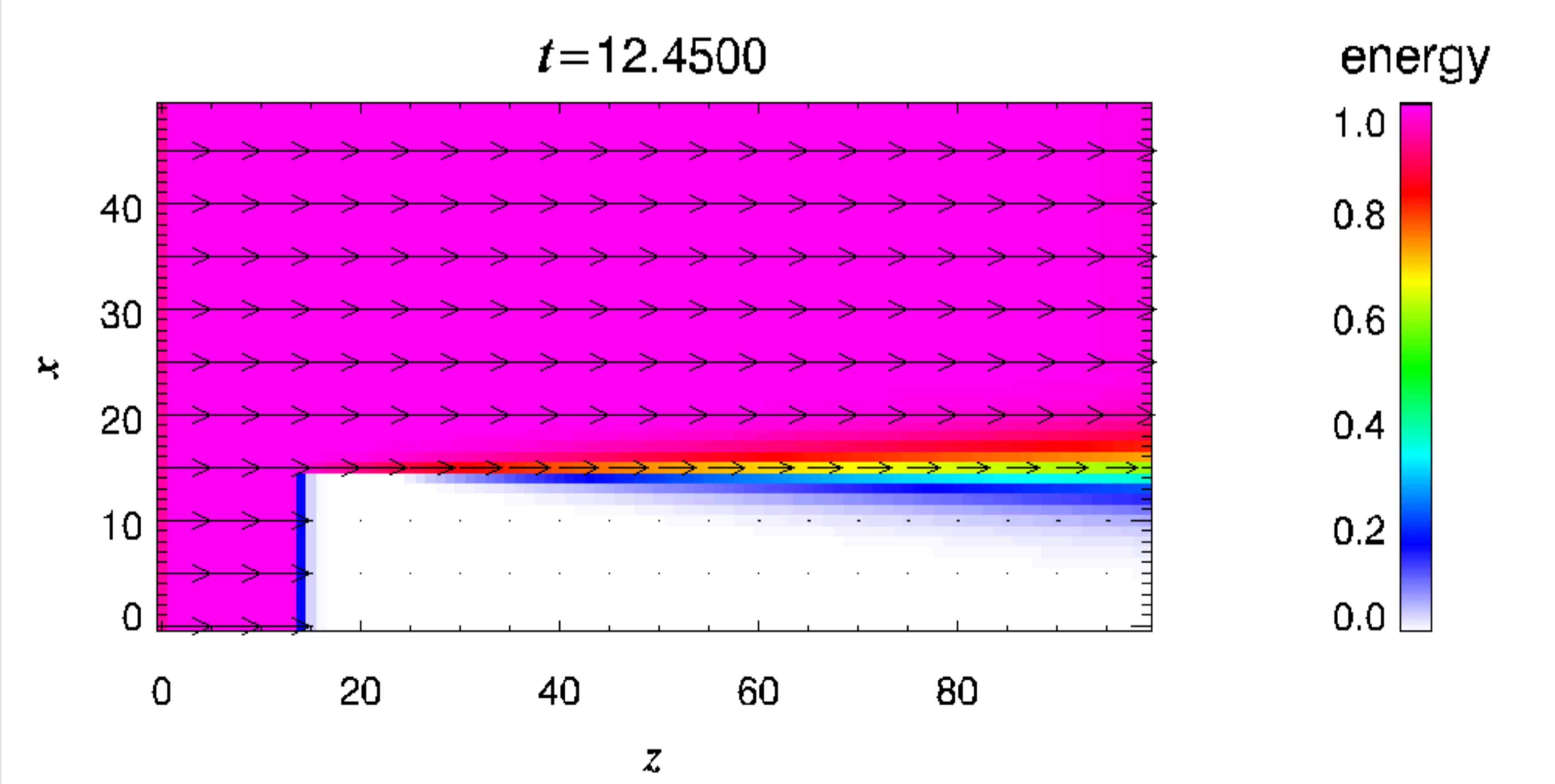
Radiative Transfer Calculation by Solving Moment Equations



Tomoyuki Hanawa, Yuji Kanno (Chiba U.)

