Finite density lattice QCD at low temperatures

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KN arXiv:1204.6480 Low-T

KN, et al PTEP01A103(2012) Canonial, Lee-Yang zero, low T

KN, AN, JHEP1204,092,(2012) EoS near Tc, Taylor, Reweighting

KN, AN, PRD83,114507(2011) Phase boundary, imaginary

KN, AN, PRD82,094027(2010) Reduction Formula for Wilson fermions

Introduction

Introduction

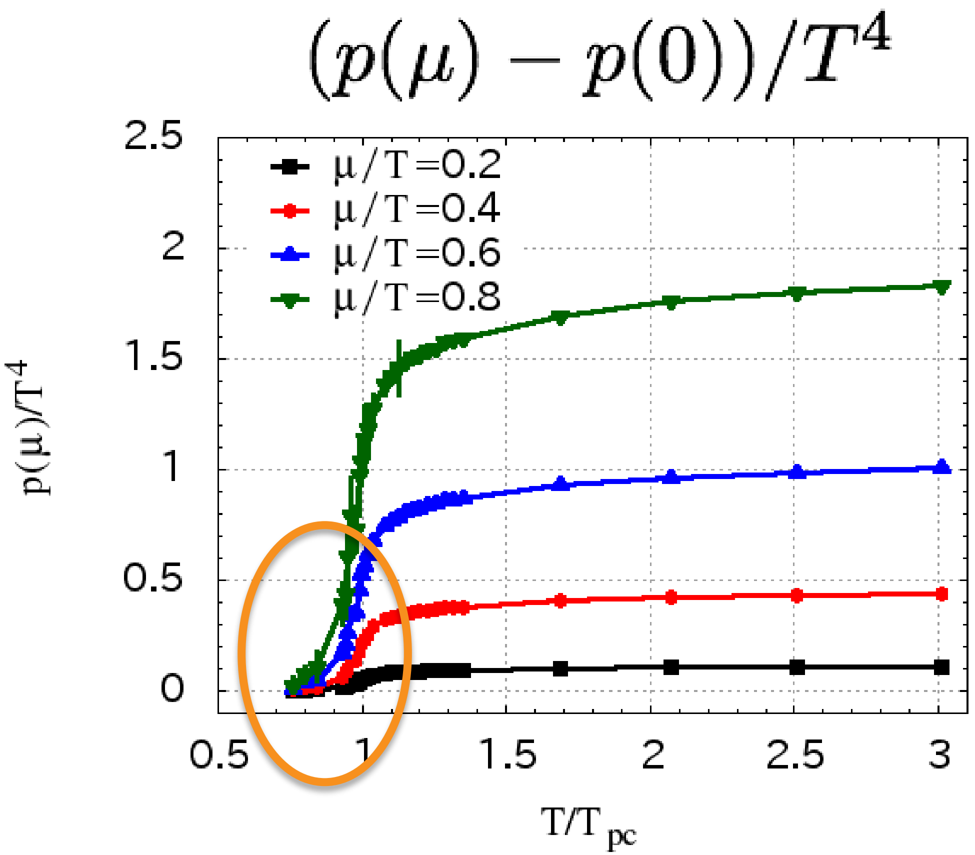
- **LQCD at finite T and μ (quark chemical potential) are desired to understand origin & properties of matter.**
- **LQCD has the sign problem at nonzero μ.**
- **There are many progress in finite density LQCD for high T and small μ.**
- **Are those methods applicable to low-T and nonzero μ region ? => two difficulties**

Introduction – small S/N ratio

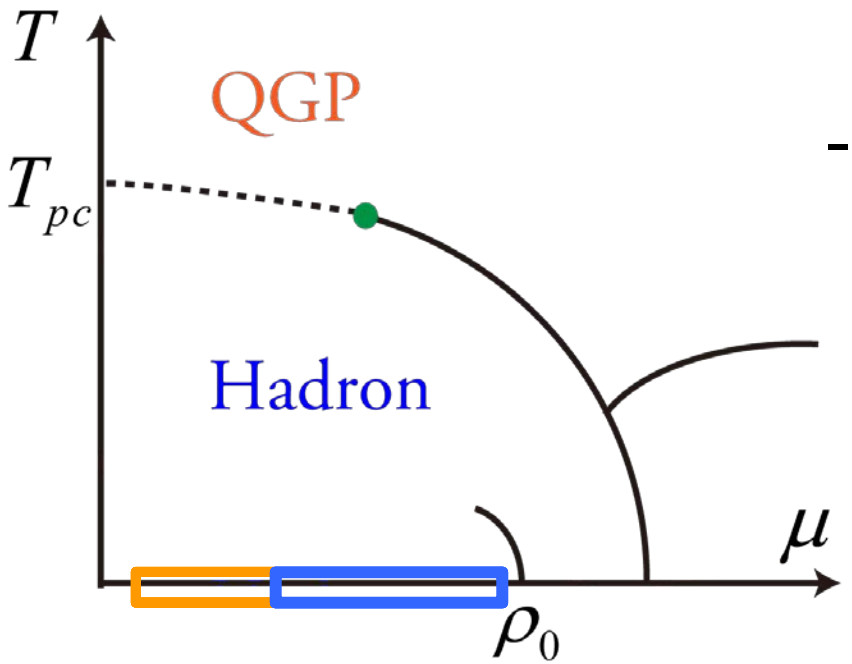
- Example of features of QCD at low-T and finite μ

- S/N ratio is small at low T.

✓ e.g., 50-100K was not enough for T/Tc=0.5 with imaginary approach [Motoki, et. al. PoS Lat2012].



Introduction - Silver Blaze of QCD



Expectatio

- baryon number density is zero for $\mu_B<m_N$ ($\mu<m_N/3$)at T=0.

– **quark number starts to increase at $\mu=m_\pi/2$ [Glasgow('90s)]**

– **Isospin Silver Blaze [Cohen, PRL91,222001('03)]**

$$\gamma_0(D+m)|\psi\rangle = \epsilon|\psi\rangle \qquad \epsilon_{\min} = m_\pi/2$$

- (No silver blaze in MDP, strong coupling)

Introduction - Scope of this talk

- **An idea is to use an analytic expression about μ-dependence with the help of a reduction formula.**

  - **μ-dependence of det Δ at low T**
  - **spectral property and physical meaning of a reduced matrix**
  - **low-T limit of det Δ**
  - **application to small μ region**

Reduction Formula

Reduction formula

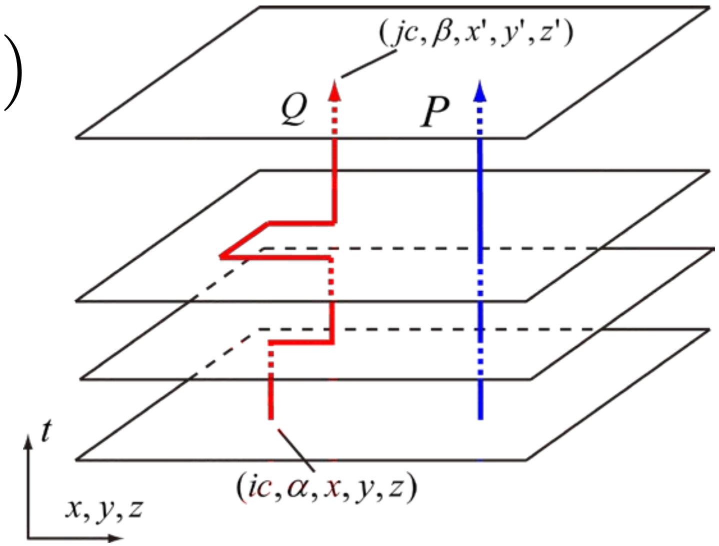
- **Fermion determinant det Δ**
  - **Locality allows to perform the temporal part of det Δ**

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0 \det(\xi + Q)$$

$$\xi = e^{-\mu/T}$$

$$N_{\rm red} = 4 N_c N_x N_y N_z$$

$$Q = A_1 A_2 \cdots A_{N_t}$$



  - **Q and C_0 are independent of μ**
  - **chemical potential and gauge fields are separated**

