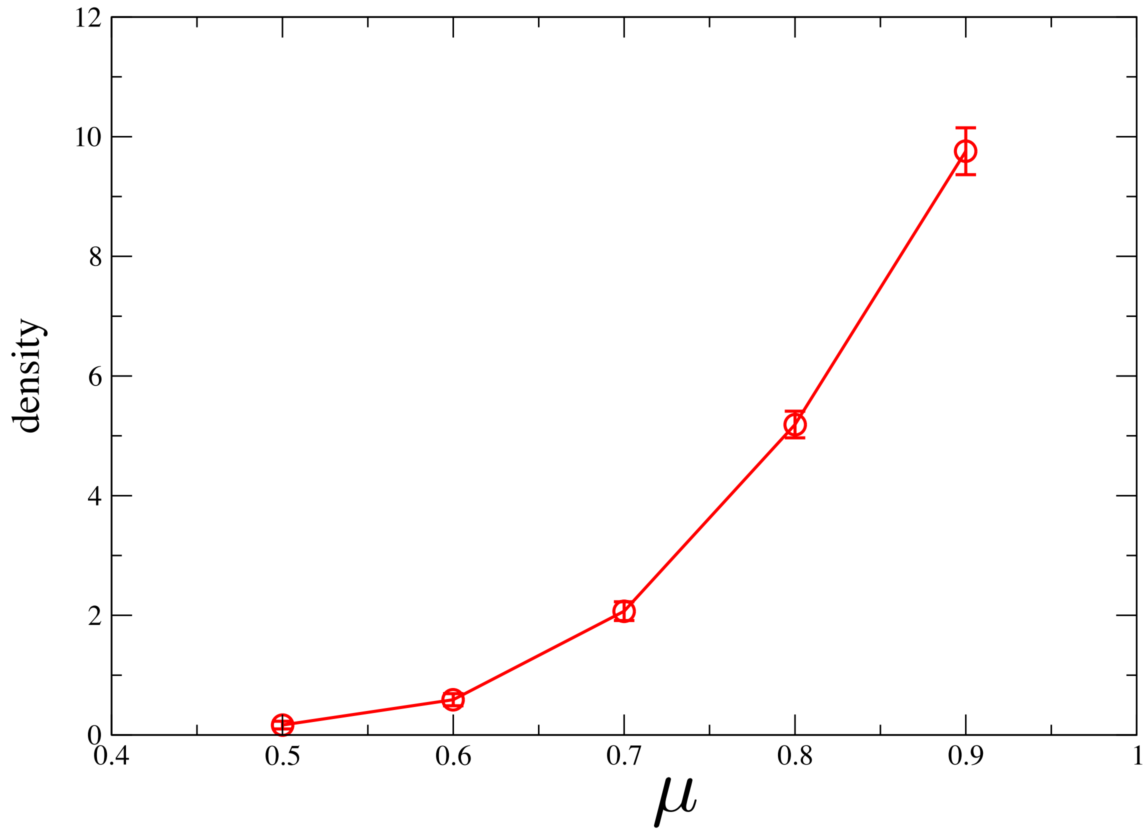
（但し裏にはFokker-Planck方程式が）

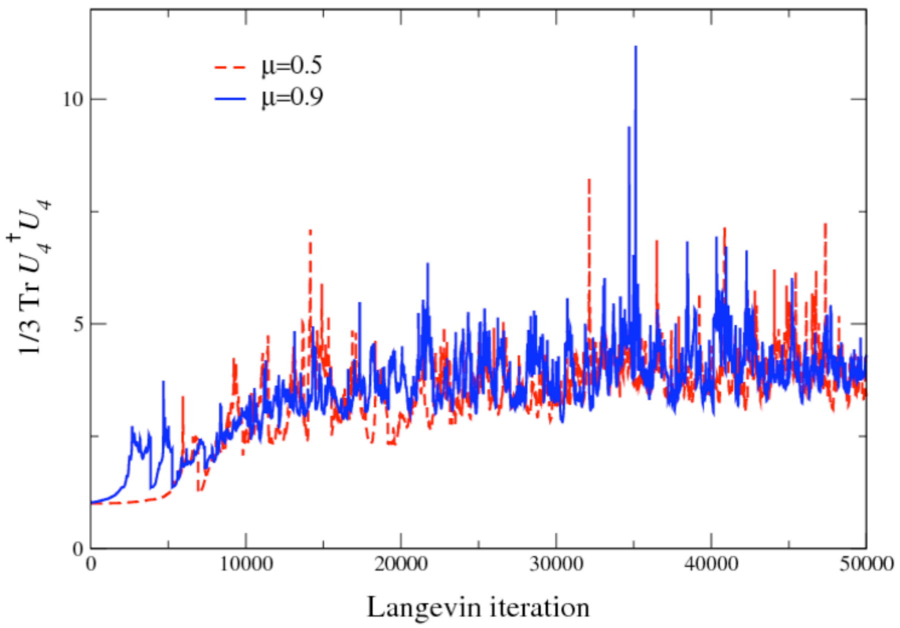
問題点：Run-Away Solutions,

Wrong Convergence