Radiative Transfer is a Key Issue in Many Astrophysical Situations

- Neutrino Emission in Core Collapse SNe
- Neutrino Emission from Merging Binary NSs
- Super Eddington Luminous BHs
- Irradiated Protoplanetary Disks
  - and more
- M1 scheme achieves an intermediate angular resolution at low cost.
  - Diffusion approximation is coarse while full radiative transfer costs much.

Contents

- M1 Moment Equations
  - Comparison with diffusion approximation and full radiative transfer
  - Reconstruction Method
    - with & without absorption
- 1D Plane Parallel Equilibrium
- 2D and 3D Examples
  - Shadow, Directional Characteristics, Propagation
- Irradiated P.-P. Disk (2 color)

Description of Radiation Field

Intensity $\quad I_\nu\,(\boldsymbol{r},\ t,\ \boldsymbol{n})$

↓

Moment

0th: $$E_\nu\,(\boldsymbol{x},\ t) \equiv \frac{1}{c}\oint I\,(\boldsymbol{x},\ t,\ \boldsymbol{n})\,d\boldsymbol{n}$$

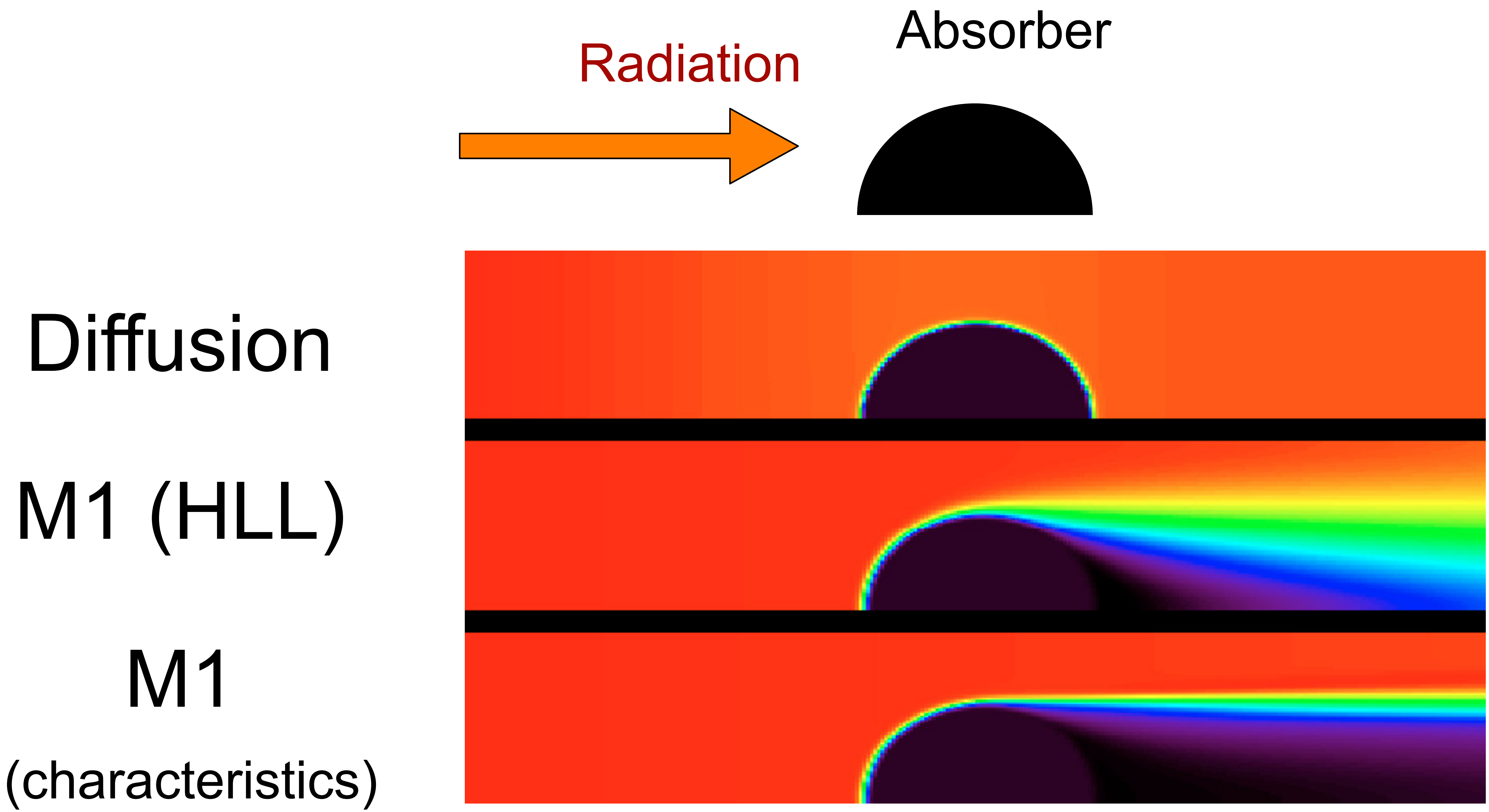
1st: $$\boldsymbol{F}_\nu(\boldsymbol{x},\ t) \equiv \oint \boldsymbol{n}\,I(\boldsymbol{x},\ t,\ \boldsymbol{n})\,d\boldsymbol{n}$$