Gibbs('86). Hasenfratz &Toussaint('92). Adams('04), Borici('04). KN&AN('10), Alexandru &Wenger('10)

Reduction formula

- **Reduction of the rank of det Δ**
  - **Δ : 4 N_c N_x N_y N_z N_t**
  - **Q : 4 N_c N_x N_y N_z**

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0 \det(\xi + Q)$$

$$= \xi^{-N_{\rm red}/2} C_0 \prod_{n=1}^{N_{\rm red}} (\xi + \lambda_n)$$

- **With the eigenvalues,**
  - **det Δ is an analytic function of μ**
  - **values of det Δ for any value of μ.**

Barbour et al. NPB557,327('99) etc, Fodor, Katz, JHEP0203, 014('0
de Forcrand, Kratochvila, PRD73, 114512('06), etc,

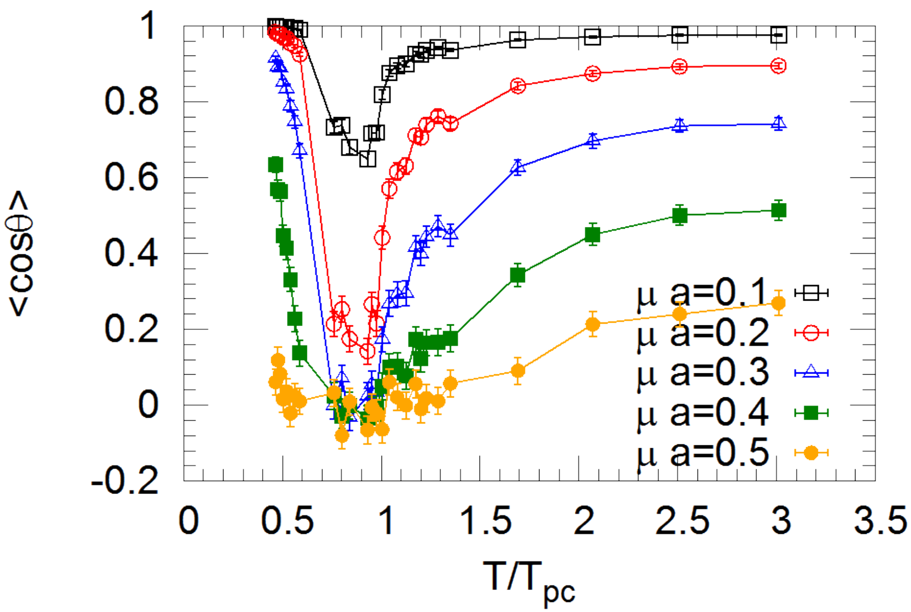
Fermion Property at Low T

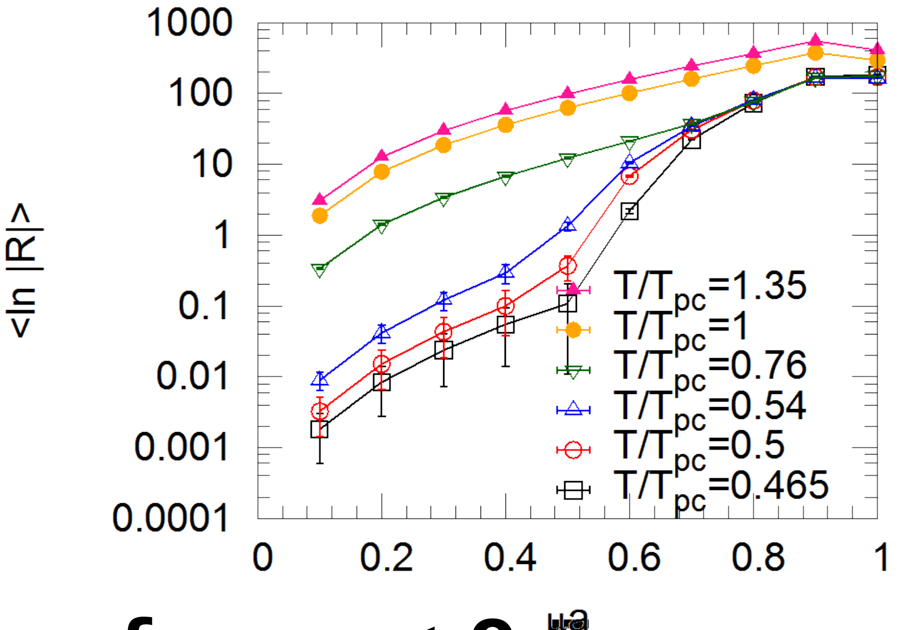
Simulation setup

- **Action (**gauge: RG-improved, fermion: clover-improved Wilson with Nf=2)
- **Mass** : mps/mV=0.8
- **Lattice sizes** : 8^3x4, 8^4 (spatial volume fixed)
- **Temperatures** : T/Tpc=0.45~3(beta = 1.5~2.4,Nt=4 and 8)
- **Configurations** :10K configurations at μ=0
- **Eigenvalue** : 400 confs. (LAPACK)

Chemical Potential Dependence at Low T

$$R = \left(\frac{\det \Delta(\mu)}{\det \Delta(0)}\right)^2 = |R|e^{i\theta}$$





- **det Δ is insensitive to μ for μa < 0.5.**
- **μ-dependence appears at μa = 0.5.**
  - **This value is close to $m_\pi/2$ in the present setup.**

Fermion Property at Low T

Meaning of Reduced matrix Q

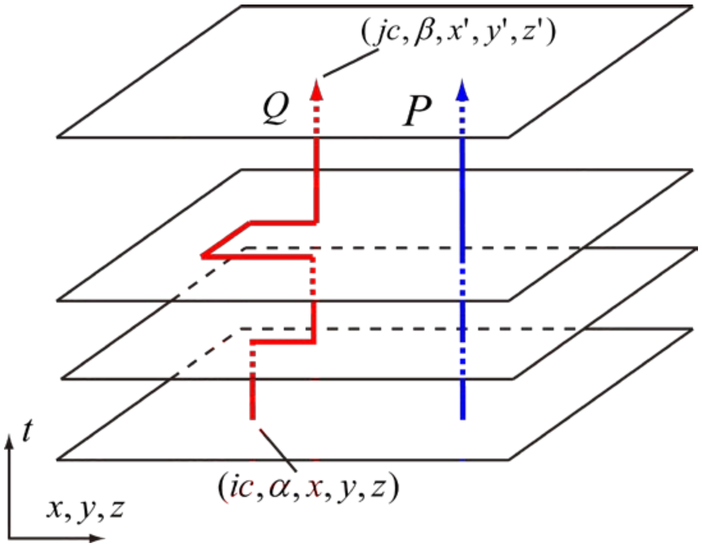
- Relation to energies of quarks

$$\lambda = e^{-\epsilon/T + i\theta}$$

  - comparison of Nt=4 & 8
  - Q and Polyakov loop

$$Q = A_1 A_2 \cdots A_{N_t}$$

$$P = \prod_{i=1}^{N_t} U_4(t_i)$$



  - zeta-regularization method [e.g. Adams, PRD70, 045002('04)]

$$Q = \exp\left(-\int_0^\beta H(\tau) d\tau \right) \qquad H = \gamma_4 \Delta(\mu) + \mu - (\partial/\partial t)$$

Meaning of Reduced matrix Q

- Quark number density

$$\det \Delta = \xi^{-N_r/2} C_0 \prod_{|\lambda|>1} (\xi + \lambda_n) \prod_{|\lambda|<1} (\xi + \lambda_n)$$

$$\hat{n} = \sum_{|\lambda|<1} \left(\frac{\lambda \xi^{-1}}{1 + \lambda \xi^{-1}} - \frac{\lambda^* \xi^{-1}}{1 + \lambda^* \xi^{-1}} \right)$$

$$\lambda = e^{-\epsilon/T + i\theta}$$

$$\xi = e^{-\mu/T}$$

$$= \sum_{|\lambda|<1} \left(\frac{1}{1 + e^{(\epsilon-\mu)/T - i\theta}} - \frac{1}{1 + e^{(\epsilon+\mu)/T + i\theta}} \right)$$

