Study of finite density lattice QCD by the histogram method

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WHOT-QCD collaboration

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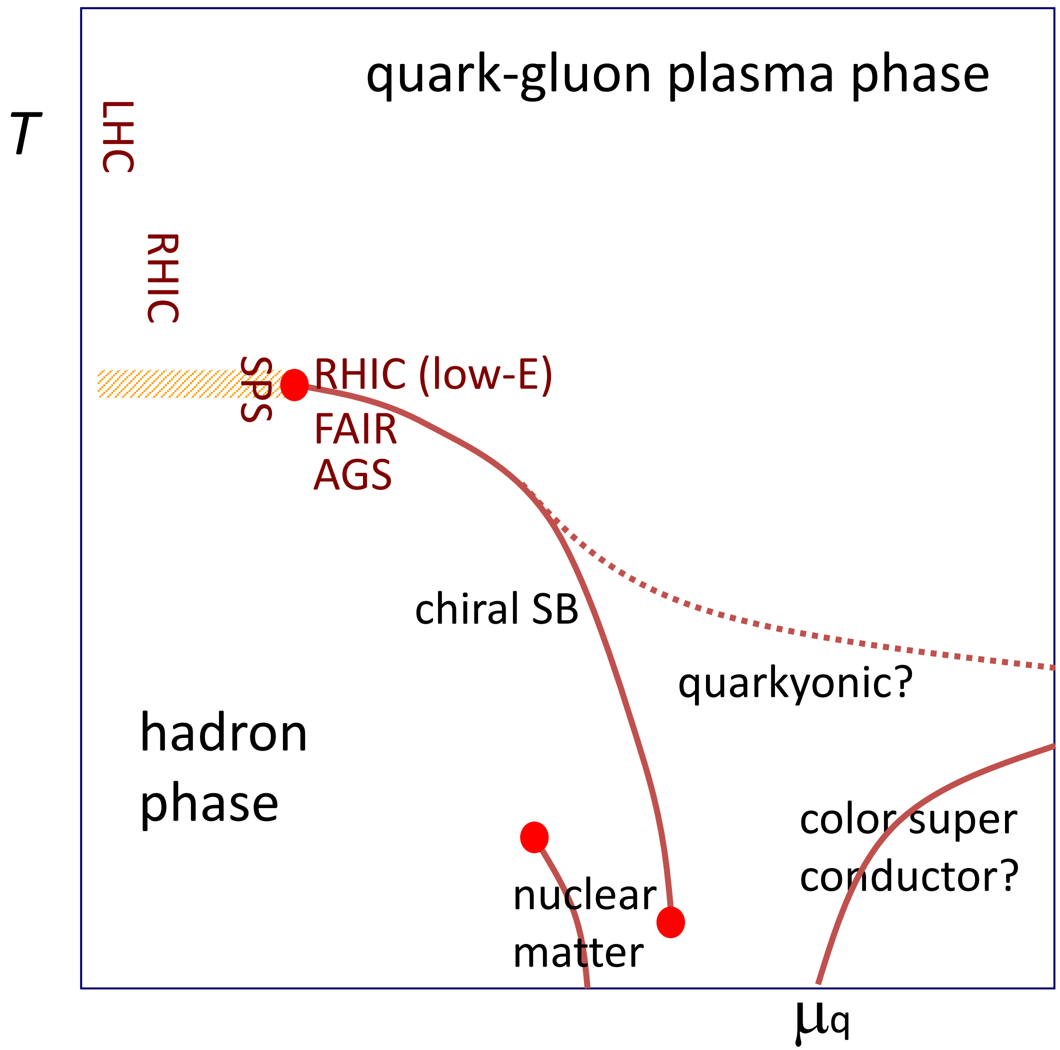
Nara, Japan, December 13-16, 2012

Phase structure of QCD at high temperature and density

- Phase transition lines
- Critical point
- Order of the transition

Lattice QCD Simulations

- Direct simulation:
  Impossible at $\mu \neq 0$.



Distribution function & the effective potential

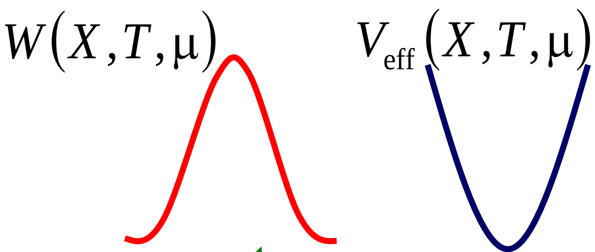
$$W(X;m,T,\mu) \equiv \int DU \delta\left(X-\hat{X}\right)\left(\det M(m,\mu)\right)^{N_f} e^{-S_g} \quad \text{(Histogram)}$$

X: order parameters, total quark number, average plaquette, etc.

$$V_{\rm eff}(X) = -\ln W(X)$$

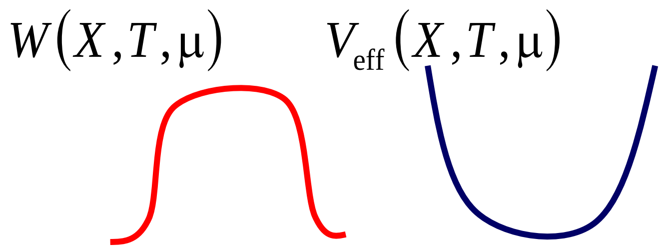
Crossover

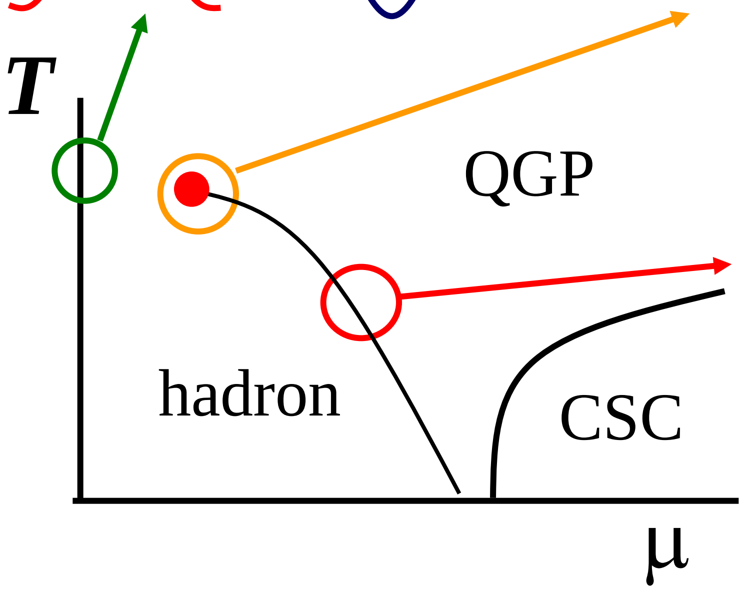
$W(X)$: Gaussian function
$V(X)$: Quadratic function