2nd: $$\overleftrightarrow{\boldsymbol{P}}_\nu(\boldsymbol{x},\ t) \equiv \frac{1}{c}\oint \boldsymbol{n}\boldsymbol{n}\,I(\boldsymbol{x},\ t,\ \boldsymbol{n})\,d\boldsymbol{n}$$

Diffusion Approx. *E*
estimate F

M1 model (*E*, ***F***)
estimate *P*

M1 Model can handle “shadow”.



Gonzalez et al. ‘06

Equation of Radiative Transfer

Transfer | Absorption

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{n}\cdot\boldsymbol{\nabla}\right) I\left(\boldsymbol{r},\ t,\ \boldsymbol{n}\right) = -\sigma_\nu I_\nu(\boldsymbol{r},\ t,\ \boldsymbol{n})$$

$$+\sigma_\nu S_\nu\left(\boldsymbol{r},\ t,\ \boldsymbol{n}\right) + \sigma_{\nu,s}\oint g\left(\boldsymbol{n},\ \boldsymbol{n}'\right) I_\nu(\boldsymbol{r},\ t,\ \boldsymbol{n}')\, d\boldsymbol{n}'$$

Emission | Scattering

Exact

M1

$$\frac{\partial E_\nu}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F}_\nu = \sigma_\nu\left(4\pi S_\nu - cE_\nu\right)$$

$$\frac{\partial \boldsymbol{F}_\nu}{\partial t} + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{P}}_\nu = -c\left(\sigma_\nu + \sigma_{\nu,s}\right)\boldsymbol{F}_\nu$$