Spectral property

- **Eigenvalues form a pair(γ_5-hermiticity)**

$$\lambda_n \leftrightarrow 1/\lambda_n^*$$

  - **$|\lambda|$< 1 for quarks**
  - **$|\lambda|$> 1 for anti-quarks**

- **The Nt-scaling law of the eigenvalues**

$$|\lambda_n| = (l_n)^{N_t}$$

XQCDJ, PTEP01A103('12)

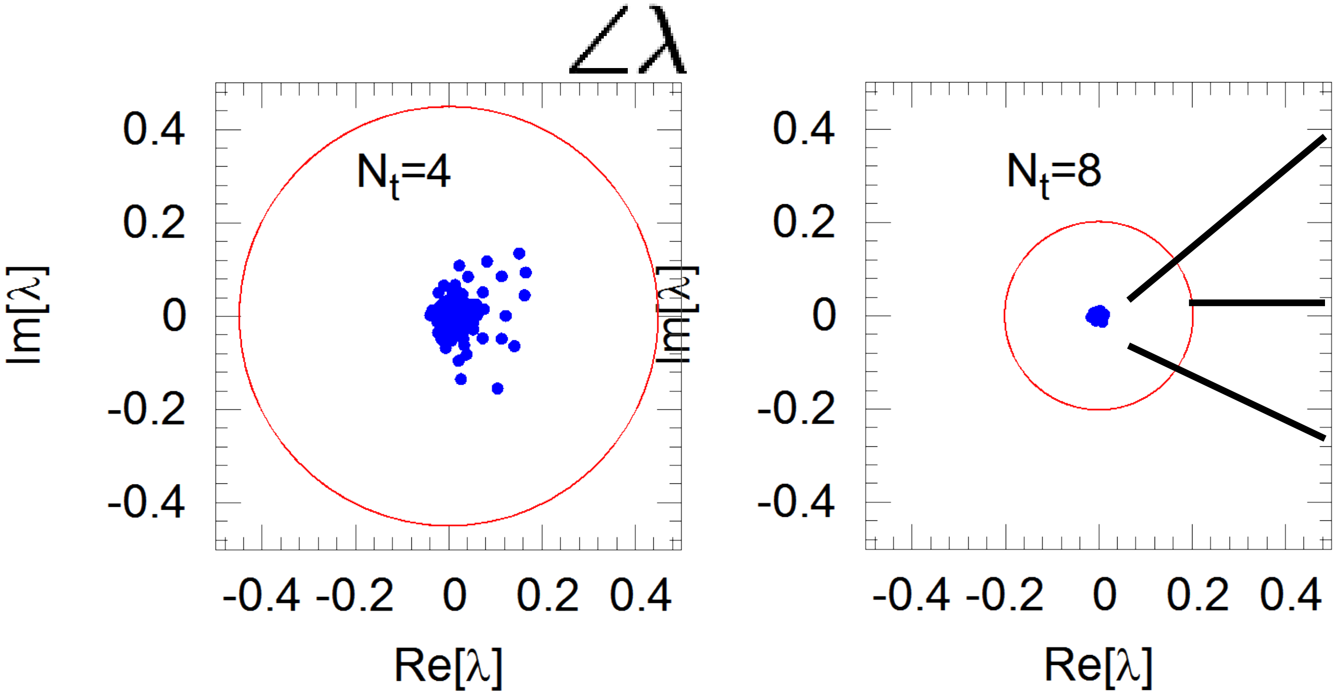
- **Ev's are related to the pion mass (lightest mass).**

$$am_\pi = -\frac{1}{N_t} \max_{|\lambda_n|<1} \ln |\lambda_n|^2$$

$$am_\pi = \lim_{N_t \to \infty} \left(-\frac{1}{N_t} \ln \left\langle c \left| \sum_{n=1}^{3V} \lambda_n \right|^2 \right\rangle \right)$$

Gibbs('86), PLB172,53('86)
Fodor, Szabo, Toth, JHEP0708,092('07)

Low Temperature Limit



$|\lambda_n| = (l_n)^{N_t}$

$e^{-\mu/T} = e^{-0.2Nt}$

$\max_{|\lambda|<1} |\lambda| = e^{-m_\pi/(2T)}$

Ev distribution is bounded

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0 \prod_{|\lambda|>1} (\xi + \lambda_n) \prod_{|\lambda|<1} (\xi + \lambda_n)$$

$$= C_0 \prod_{|\lambda|>1} (\lambda_n)$$

$$\mu < m_\pi/2$$

Meaning of Reduced matrix Q

- Quark number density

$$\det \Delta = \xi^{-N_r/2} C_0 \prod_{|\lambda|>1} (\xi + \lambda_n) \prod_{|\lambda|<1} (\xi + \lambda_n)$$

$$\lambda = e^{-\epsilon/T + i\theta}$$

$$\xi = e^{-\mu/T}$$

$$\hat{n} = \sum_{|\lambda|<1} \left(\frac{\lambda \xi^{-1}}{1 + \lambda \xi^{-1}} - \frac{\lambda^* \xi^{-1}}{1 + \lambda^* \xi^{-1}} \right)$$

$$= \sum_{|\lambda|<1} \left(\frac{1}{1 + e^{(\epsilon - \mu)/T - i\theta}} - \frac{1}{1 + e^{(\epsilon + \mu)/T + i\theta}} \right)$$

Free Energy in RF

- Free energy

$$Z(\mu) = \left\langle \frac{\det \Delta(\mu)}{\det \Delta(0)} \right\rangle = \left\langle N_{\max} \frac{\mu}{T} + \sum_{n=1}^{N_{\mathrm{red}}} \ln \frac{\lambda_n + \xi}{\lambda_n + 1} \right\rangle$$

$$\ln Z(\mu) = \sum_{m=1}^{\infty} \frac{\langle A^m \rangle_c}{m!} = \left\langle N_{\max} \frac{\mu}{T} + \sum_{n=1}^{N_{\mathrm{red}}} \ln \frac{\lambda_n + \xi}{\lambda_n + 1} \right\rangle$$

- cf

$$\ln Z = d_F V \int d^3k \ln(1 + e^{-(E-\mu)/T})$$

Summary

- We have studied QCD at low T & finite density.
  - approach to low-T region with reduction formula
  - quark number density is zero for $\mu<m_\pi/2$ at $T=0$ *for any configuration,* assuming fixed lattice size
  - quark number density starts to increase at $\mu=m_\pi/2$ *for each configuration.*

  - confirmation: Nt-scaling law, volume dependence, quark mass dependence, etc

Summary

- To obtain EoS, we have studied/will study
- for low T & small μ (small S/N ratio)
  - expression of free energy in reduction formula
- for large μ,
  - Improvement of overlap : DoS, high density limit, imaginary, isospin, etc, multi-ensemble reweighting
  - Silver Blaze : finite size effect, increase statistics etc
  - (MDP, strong coupling ...)

Buckup Slides

Free energy

$$Z(\mu) = \left\langle \frac{\det \Delta(\mu)}{\det \Delta(0)} \right\rangle = \left\langle N_{\max} \frac{\mu}{T} + \sum_{n=1}^{N_{\rm red}} \ln \frac{\lambda_n + \xi}{\lambda_n + 1} \right\rangle$$