Aarts and Stamatescu
JHEP 0809:018,2008 (arXiv:0807.1597)







できた！？

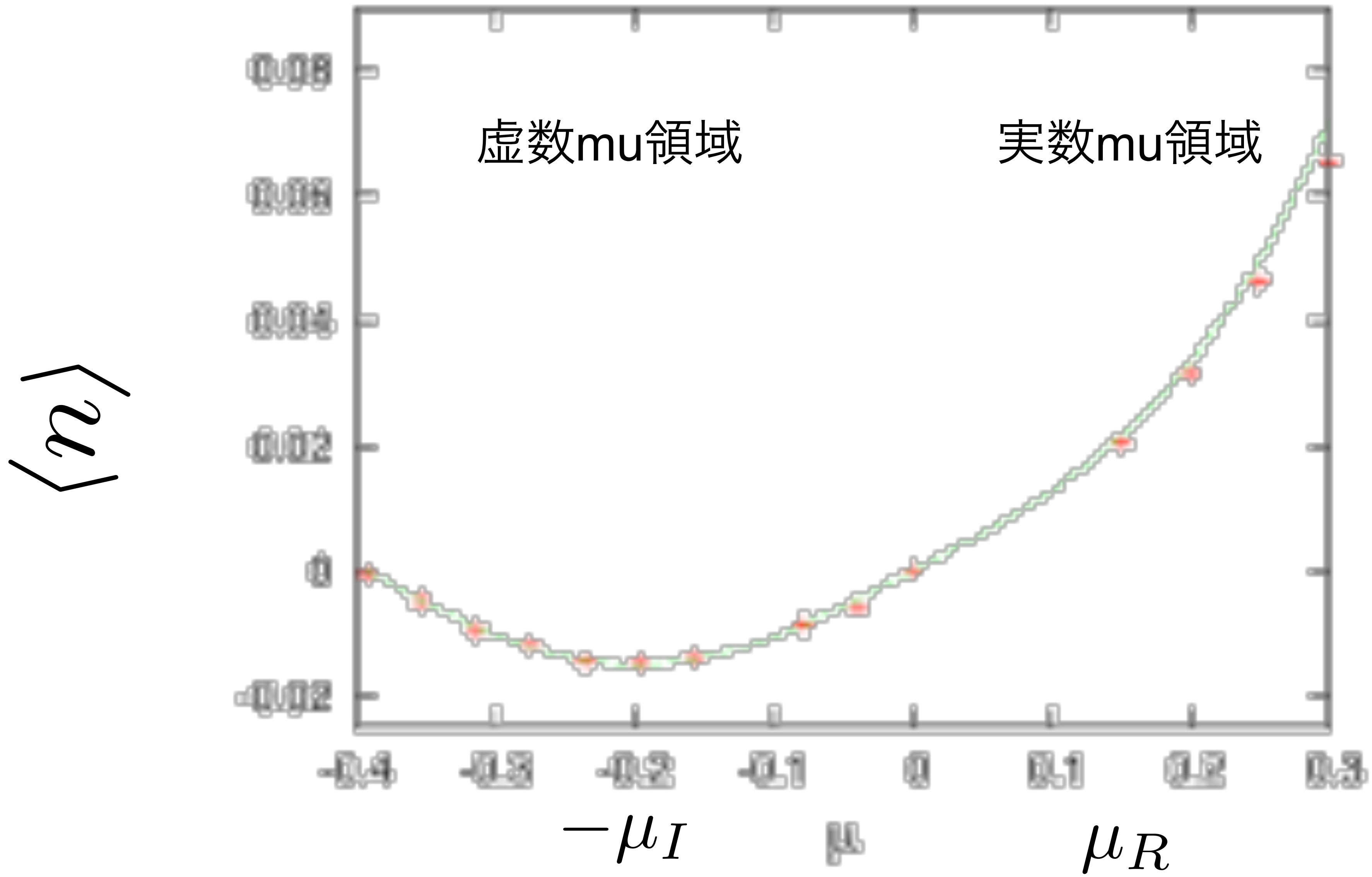
なぜ？

Adaptive step size？
本当？

モデルが簡単？(Hopping展開)