closure relation

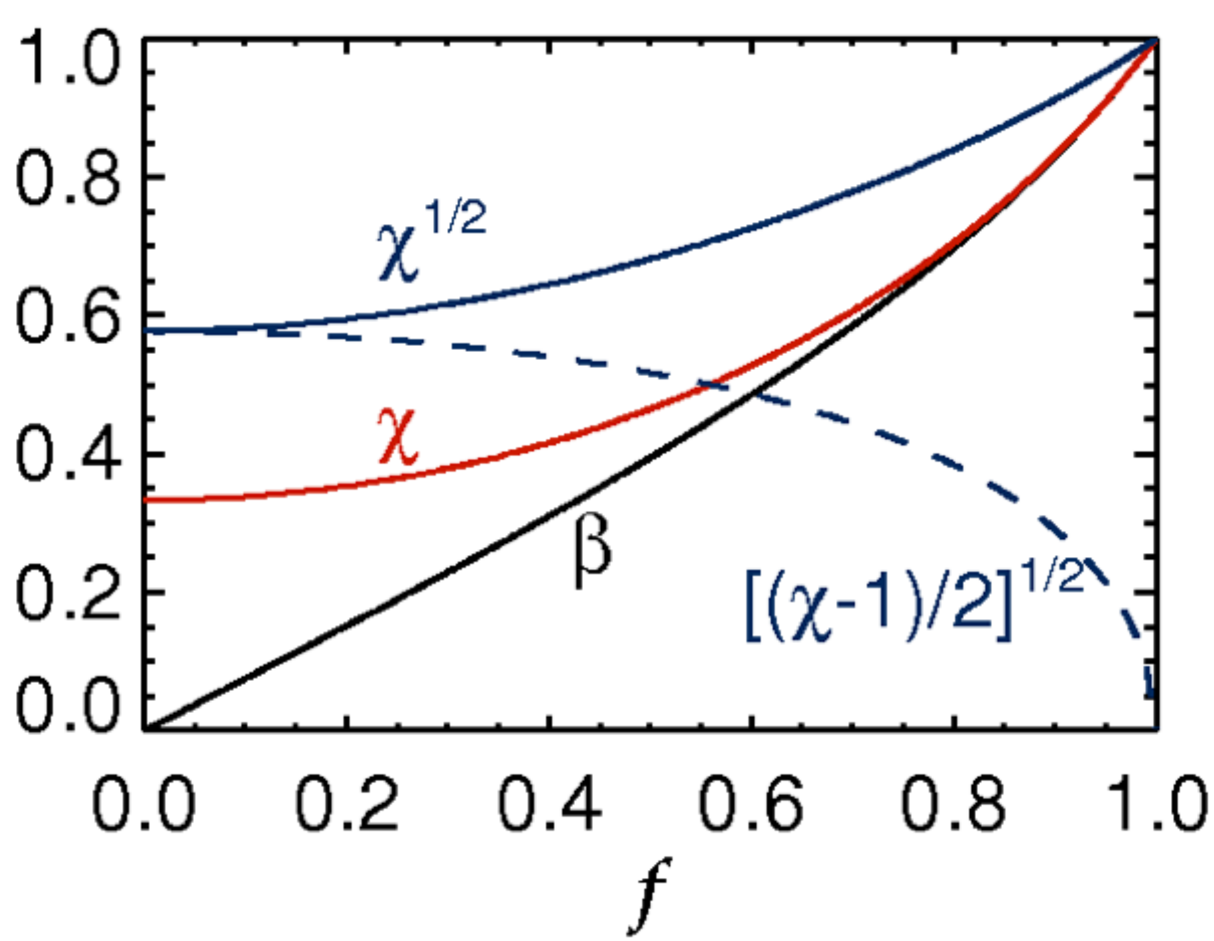
$$\overleftrightarrow{\boldsymbol{P}}_\nu = \left(\frac{1-\chi}{2}\overleftrightarrow{\boldsymbol{I}} + \frac{3\chi-1}{2}\boldsymbol{n}\boldsymbol{n}\right)E_\nu,\quad \boldsymbol{n} = \frac{\boldsymbol{F}_\nu}{\boldsymbol{F}_\nu},\quad \chi = \frac{3+4f^2}{5+2\sqrt{4-3f^2}},\quad f = \frac{|\boldsymbol{F}_\nu|}{E_\nu}$$