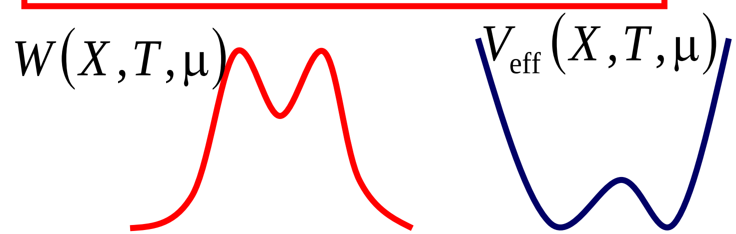
Critical point

W(X): Flat
V(X): Curvature: Zero





1st order phase transition

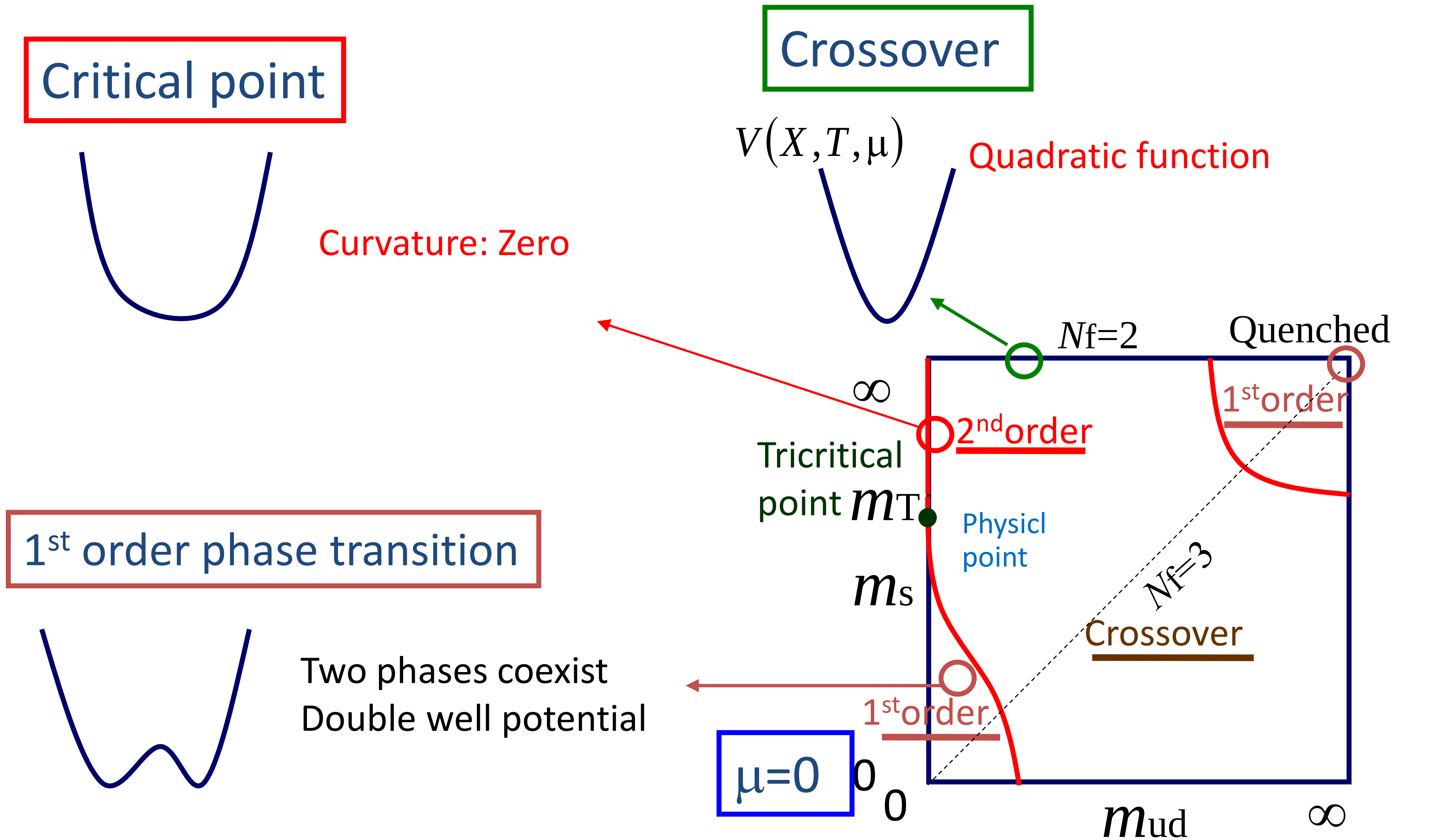


W(X): Two phases coexist
V(X): Double well potential

Mass-dependence of the effective potential

$$W(X',m,T,\mu) \equiv \int DU \delta(X\text{-}X') \sum_{f=1}^{N_f} \det M(m_f,\mu_f) e^{-S_g}, \quad V_{\rm eff}(X) = -\ln W(X)$$

X: order parameters, total quark number, average plaquette etc.



(β, m, μ)-dependence of the Distribution function

$$W(X, \beta, m, \mu) \equiv \int DU \delta(\hat{X} - X)(\det M(m, \mu))^{N_f} \; e^{6 N_{site} \beta \hat{P}} \qquad (\beta = 6/g^2)$$

plaquette P (1x1 Wilson loop for the standard action)

$$R(X, \beta, \beta_0 m, m_0, \mu) \equiv W(X, \beta, m, \mu) / W(X, \beta_0, m_0, 0)$$ (Reweight factor)

$$R(X) = \frac{\left\langle \delta(\hat{X} - X) e^{6 N_{site} (\beta - \beta_0) \hat{P}} \left(\frac{\det M(m, \mu)}{\det M(m_0, 0)} \right)^{N_f} \right\rangle_{(\beta_0, \mu = 0)}}{\left\langle \delta(\hat{X} - X) \right\rangle_{(\beta_0, \mu = 0)}} \equiv \left\langle e^{6 N_{site} (\beta - \beta_0) \hat{P}} \left(\frac{\det M(m, \mu)}{\det M(m_0, 0)} \right)^{N_f} \right\rangle_{X: \text{fixed}}$$

Effective potential:

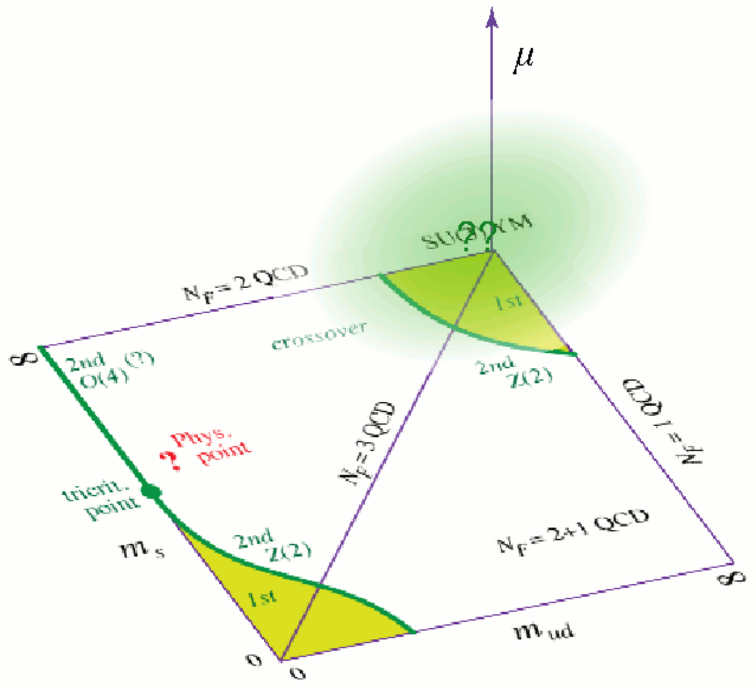
$$V_{eff}(X, \beta, m, \mu) = -\ln[W(X, \beta m, \mu)] = V_{eff}(X, \beta_0, m_0, 0) - \ln R(X, \beta, \beta_0 m, m_0, \mu)$$

$$\ln R(X) = \ln \left\langle \exp[6 N_{site} (\beta - \beta_0) \hat{P}] \left(\frac{\det M(m, \mu)}{\det M(m_0, 0)} \right)^{N_f} \right\rangle_{X: \text{fixed}}$$

Performing simulations at various β, combine the data by multi-β reweighting

(Ferrenberg &Swendsen , 89)

Distribution function in the heavy quark region



- We study the properties of $W(X)$ in the heavy quark region.
- Performing quenched simulations + Reweighting.
- Standard Wilson quark action + plaquette gauge action, $S_g = -6N_{\text{site}}\beta P$
- lattice size: $N_s^3 \times N_t = 24^3 \times 4$
- 5 simulation points; β=5.68-5.70.