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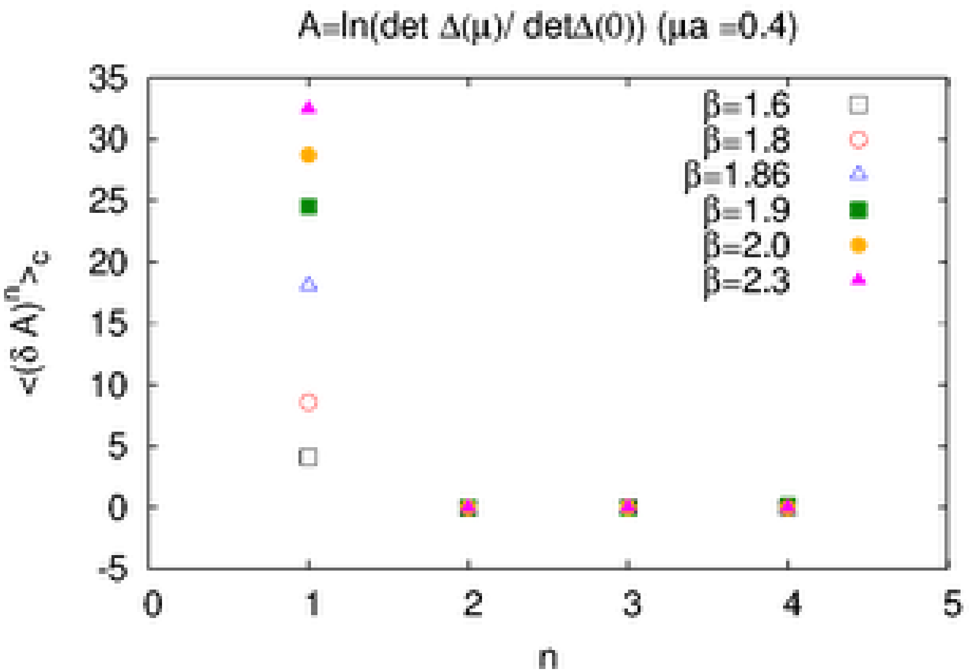
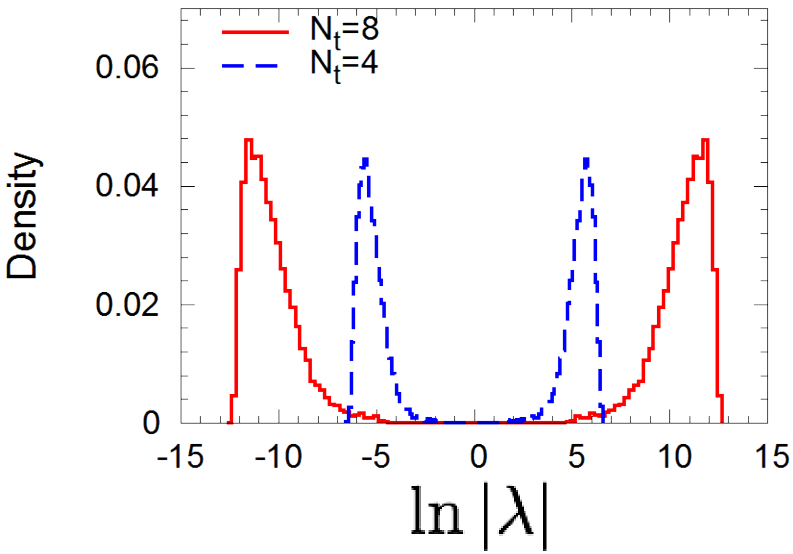
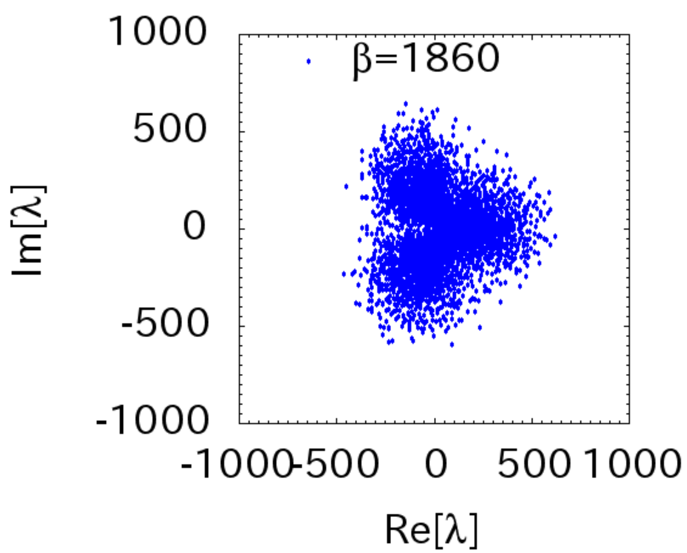
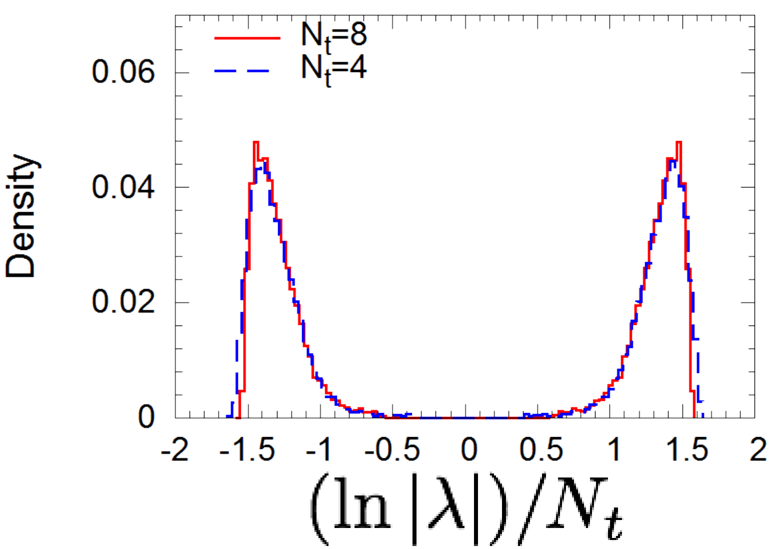


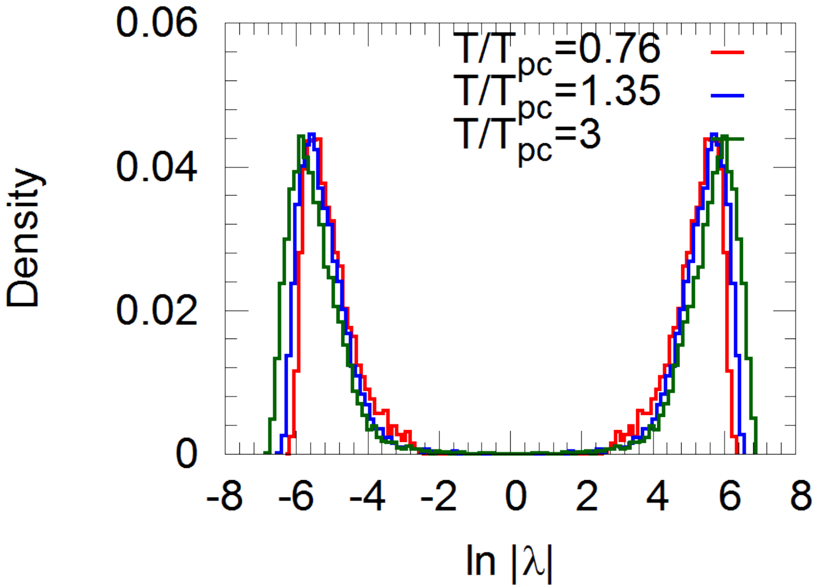
Spectral property - spectral density







Spectral property - Symmetry



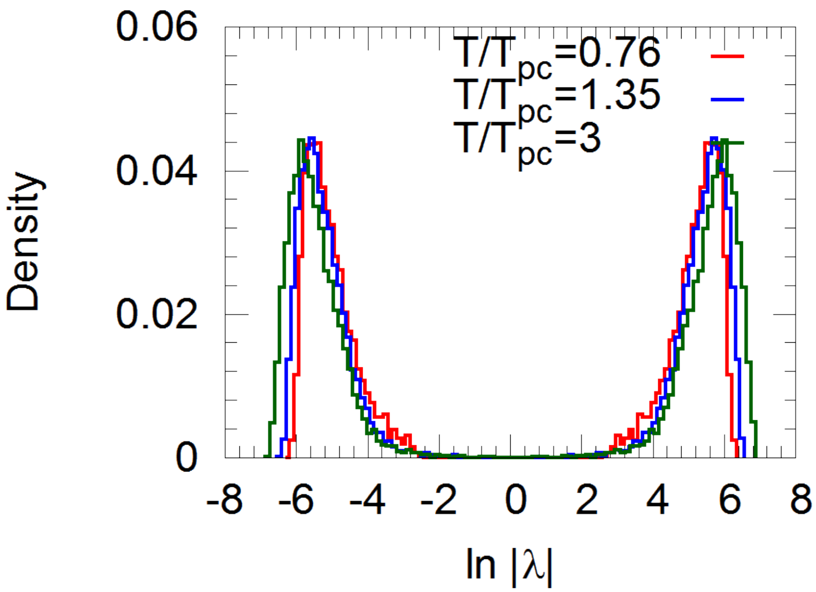
- γ_5-hermiticity
- **Eigenvalues form a pair**