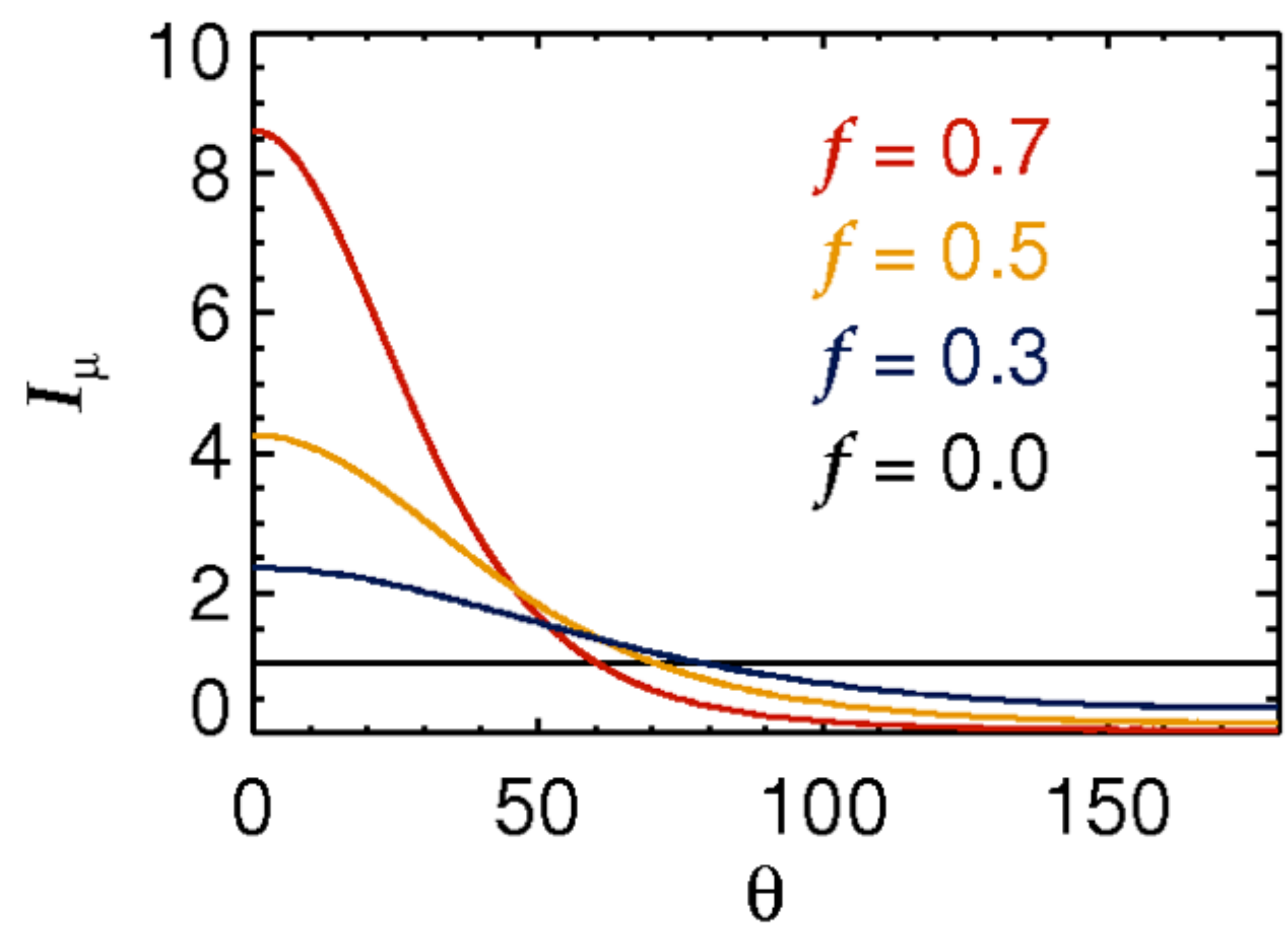
Reconstructed Radiation Field

$$\overleftrightarrow{\boldsymbol{P}}_\nu = \left(\frac{1-\chi}{2} \overleftrightarrow{\boldsymbol{I}} + \frac{3\chi - 1}{2} \boldsymbol{n}\boldsymbol{n} \right) E_\nu, \quad \boldsymbol{n} = \frac{\boldsymbol{F}_\nu}{\boldsymbol{F}_\nu}, \quad \chi = \frac{3 + 4f^2}{5 + 2\sqrt{4 - 3f^2}}, \quad f = \frac{|\boldsymbol{F}_\nu|}{E_\nu}$$

closure relation

$$I\,(\theta) = \frac{3\,(1 - \beta^2)^3\,E}{8\pi\,(3 + \beta^2)}\,(1 - \beta\cos\theta)^{-4}$$