Color SU(2)

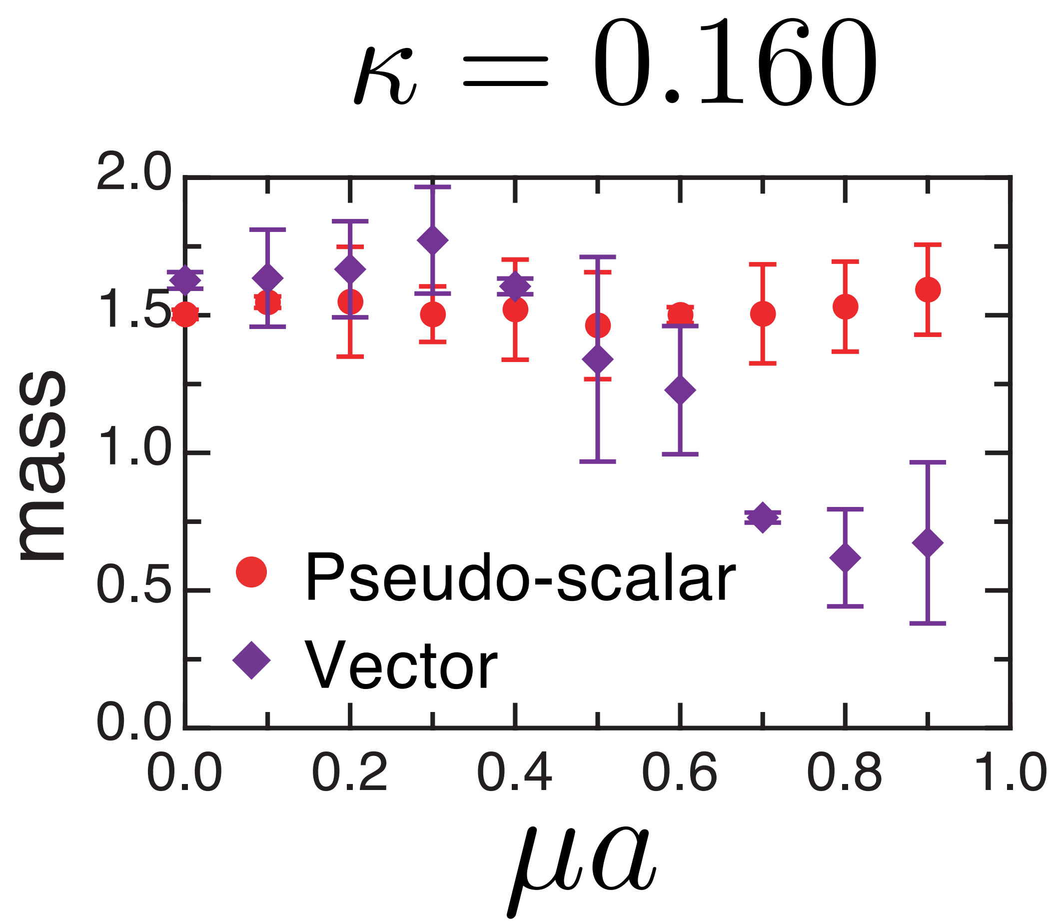
- 符号問題が無い理論 ⇒ MC可能
- 閉じ込め/非閉じ込め相を持つ
- メソンと中間子の区別はつかない
- 有限密度場の理論の非摂動的振る舞いを調べる実験場
- 理論的な解析(強結合、ランダム行列、カイラル摂動、PNJL、・・・)と合わせると、現実への情報も
  – 何がSU(2)固有で、何がそうでないか