$$\lambda_n \leftrightarrow 1/\lambda_n^*$$

- **Small & large eigenvalues correspond to**
  - $|\lambda| < 1$ for quarks
  - $|\lambda| > 1$ for anti-quarks

$$Q \sim P = e^{-F/T}$$

Spectral property - Gap



- **Eigenvalues form a gap**

$$|\lambda| \sim 1$$

- **Ev's are related to the pion mass (lightest mass).**

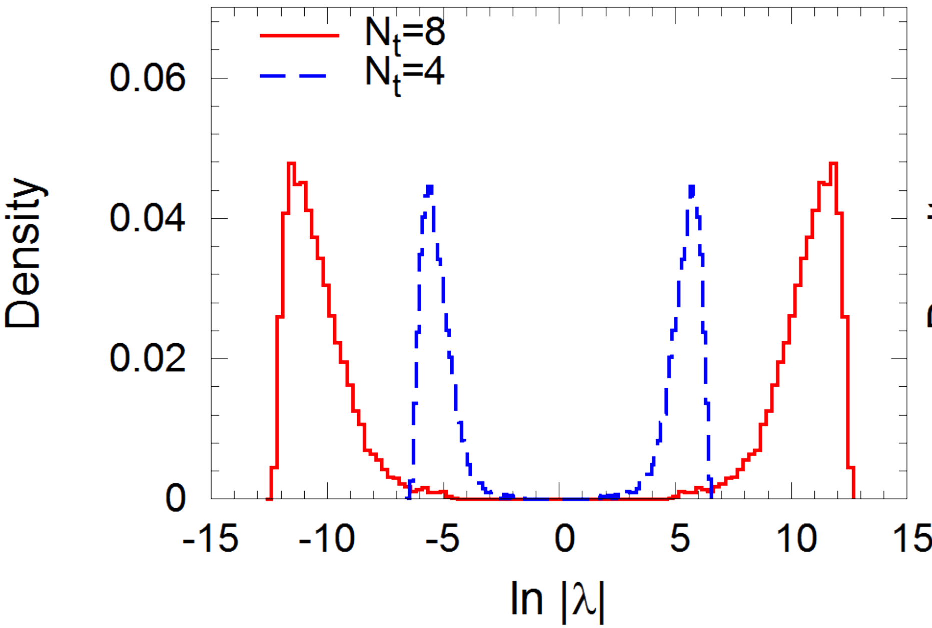
$$am_\pi = -\frac{1}{N_t} \max_{|\lambda_n|<1} \ln |\lambda_n|^2$$

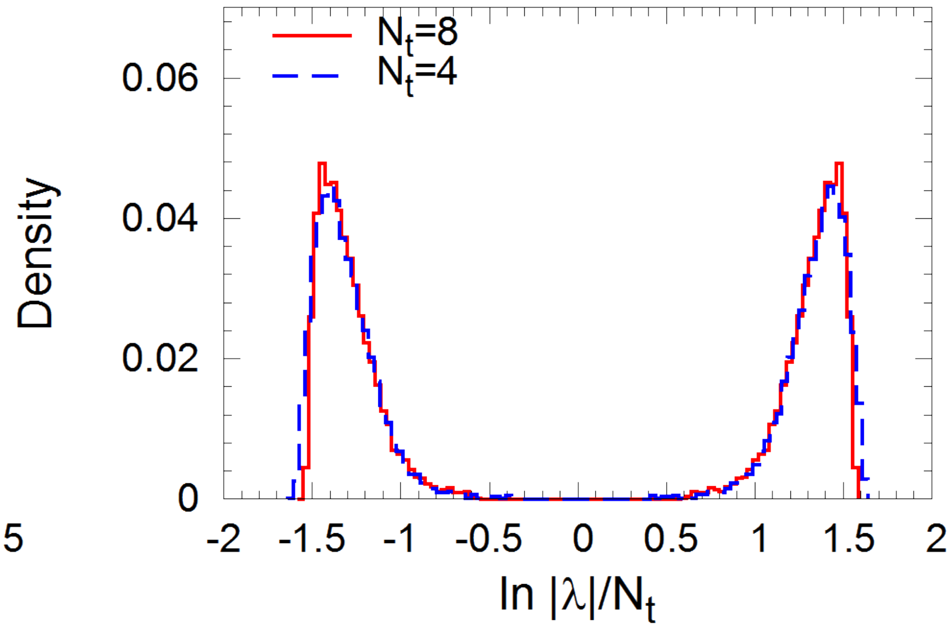
Gibbs('86), PLB172,53('86)

$$am_\pi = \lim_{N_t \to \infty} \left(-\frac{1}{N_t} \ln \left\langle c \left| \sum_{n=1}^{3V} \lambda_n \right|^2 \right\rangle \right)$$

Fodor, Szabo, Toth, JHEP0708,092('07)

Spectral property - Nt scaling law





- **The Nt-scaling law of the eigenvalues**

$$|\lambda_n| = (l_n)^{N_t}$$

$$Q = A_1 A_2 \cdots A_{N_t}$$

$$Q = \exp\left(-\int_0^\beta H(\tau)d\tau\right)$$

KN et.al. (XQCD-J), ('12)

Free Energy in RF

- Free energy

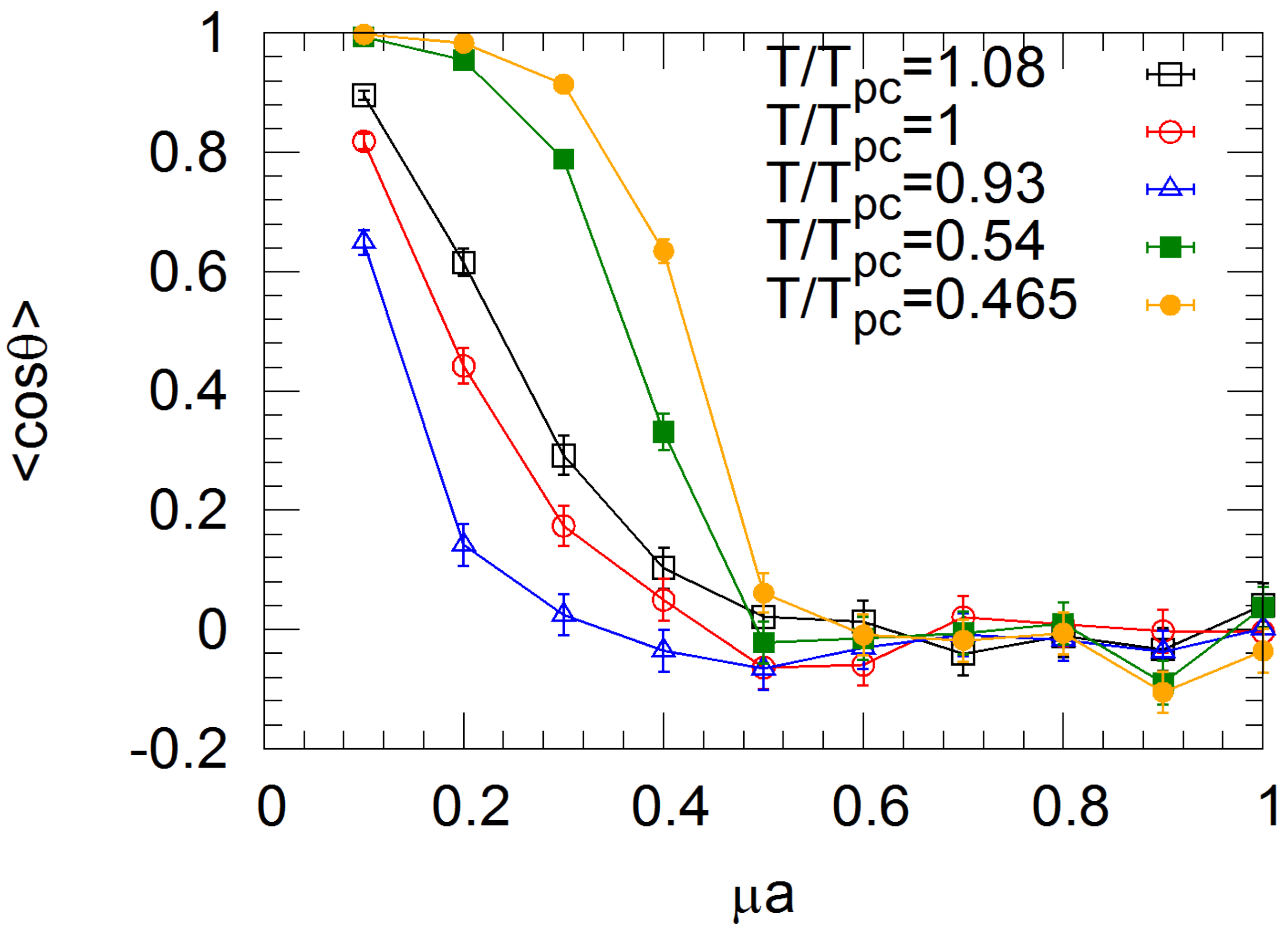
$$Z(\mu) = \left\langle \frac{\det \Delta(\mu)}{\det \Delta(0)} \right\rangle = \left\langle N_{\rm max}\frac{\mu}{T} + \sum_{n=1}^{N_{\rm red}} \ln \frac{\lambda_n + \xi}{\lambda_n + 1} \right\rangle$$