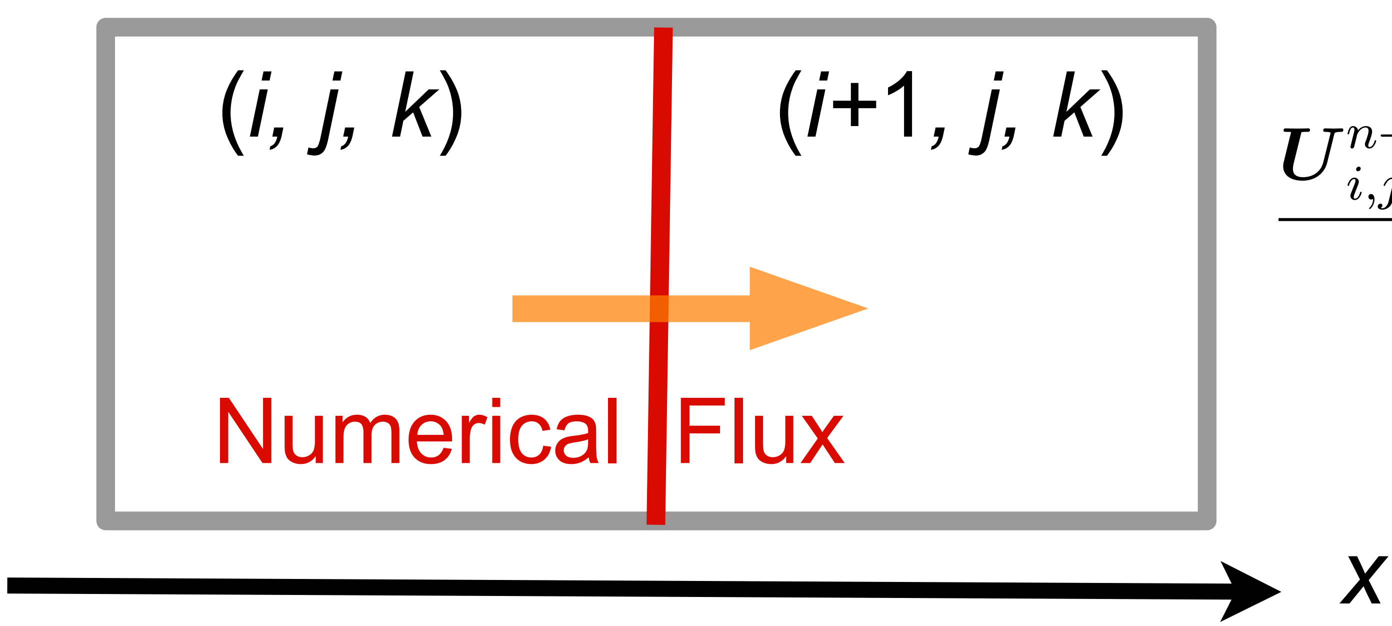
Numerical Integration of M1 Model

Conservation Form

$$\frac{\partial}{\partial t}\boldsymbol{U} + \frac{\partial}{\partial x}\boldsymbol{F}_x + \frac{\partial}{\partial y}\boldsymbol{F}_y + \frac{\partial}{\partial z}\boldsymbol{F}_z = \boldsymbol{S}$$

$$\boldsymbol{U} = \begin{pmatrix} E_\nu \\ F_{x,\nu} \\ F_{y,\nu} \\ F_{z,\nu} \end{pmatrix}, \quad \boldsymbol{F}_x = \begin{pmatrix} F_{x,\nu} \\ c^2 P_{xx,\nu} \\ c^2 P_{xy,\nu} \\ c^2 P_{xz,\nu} \end{pmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \sigma_\nu \left(4\pi S_\nu - cE_\nu\right) \\ -c\left(\sigma_\nu + \sigma_{\nu,s}\right)F_{x,\nu} \\ -c\left(\sigma_\nu + \sigma_{\nu,s}\right)F_{y,\nu} \\ -c\left(\sigma_\nu + \sigma_{\nu,s}\right)F_{z,\nu} \end{bmatrix}$$

$$\boldsymbol{F}_{x,i+1/2,j,k} = \boldsymbol{F}_{x,i+1/2,j,k}\left(\boldsymbol{U}_{x,i,j,k}, \boldsymbol{U}_{x,i+1,j,k}\right)$$



$$\frac{\boldsymbol{U}^{n+1}_{i,j,k} - \boldsymbol{U}^{n}_{i,j,k}}{\Delta t} + \frac{\boldsymbol{F}^{n}_{i+1/2,j,k} - \boldsymbol{F}^{n}_{i-1/2,j,k}}{\Delta x} = \boldsymbol{S}^{n}_{i,j,k}$$