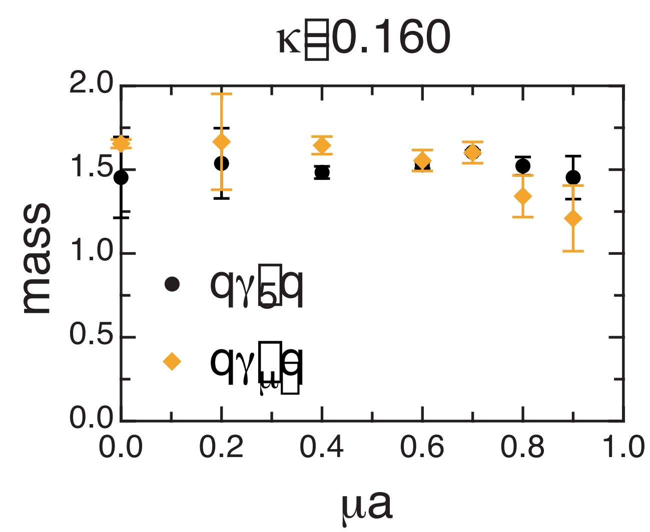


境

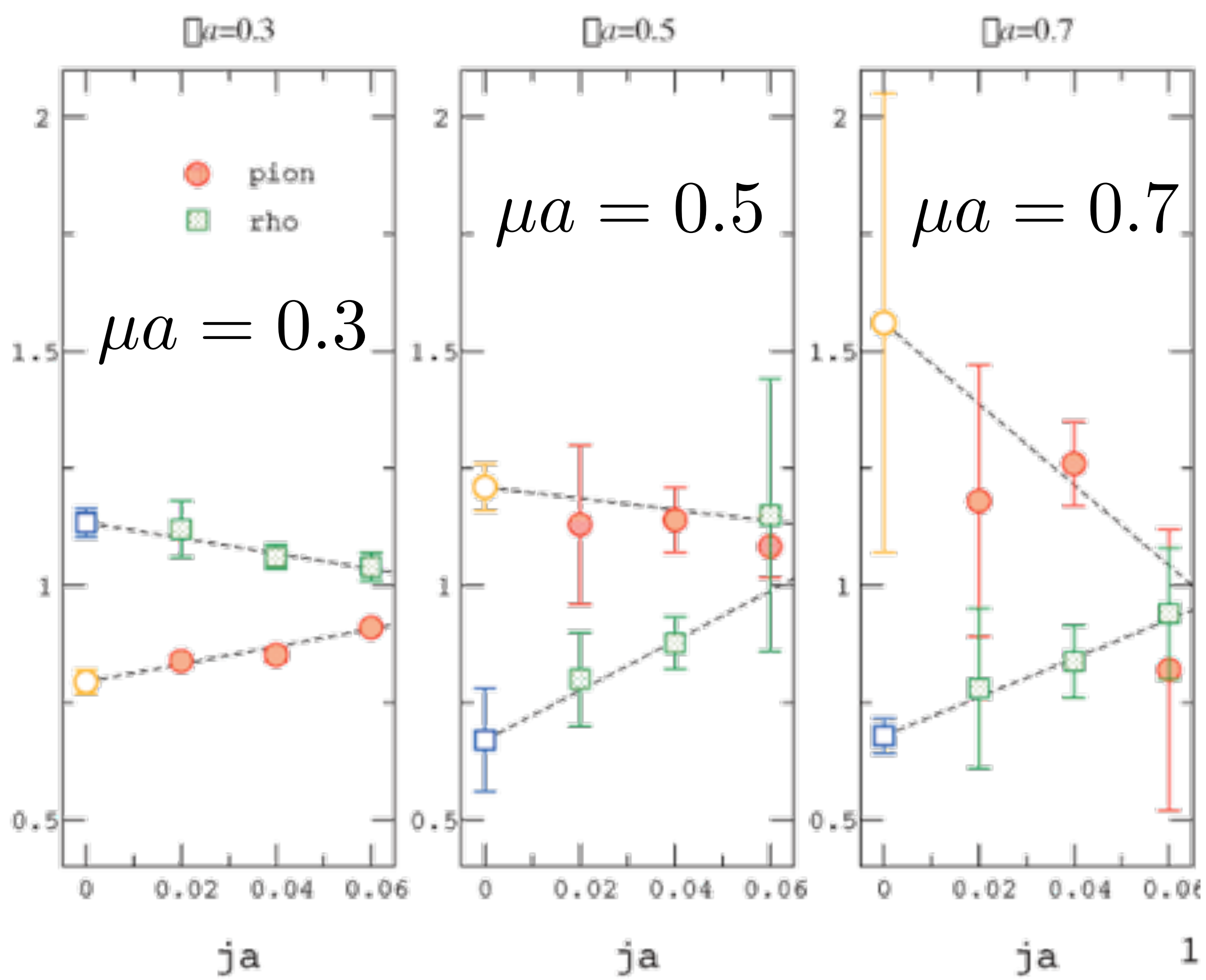
$$Z = C_0 + C_2(\xi^2 + \xi^{-2}) + C_4(\xi^4 + \xi^{-4})$$

$$\langle n \rangle = \xi \frac{\partial}{\partial \xi} Z = \frac{2C_2(\xi^2 - \xi^{-2}) + 4C_4(\xi^4 - \xi^{-4})}{C_0 + C_2(\xi^2 + \xi^{-2}) + C_4(\xi^4 + \xi^{-4})}$$