$$\ln Z(\mu) = \sum_{m=1}^{\infty} \frac{\langle A^m \rangle_c}{m!} = \left\langle N_{\rm max}\frac{\mu}{T} + \sum_{n=1}^{N_{\rm red}} \ln \frac{\lambda_n + \xi}{\lambda_n + 1} \right\rangle$$

- This form may be compared to

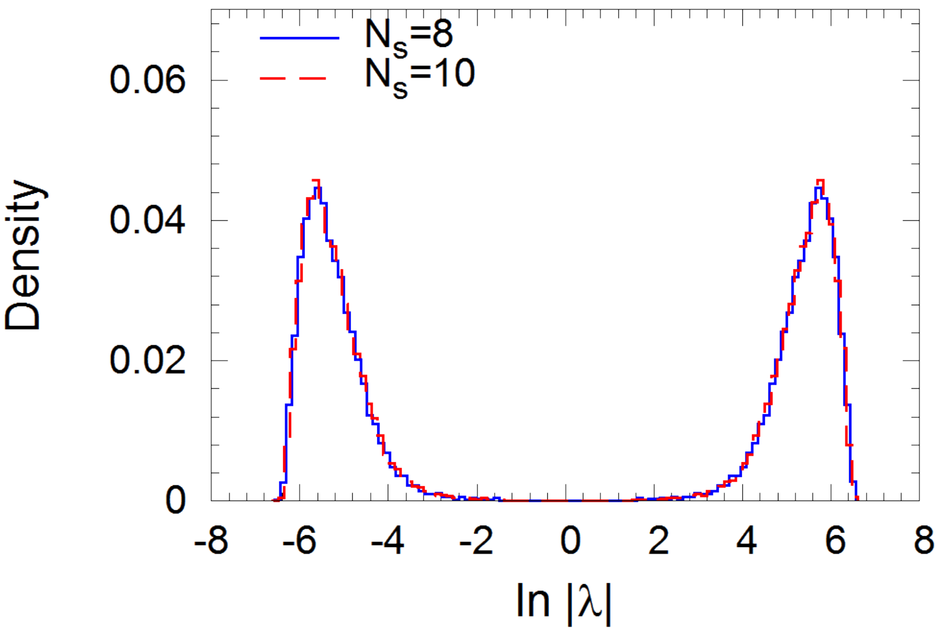
$$\ln Z = d_F V \int d^3k \ln(1 + e^{-(E-\mu)/T})$$