(WHOT-QCD, Phys.Rev.D84, 054502(2011))

Hopping parameter expansion $(\kappa \sim 1/m)$

$$N_{\text{f}} \ln\left(\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right) = N_{\text{f}}\left(288N_{\text{site}}\kappa^4 P + 12\cdot 2^{N_t} N_s^3 \kappa^{N_t}\left(\cosh(\mu/T)\Omega_R + \underline{\underline{i\sinh(\mu/T)\Omega_I}}\right) + \cdots\right)$$

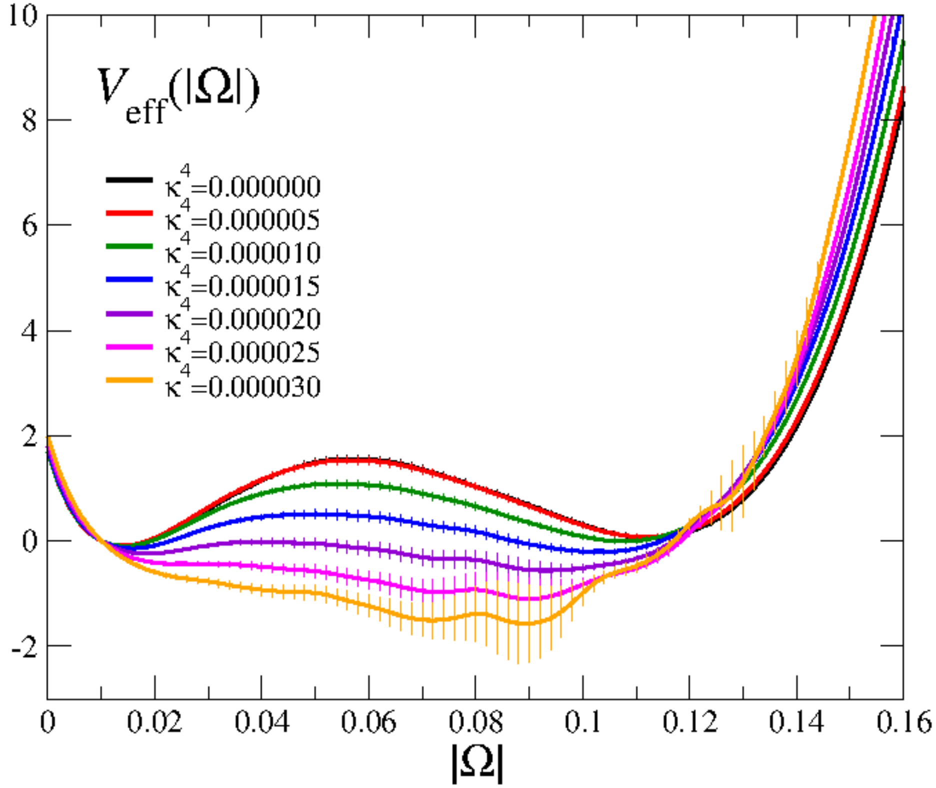
phase

P: plaquette, $\Omega=\Omega_R+i\Omega_I$: Polyakov loop (order parameter)

The plaquette term can be absorbed into the gauge action by redefining β.

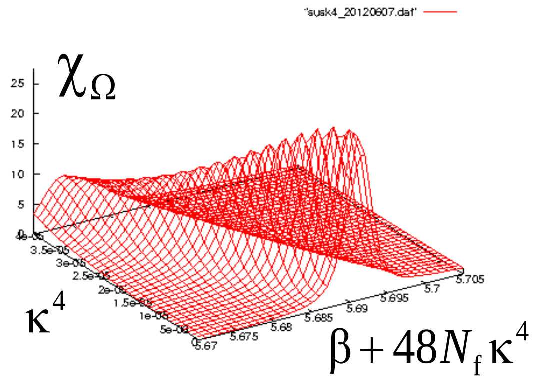
Order of the phase transition
Polyakov loop distribution
(order parameter of confinement)

Effective potential of |Ω|
on the pseudo-critical line at μ=0



$\text{Critical point}: \kappa^4 \approx 2.0\times 10^{-5}$

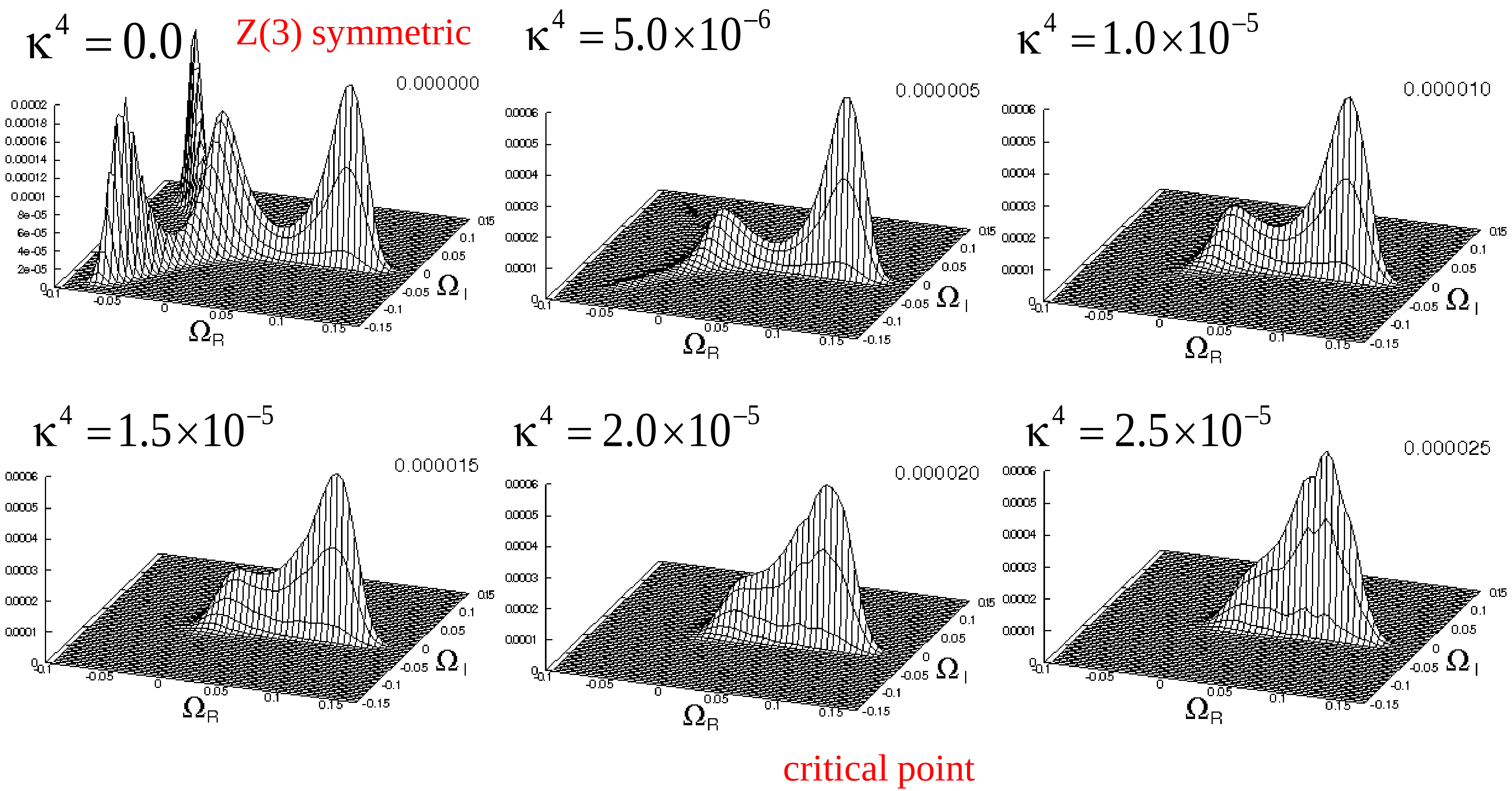
- The pseudo-critical line is determined by χ_Ω peak.