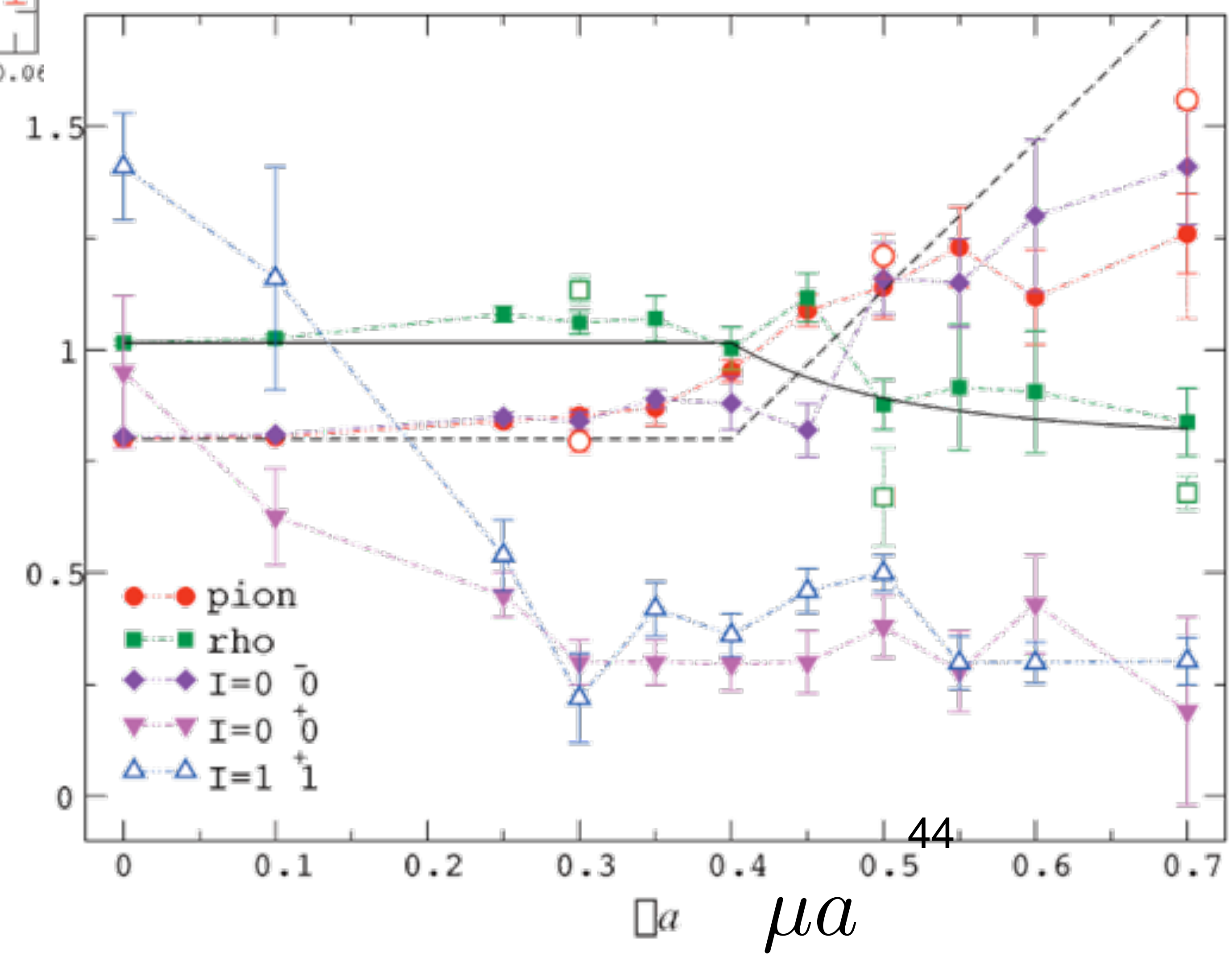




Muroya, A.N. and Nonaka
Phys.Lett. B551 (2003) 305
(hep-lat/0211010)



di-quark sourceの強さ



Hands, Sitch and Skullerud
Phys.Lett. B662, (2008)405
(arXiv:0710.1966)