Upwind (Characteristics)

(simple) HLL

$$\boldsymbol{F}^{(\mathrm{HLL})}_{i+1/2,j,k} = \frac{\lambda_{\mathrm{R}}\boldsymbol{F}_{i,j,k} - \lambda_{\mathrm{L}}\boldsymbol{F}_{i+1,j,k} + \lambda_{\mathrm{R}}\lambda_{\mathrm{L}}\left(\boldsymbol{U}_{i+1,j,k} - \boldsymbol{U}_{i,j,k}\right)}{\lambda_{\mathrm{R}} - \lambda_{\mathrm{L}}}$$

$\lambda_{\mathrm{R}} = c, \quad \lambda_{\mathrm{L}} = -c$ Max. Signal Speed, Safe but Diffusive

Godunov (characteristics and eigen modes)
Less diffusive but costs more. (mean characteristics are not well defined.)

Reconstruction (this work)

$$\boldsymbol{U} = \begin{pmatrix} E_\nu \\ F_{x,\nu} \\ F_{y,\nu} \\ F_{z,\nu} \end{pmatrix} \longrightarrow \begin{aligned} I_\nu(\boldsymbol{n}) &= \frac{3E_\nu}{8\pi}\frac{(1-\beta^2)^3}{3+\beta^2}\left(1-\boldsymbol{\beta}\cdot\boldsymbol{n}\right)^{-4} \\ \beta &= \frac{3f}{2+\sqrt{4-3f^2}}, \quad \boldsymbol{\beta} = \beta\,\frac{\boldsymbol{F}}{|\boldsymbol{F}|} \end{aligned}$$

consistent with the closure relation

Upwind Reconstruction

$$I^*_{\nu,i+1/2,j,k}(\boldsymbol{n}) = \begin{cases} I_{\nu,i,j,k}(\boldsymbol{n}) & (n_x > 0) \\ I_{\nu,i+1,j,k}(\boldsymbol{n}) & (n_x < 0) \end{cases}$$

$I_{j+1}(\boldsymbol{n})$ ← → $I_j(\boldsymbol{n})$

$$\begin{pmatrix} E_j \\ \boldsymbol{F}_j \end{pmatrix} = \begin{pmatrix} E_j \\ \boldsymbol{f}_j E_j \end{pmatrix} \qquad \begin{pmatrix} E_{j+1} \\ \boldsymbol{F}_{j+1} \end{pmatrix} = \begin{pmatrix} E_{j+1} \\ \boldsymbol{f}_{j+1} E_{j+1} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{F}^*_{j+1/2} = \boldsymbol{F}^{(+)}_j + \boldsymbol{F}^{(-)}_{j+1} \\ \mathrm{P}^*_{j+1/2} = \mathrm{P}^{(+)}_j + \mathrm{P}^{(-)}_{j+1} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{F}^*_{j+1/2} \\ \mathrm{P}^*_{j+1/2} \end{pmatrix}$$

$$I_\nu(\boldsymbol{n}) = \frac{3E_\nu}{8\pi}\frac{(1-\beta^2)^3}{3+\beta^2}(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^{-4}$$

$$\beta = \frac{3f}{2+\sqrt{4-3f^2}}, \qquad \boldsymbol{\beta} = \beta\frac{\boldsymbol{F}}{|\boldsymbol{F}|}$$

Consistent with M1 closure

Reconstructed Numerical Flux

$$F_{\nu,x,i+1/2,j,k} = F^{(+)}_{\nu,x,i+1/2,j,k} + F^{(-)}_{\nu,x,i+1/2,j,k}$$

$$F^{(+)}_{\nu,x,i+1/2j,k} = \oint_{n_x > 0} n_x \, I^{*}_{i+1/2,j,k}(\boldsymbol{n}) \, d\boldsymbol{n}$$

$$F^{(-)}_{\nu,x,i+1/2j,k} = \oint_{n_x < 0} n_x \, I^{*}_{i+1/2,j,k}(\boldsymbol{n}) \, d\boldsymbol{n}$$