- Double-well at small κ
  - First order transition
- Single-well at large κ
  - Crossover

Polyakov loop distribution in the complex plane

(μ=0)



- on β_{pc} measured by the Polyakov loop susceptibility.

Distribution function of Ω_R at finite density

$$W(\Omega_R;\beta,\kappa,\mu) = \int DU\ \delta(\Omega_R - \hat{\Omega}_R) e^{6N_{\rm site}\hat{P}} (\det M(\kappa))^{N_{\rm f}}$$
$$= \int DU\ \delta(\Omega_R - \hat{\Omega}_R) e^{6N_{\rm site}\hat{P}} |\det M(\kappa)|^{N_{\rm f}} e^{i\theta}$$
$$= W_0(\Omega_R;\beta,\kappa,\mu) \langle e^{i\theta} \rangle_{\Omega_R}$$

Phase-quenched simulation: $W_0(\Omega_R) = \int DU\ \delta(\Omega_R - \hat{\Omega}_R) |\det M|^{N_{\rm f}}\ e^{6N_{\rm site}\beta\hat{P}}$

$$\left(\theta = 12 \cdot 2^{N_t} N_s^3 N_{\rm f} \kappa^{N_t} \sinh(\mu/T) \hat{\Omega}_I\right)$$

- Heavy quark region
- Effective potential: $V_{\rm eff}(\Omega_R) = -\ln W(\Omega_R)$, $V_0(\Omega_R) = -\ln W_0(\Omega_R)$

(Phase-quenched part)

$$V_{\rm eff}(\beta,\kappa,\mu) = V_0(\beta,\kappa,\mu) - \underline{\ln\langle e^{i\theta}\rangle_{\Omega_R}}$$

Phase average

- V_0 is equal to $V_{\rm eff}$ (μ=0) when we replace $\underline{\kappa^{N_t} \Rightarrow \kappa^{N_t}\cosh(\mu/T)}$
  – Critical point (phase-quenched)

$$\kappa_{\rm cp}^{N_t}(0) = \kappa_{\rm cp}^{N_t}(\mu)\cosh(\mu/T)$$

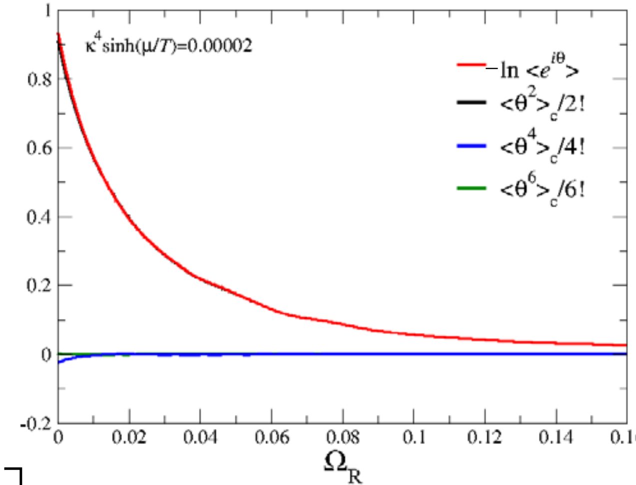
Nf=2 at μ=0: $\kappa_{\rm cp}$=0.0658(3)(8)

(WHOT-QCD, Phys.Rev.D84, 054502(2011))

Avoiding the sign problem

(SE, Phys.Rev.D77,014508(2008), WHOT-QCD, Phys.Rev.D82, 014508(2010))

θ: complex phase $\theta \equiv \mathrm{Im} \ln \det M \approx 12 \cdot 2^{N_t} N_s^3 N_f \kappa^{N_t} \sinh(\mu/T) \Omega_I$



- Sign problem: If $e^{i\theta}$ changes its sign,

$$\left\langle e^{i\theta} \right\rangle_{\Omega_R \text{ fixed}} \ll \text{(statistical error)}$$

- Cumulant expansion

$$\left\langle e^{i\theta} \right\rangle_{\Omega_R} = \exp\left[i\langle\theta\rangle_C - \frac{1}{2}\langle\theta^2\rangle_C - \frac{i}{3!}\langle\theta^3\rangle_C + \frac{1}{4!}\langle\theta^4\rangle_C + \cdots \right]$$

$\langle\theta\rangle_C \to 0$, $\langle\theta^3\rangle_C \to 0$

$\kappa^4(\mu)\sinh(\mu/T) = 0.00002$

$\beta = 5.69$

cumulants

$$\langle\theta\rangle_C = \langle\theta\rangle_{\Omega_R}, \quad \langle\theta^2\rangle_C = \langle\theta^2\rangle_{\Omega_R} - \langle\theta\rangle_{\Omega_R}^2, \quad \langle\theta^3\rangle_C = \langle\theta^3\rangle_{\Omega_R} - 3\langle\theta^2\rangle_{\Omega_R}\langle\theta\rangle_{\Omega_R} + 2\langle\theta\rangle_{\Omega_R}^3, \quad \langle\theta^4\rangle_C = \cdots$$

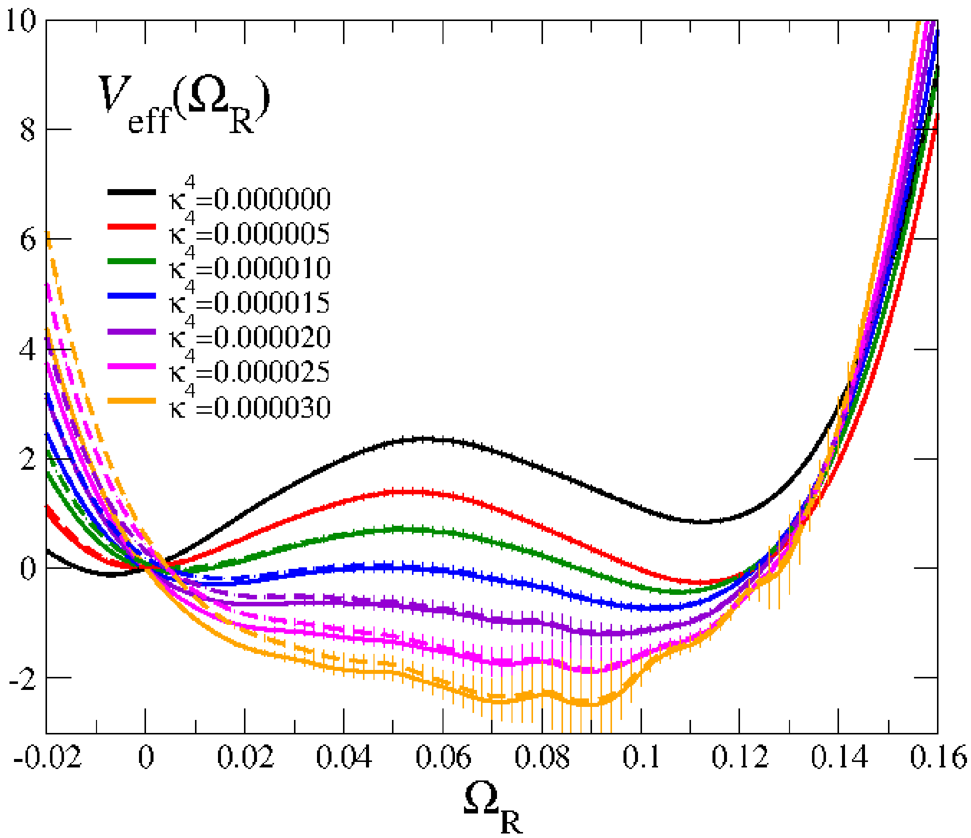
– Odd terms vanish from a symmetry under $\mu \leftrightarrow -\mu$ ($\theta \leftrightarrow -\theta$)

Source of the complex phase

If the cumulant expansion converges, No sign problem.