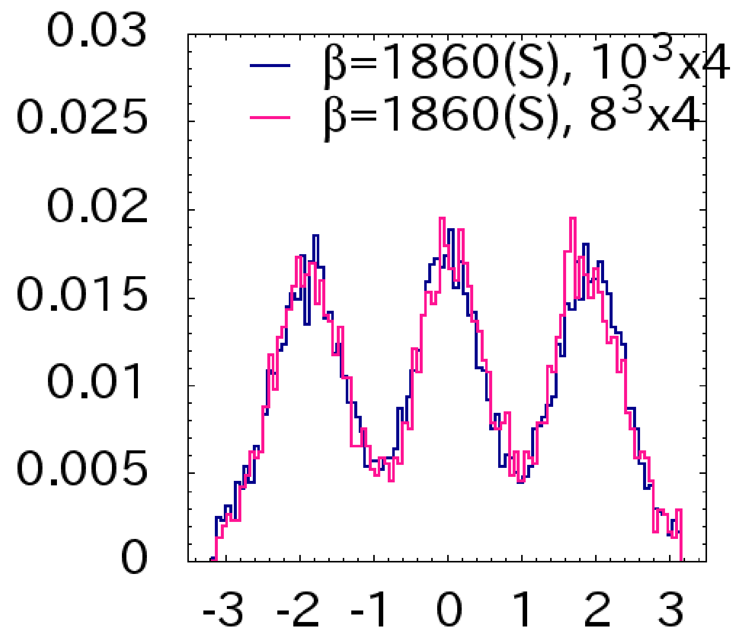
Average phase factor vs μ



Spectral property - Volume

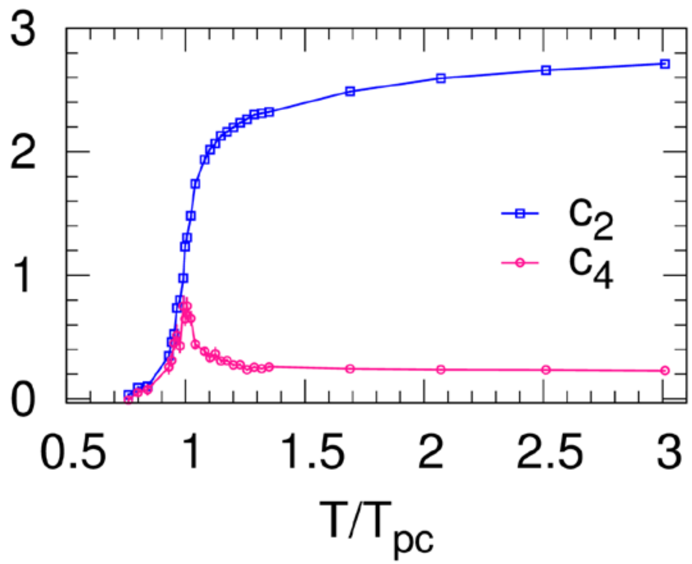
- Volume dependence

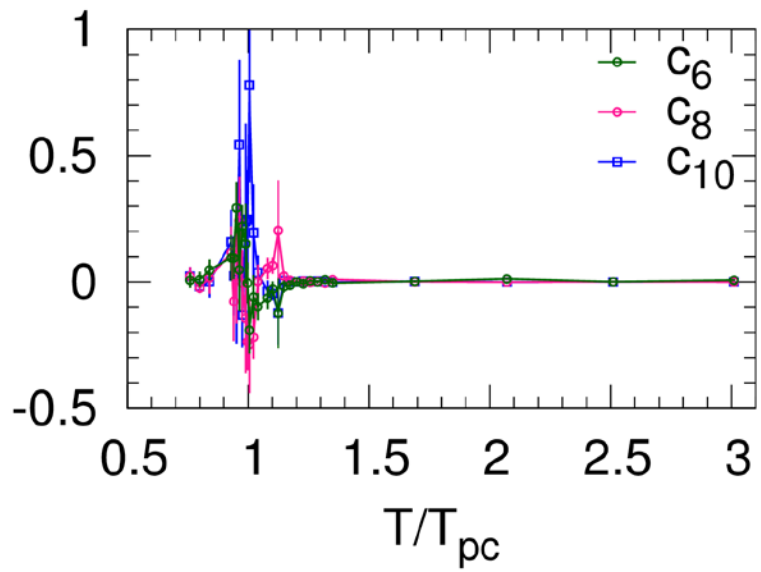




Taylor coefficients of EoS

$$f(\mu) - f(0) = \sum_{n=1} c_{2n} (\mu/T)^{2n}$$





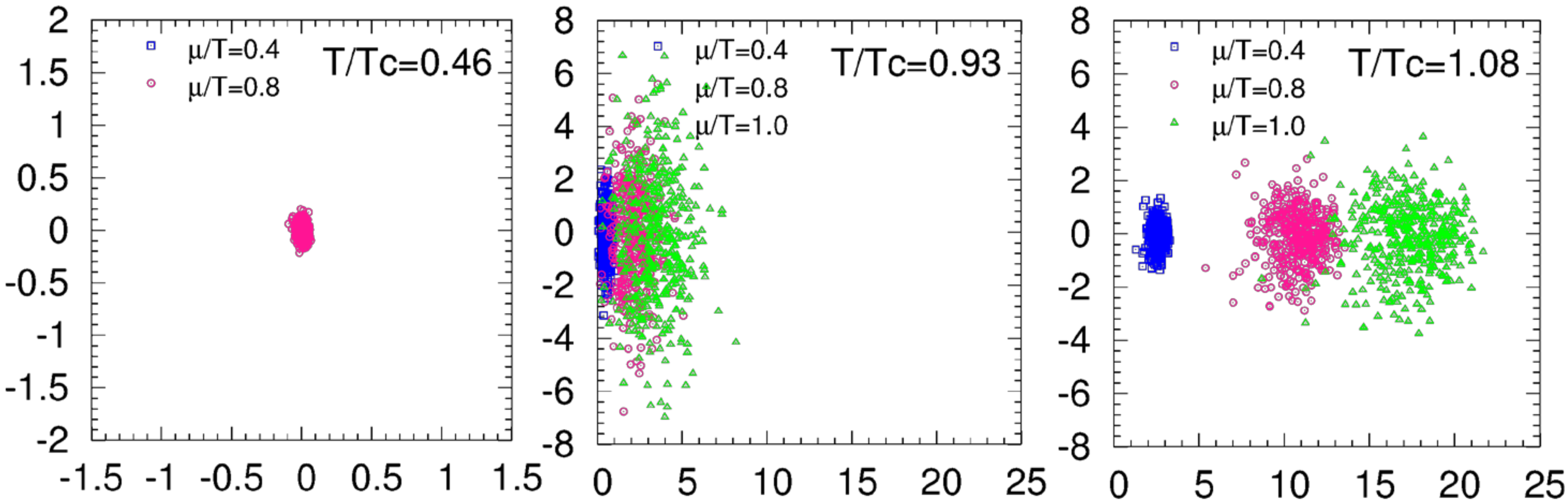
– slow convergence of the Taylor series of EoS

– small S/N ratio of chemical potential dependence

Fermion properties at low T

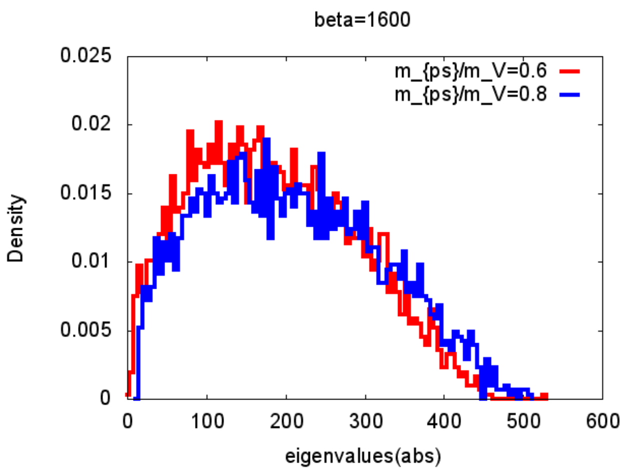
- Fermion determinants are less sensitive to mu at lower temperatures.

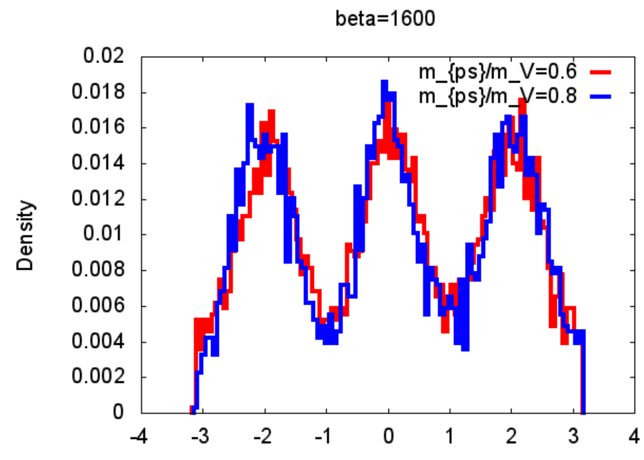
$$R = \frac{\det \Delta(\mu)}{\det \Delta(0)}$$

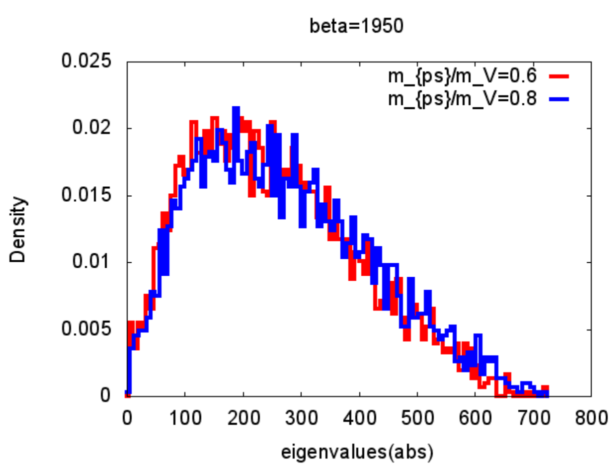


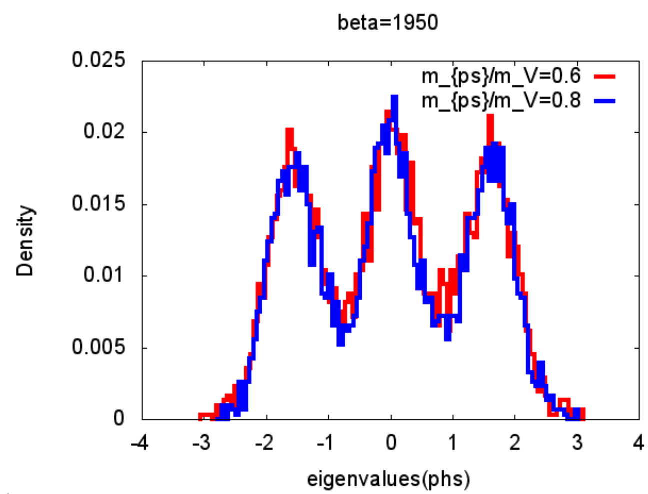
Quark mass dependence

mps / mv = 0.6 (red), 0.8 (blue)
Histograms : |ev| (Left), arg(ev) (Right)
confinement (top), deconfinement(bottom)