Effect from the complex phase factor

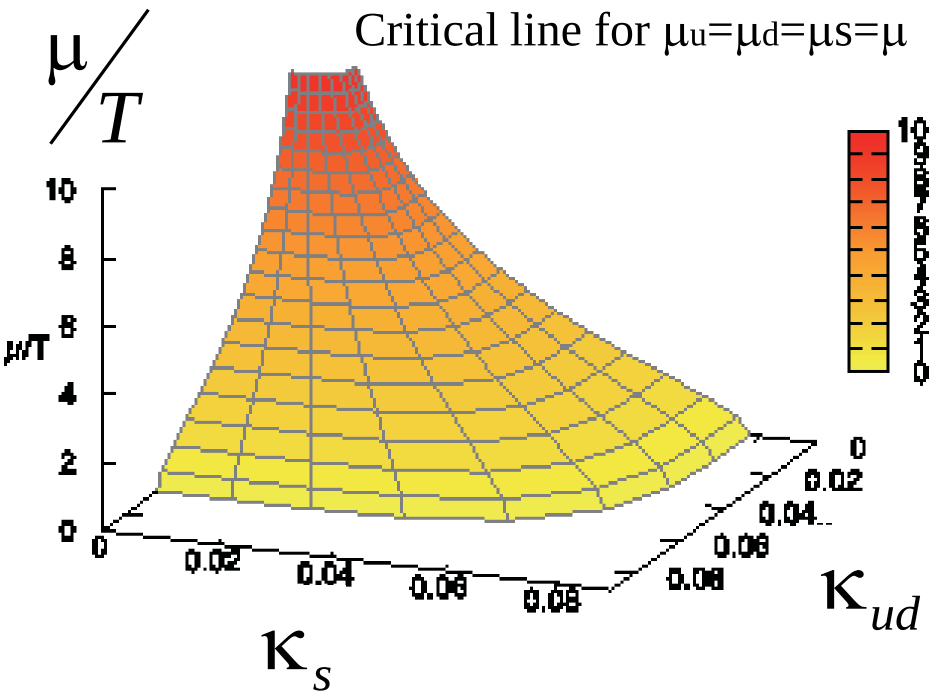
- Polyakov loop effective potential at various $\kappa^{N_t}\cosh(\mu/T)$ at the transition point.
  - Solid lines: μ=0 , i.e., $\cosh(\mu/T)=1, \quad \sinh(\mu/T)=0$
  - Dashed lines: $\mu=\infty$, i.e, $\sinh(\mu/T)=\cosh(\mu/T)$

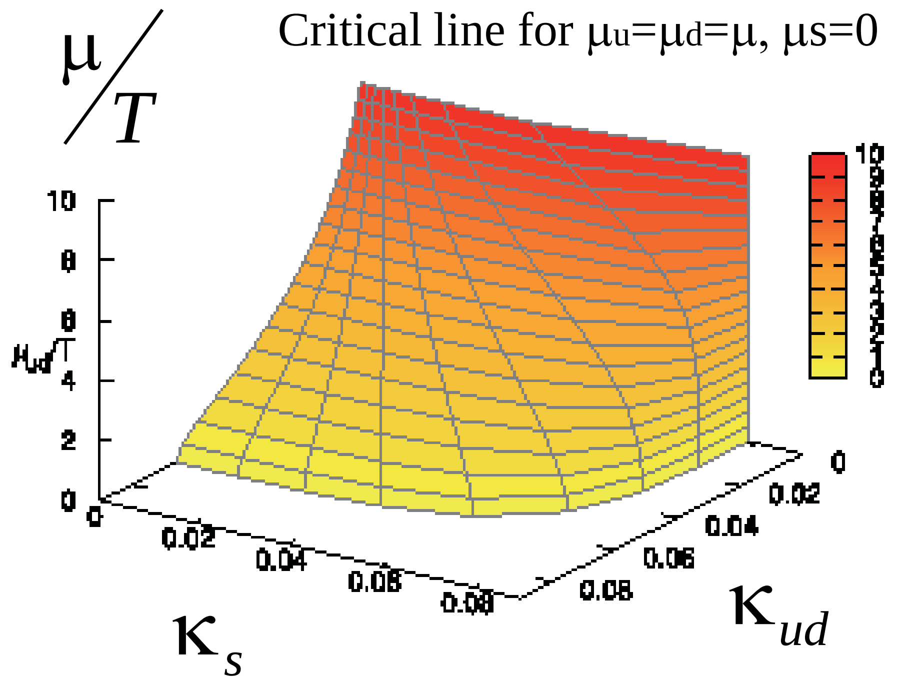


The effect from the complex phase factor is very small except near Ω_R=0.

Critical surface in 2+1-flavor finite density QCD in the heavy quark region

- The effect from the complex phase is very small for the determination of κ_{cp}.
- The phase effect is neglected.





Probability distribution function in the light qurk region ➡ Nakagawa's poster

- We perform phase quenched simulations with the weight:

$$\left|\det M(m,\mu)\right|^{N_f} e^{-S_g}$$

$$W(P',F',\beta,m,\mu) = \int DU\, \delta(\hat{P}-P')\delta(\hat{F}-F')(\det M(m,\mu))^{N_f}\ e^{6N_{site}\beta\hat{P}}$$

$$= \int DU\, \delta(\hat{P}-P')\delta(\hat{F}-F')e^{i\theta}\left|\det M(m,\mu)\right|^{N_f}\ e^{6N_{site}\beta\hat{P}}$$

$$= \left\langle e^{i\theta}\right\rangle_{P',F'} \times W_0(P',F',\beta,m,\mu)$$

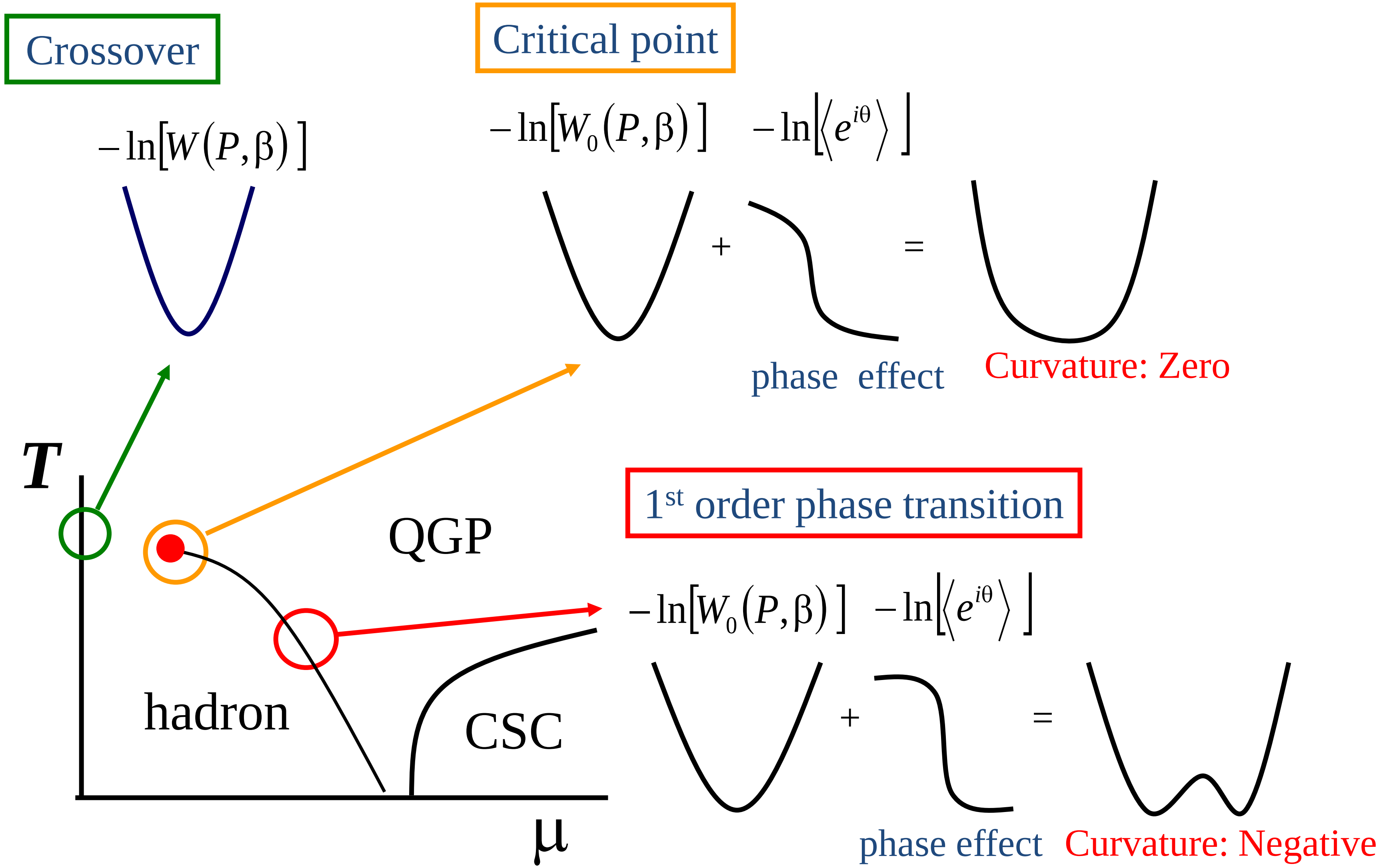
expectation value with fixed *P,F* histogram

P: plaquette $F(\mu) = \frac{N_f}{N_{site}} \ln\left|\frac{\det M(\mu)}{\det M(0)}\right|$ $\theta \equiv N_f \operatorname{Im}\ln\det M$

Distribution function of the phase quenched. $W_0(P',F') = \int DU\, \delta(\hat{P}-P')\delta(\hat{F}-F')\left|\det M\right|^{N_f}\ e^{6N_{site}\beta\hat{P}}$

μ-dependence of the effective potential

Curvature of the effective potential



Curvature of the effective potential

- Assuming the distribution is Gaussian,

$$W_0(P,F) \approx \sqrt{\frac{6N_{\rm site}}{2\pi\chi_P}} \exp\left[-\frac{6N_{\rm site}}{2\chi_P}\left(P-\langle P\rangle\right)^2\right] \times \sqrt{\frac{N_{\rm site}}{2\pi\chi_F}} \exp\left[-\frac{N_{\rm site}}{2\chi_F}\left(F-\langle F\rangle\right)^2\right]$$

$$\chi_P = 6N_{\rm site}\left\langle\left(P-\langle P\rangle\right)^2\right\rangle \qquad \chi_F = N_{\rm site}\left\langle\left(F-\langle F\rangle\right)^2\right\rangle$$

$$\frac{\partial^2(-\ln W_0)}{\partial P^2}\left(\langle P\rangle,\langle F\rangle\right) = \frac{6N_{\rm site}}{\chi_P} \qquad \frac{\partial^2(-\ln W_0)}{\partial F^2}\left(\langle P\rangle,\langle F\rangle\right) \approx \frac{N_{\rm site}}{\chi_F}$$

at the peak of the distribution

Cumulant expansion

$$W(P,F) = W_0(P,F)\left\langle e^{i\theta}\right\rangle_{P,F:\text{fixed}} \approx W_0(P,F)\exp\left[\frac{1}{2}\left\langle\theta^2\right\rangle_{P,F:\text{fixed}}\right]$$

$$\left\langle\theta^2\right\rangle_{P,F:\text{fixed}} \approx \left\langle\theta^2\right\rangle\left(\langle P\rangle,\langle F\rangle\right)$$

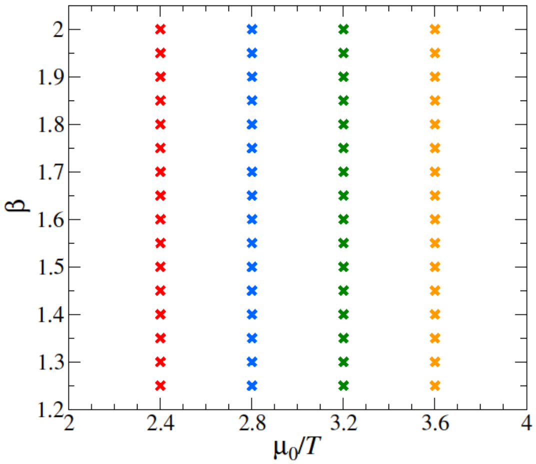
⇨ Curvature of $\left\langle\theta^2\right\rangle$

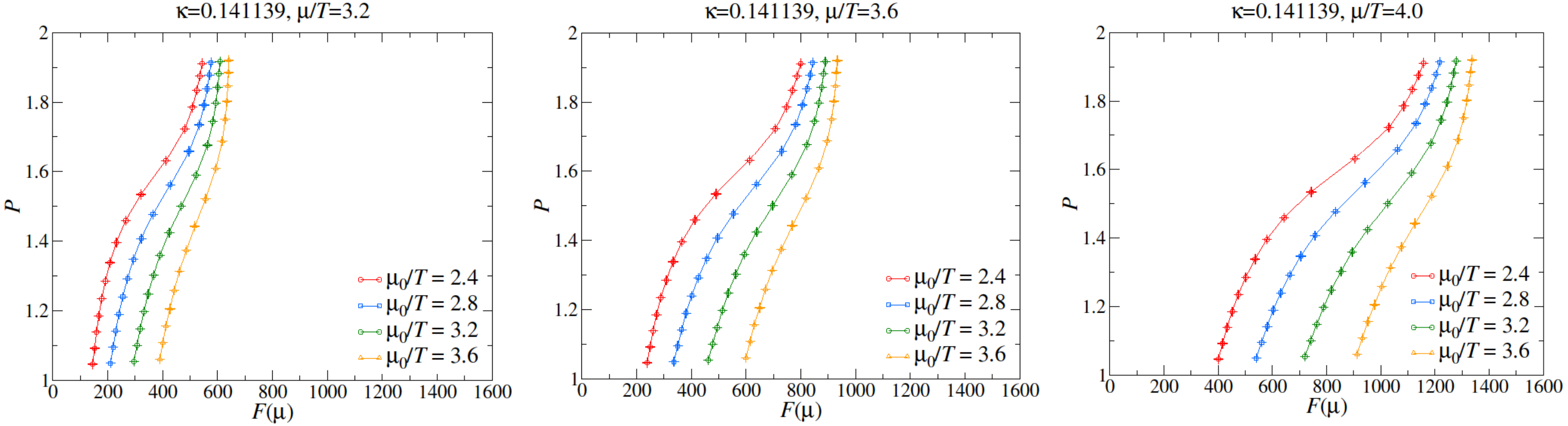
Simulations

2-flavor QCD Iwasaki gauge
+ clover Wilson quark action
Random noise method is used.