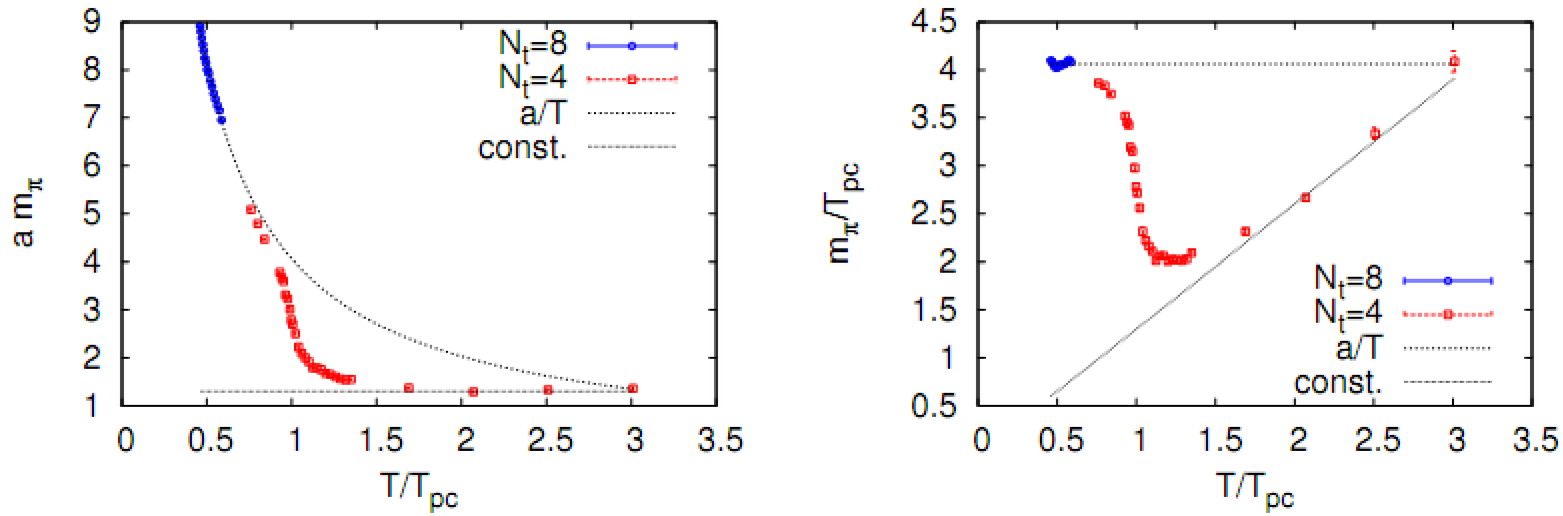


Gap is related to pion mass

$$am_\pi = -\frac{1}{Nt}\ln \max_{|\lambda_n|<1} |\lambda_n|^2$$

Gibbs('86). Eigenvalues and mpi

See also, Fodor, Szabo, Toth ('06). Eigenvalues and hadron spectrum



- *At low T, mpi/T is well fitted with a/T, a = 4 Tpc (mq heavy)*
- *At high T, mpi approaches to a constant*

Low Temperature Limit

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0 \det(\xi + Q)$$

$$= \xi^{-N_{\rm red}/2} C_0 \prod_{n=1}^{N_{\rm red}} (\xi + \lambda_n)$$

- **Eigenvalues form a pair** $\lambda_n \leftrightarrow 1/\lambda_n^*$
  - **gamma5-hermiticity**
  - **|λ|< 1 for quarks**
  - **|λ|> 1 for anti-quarks**

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0 \prod_{|\lambda_n|<1} (\xi + \lambda_n) \prod_{|\lambda_n|>1} (\xi + \lambda_n)$$

Low Temperature Limit

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0 \prod_{|\lambda_n|<1} (\xi + \lambda_n) \prod_{|\lambda_n|>1} (\xi + \lambda_n)$$

- **The Nt-scaling law of the eigenvalues**

$$|\lambda_n| = (l_n)^{N_t}$$

$$\det \Delta = C_0 \xi^{-N_{\rm Nred}} \prod_{|\lambda_n|<1} (e^{-\mu/T} + l_n^{-N_t} e^{i\theta_n}) \prod_{|\lambda_n|>1} (e^{-\mu/T} + l_n^{N_t} e^{i\theta_n})$$

- Condition of low-T & small mu limit

$$\xi > \max_{|\lambda_n|<1} |\lambda| > \text{others}$$

Low Temperature Limit

- **Condition & pion mass**

$$am_\pi = -\frac{1}{N_t} \max_{|\lambda_n|<1} \ln |\lambda_n|^2$$

$$am_\pi = \lim_{N_t\to\infty} -\frac{1}{N_t} \ln\left\langle c\left|\sum_{k=1}^{3V} \lambda_k\right|^2\right\rangle$$

$$\xi = e^{-\mu/T} > \max_{|\lambda|<1} |\lambda| = e^{-m_\pi/(2T)}$$

- **Low T and small m limit.**

$$\det\Delta = C_0 \xi^{-N_{\rm Nred}/2} \prod_{|\lambda_n|<1} (e^{-\mu/T} + l_n^{-N_t} e^{i\theta_n}) \prod_{|\lambda_n|>1} (e^{-\mu/T} + l_n^{N_t} e^{i\theta_n})$$

$$= C_0 \prod_{|\lambda_n|>1} \lambda_n, \quad \text{for } N_t \to \infty,\ \xi > \max_{|\lambda|<1} |\lambda|$$

Low-T limit

a. *small μ*

$$\det \Delta(\mu) = C_0 \prod_{i=1}^{N_r/2} \lambda_n^L$$

$$\langle n \rangle = 0$$

a. *large μ*

$$\det \Delta(\mu) = C_0 \xi^{-N_r/2} \det Q$$

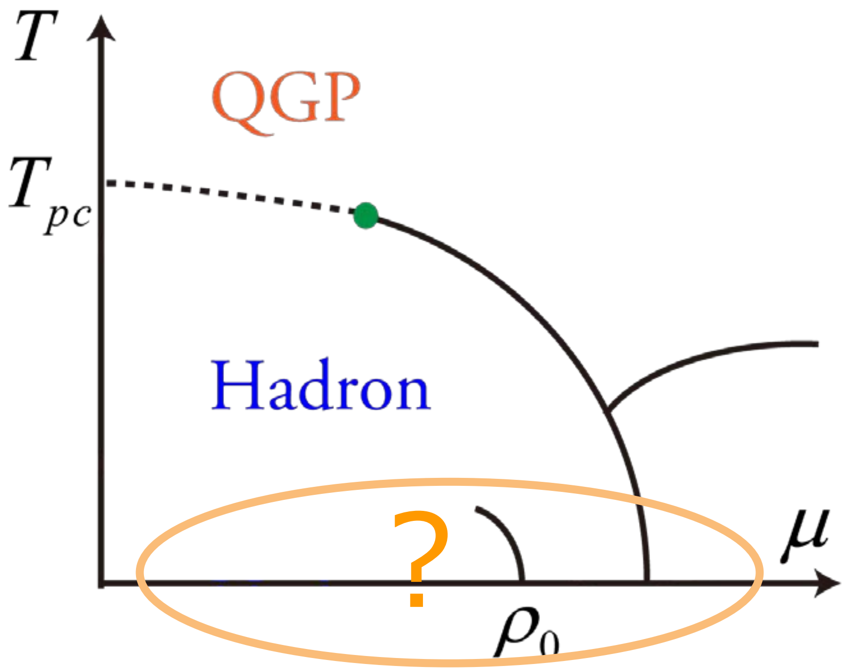
$$= e^{2N_c N_s^3 \mu/T} \prod_{i=1}^{N_t} \det(B_i r_+ - 2\kappa r_-)$$

$$\langle n \rangle = 2N_c N_f$$

- ✓ *Quarks move on a fixed time slice, there is no propagation in t-direction(t-link vanishes)*
- ✓ *Z3 invariant*
- ✓ *ev's are not needed.*

Buckup Slides

Introduction-LQCD with chemical potential



- **High T, small μ (μ/T<1)**
  - **sign fluctuation is mild**
  - **μ-dependence is measurable**
- **Low T, finite μ**
  - **S/N ratio is small**

- **Calculation of quantities at low T & finite μ requires large statistics.**

Approach to low temperature

- Chemical potential is included in the fermion determinant, $(\det \Delta)^{N_f} = \int \mathcal{D}[q\bar{q}]e^{-S_F}$
- Calculation of det D seems to be inevitable at low temperatures, e.g.
  - slow convergence of Taylor series
  - small S/N ratio
- A reduction formula provides
  - a way to evaluate det D for large Nt
  - physical implications & applications