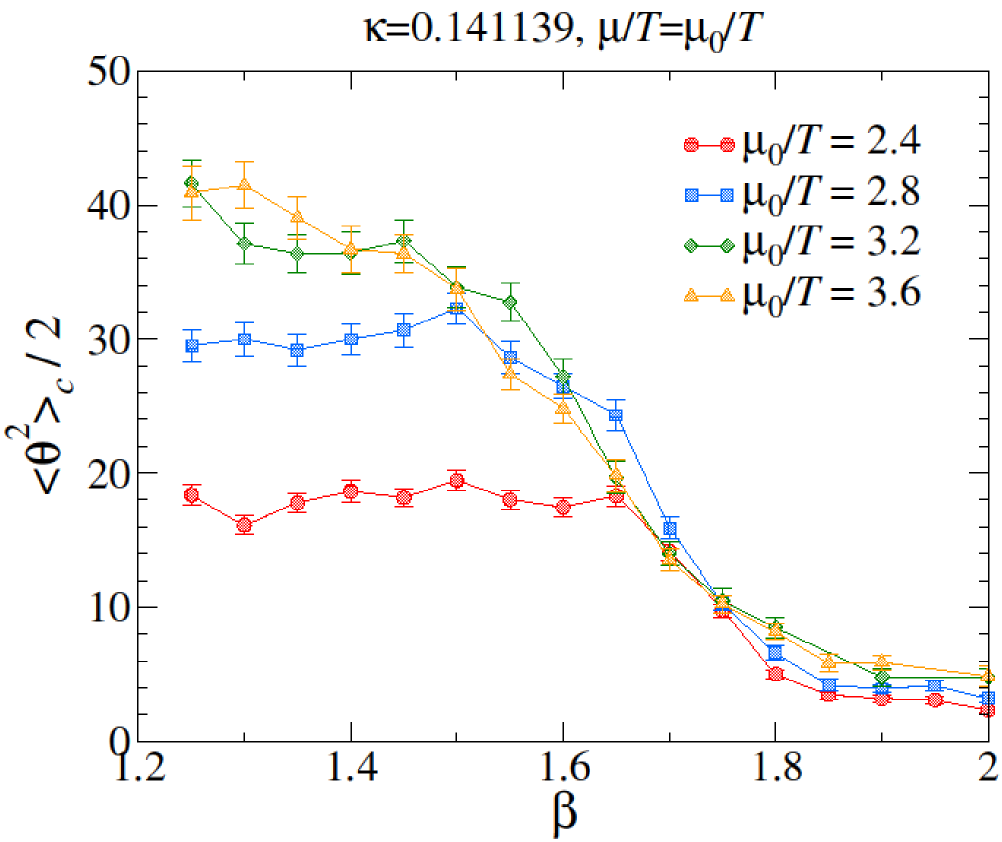
$8^3 \times 4$ lattice $m_\pi / m_\rho \approx 0.8$

- Simulation points in the (β, μ_0/T)
- Peak of $W_0(P,F)$ for each μ

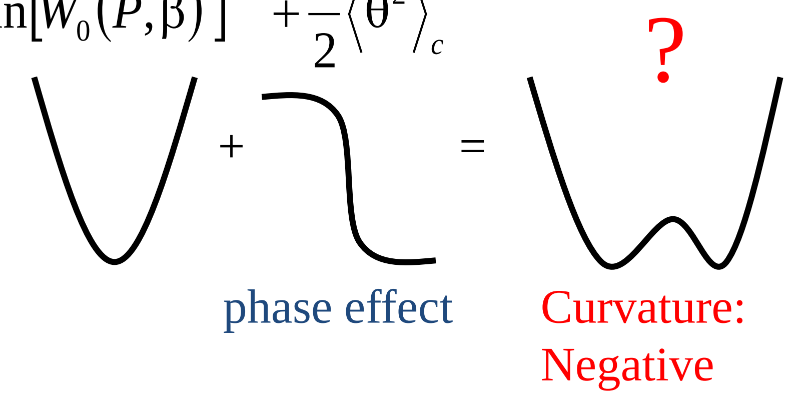




Effect from the complex phase

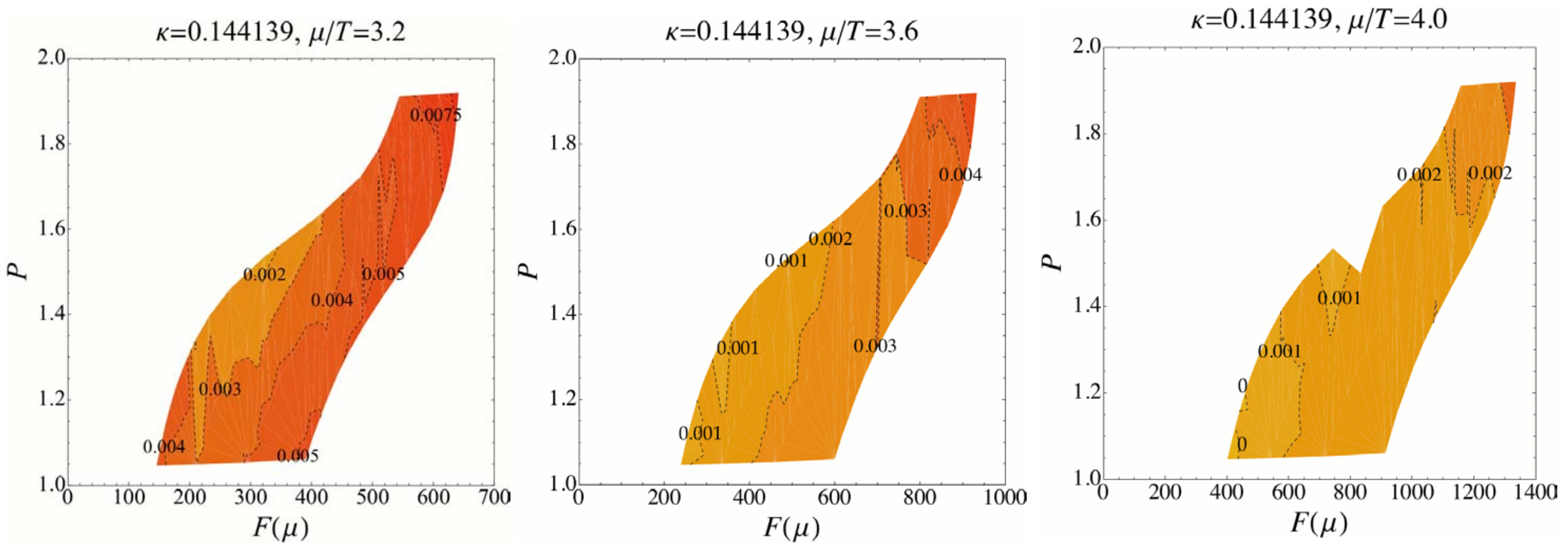


$$-\ln\left[W_0(P,\beta)\right] \quad + \frac{1}{2}\left\langle \theta^2 \right\rangle_c$$



- Rapidly changes around the pseudo-critical point.
- Strong curvature in $\left\langle \theta^2 \right\rangle_c / 2$

Curvature of the effective potential for *F*-direction

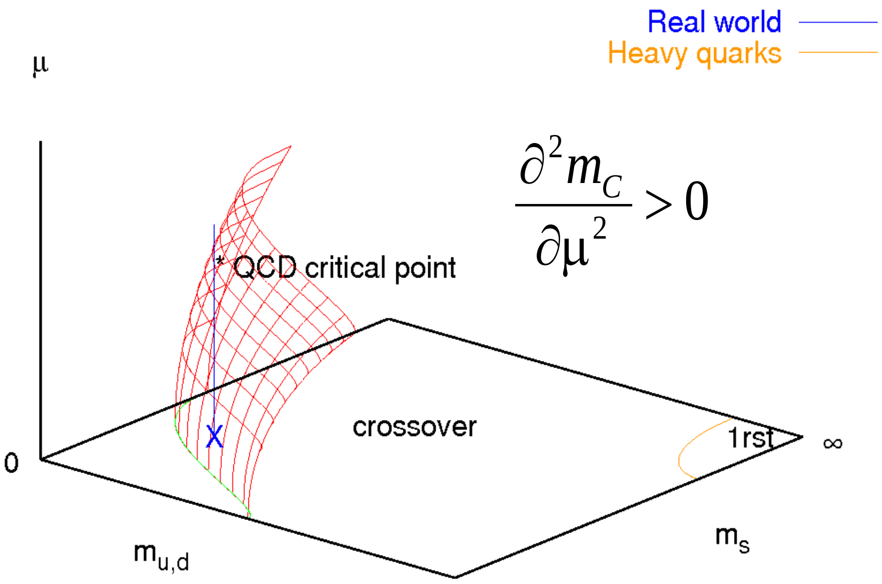


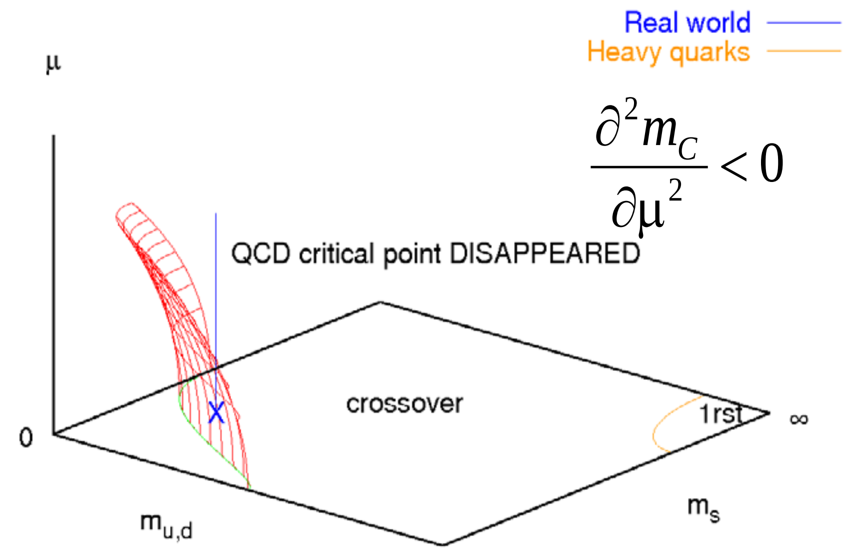
zero curvature
Critical point

- Appearance of the critical point: suggested at μ/T=4.0

(The quark mass is much heavier than the physical mass.)

Curvature of the critical surface





- Usual expectation
- Critical point: exists

- de Forcrand - Philipsen,
  JHEP01(2007)077; PoS(LAT2007)178
  - Curvature: slightly negative.
    (3-flavor staggered, 8^3x4 lattice)

New approach 2+N_f-flavor QCD (large N_f) ➡ Yamada's talk
(2 light quarks + N_f heavy quarks)
➡ Curvature: positive.

2+N_f-flavor QCD ($N_f \gtrsim 10$)

(Ejiri, Yamada, 2012, in preparation)

$$h = 2N_f \left(2\kappa_h\right)^{N_t}$$

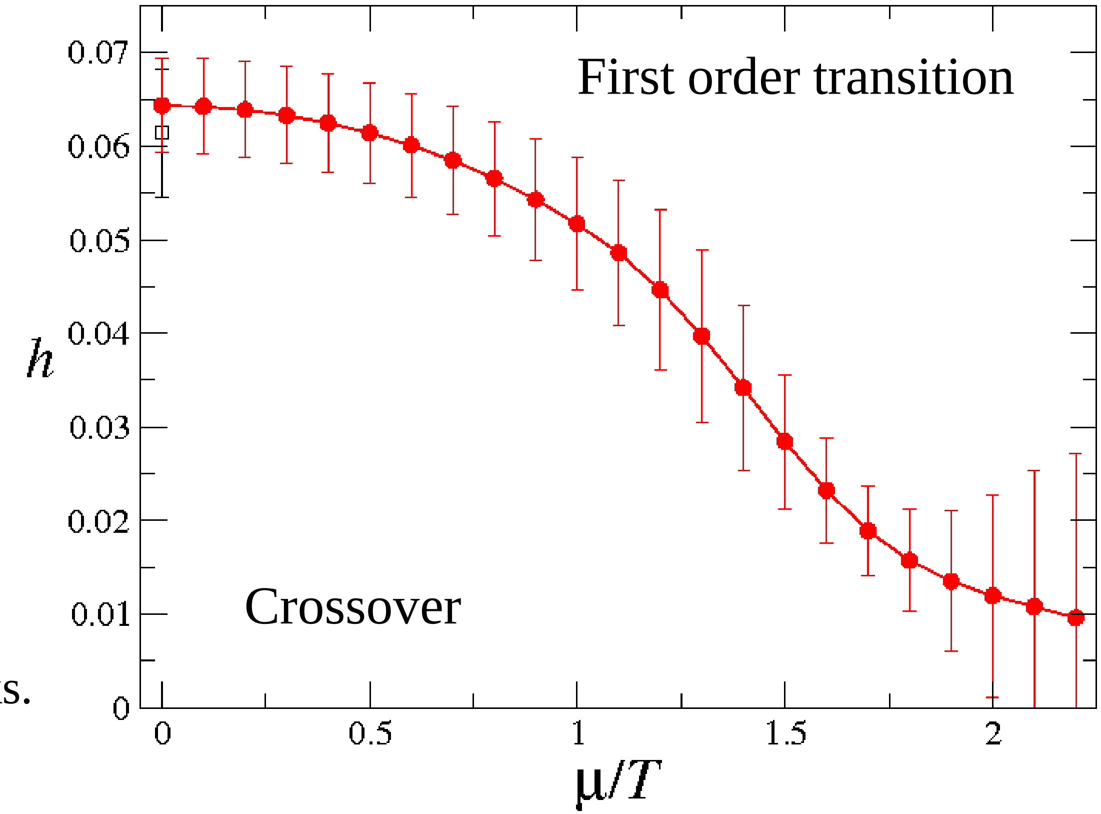
for Wilson quarks

$$h = N_f \Big/ \left(4\left(2m_h\right)^{N_t}\right)$$

for staggered quarks

2-flavor dynamical simulation
with p4-impleved staggered quarks
and reweighting for the heavy quarks.

$$m_\pi / m_\rho \approx 0.7$$



- The critical mass: larger with N_f.
- For large N_f, the critical mass is in the heavy quark region.
- First order transition region: wider as increasing μ.

Summary

- We studied the quark mass and chemical potential dependence of the nature of QCD phase transition.
- Heavy quark region: The shape of the probability distribution function changes as a function of the quark mass and chemical potential.
- To avoid the sign problem, the method based on the cumulant expansion of θ is useful.
- Phase quenched simulations: The effective potential at large μ suggests the the existence of the critical point.
- 2+N_f-flavor QCD: First order transition region: wider with μ.
- To find the critical point at finite density, further studies in light quark region are important applying this method.