$$P_{\nu,xx,i+1/2,j,k} = P^{(+)}_{\nu,xx,i+1/2,j,k} + P^{(-)}_{\nu,xx,i+1/2,j,k}$$

$$P_{\nu,xy,i+1/2,j,k} = P^{(+)}_{\nu,xy,i+1/2,j,k} + P^{(-)}_{\nu,xy,i+1/2,j,k}$$

$$P_{\nu,xz,i+1/2,j,k} = P^{(+)}_{\nu,xz,i+1/2,j,k} + P^{(-)}_{\nu,xz,i+1/2,j,k}$$

$$I^{*}_{\nu,i+1/2,j,k}(\boldsymbol{n}) = \begin{cases} I_{\nu,i,j,k}(\boldsymbol{n}) & (n_x > 0) \\ I_{\nu,i+1,j,k}(\boldsymbol{n}) & (n_x < 0) \end{cases}$$

without absorption

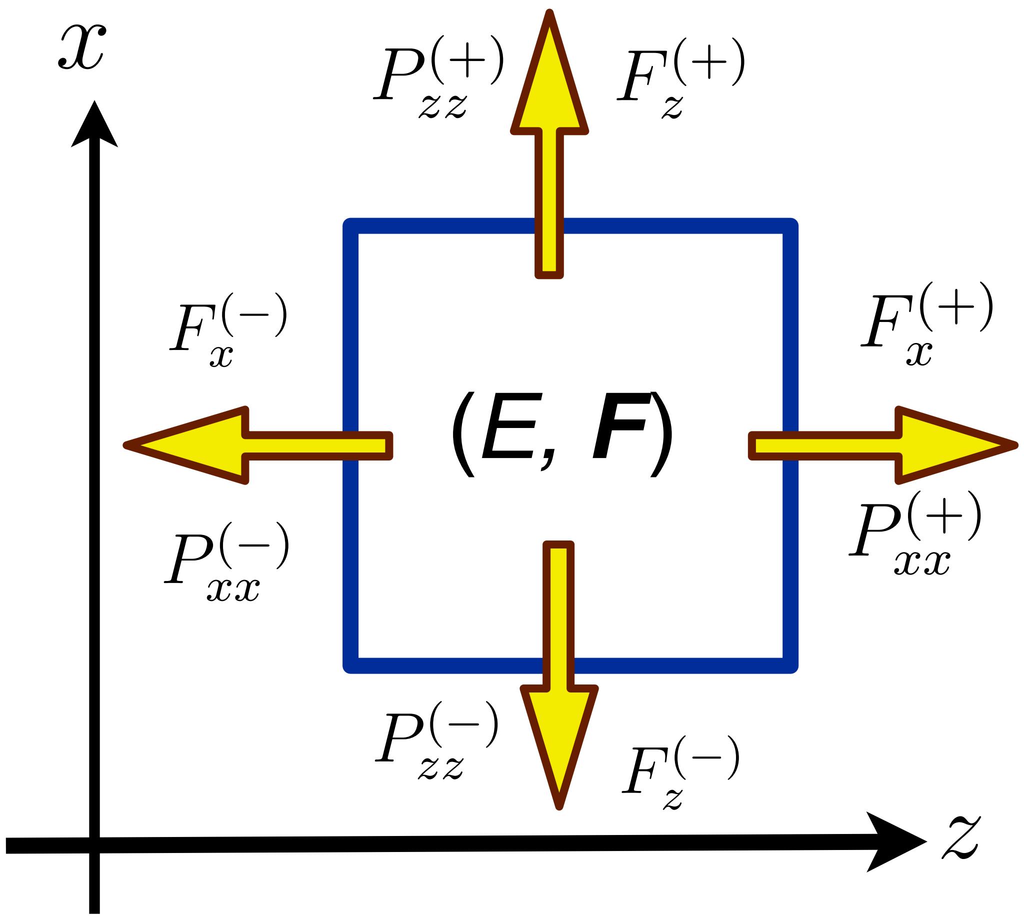
Numerical fluxes are explicit functions of *E* and ***F***.

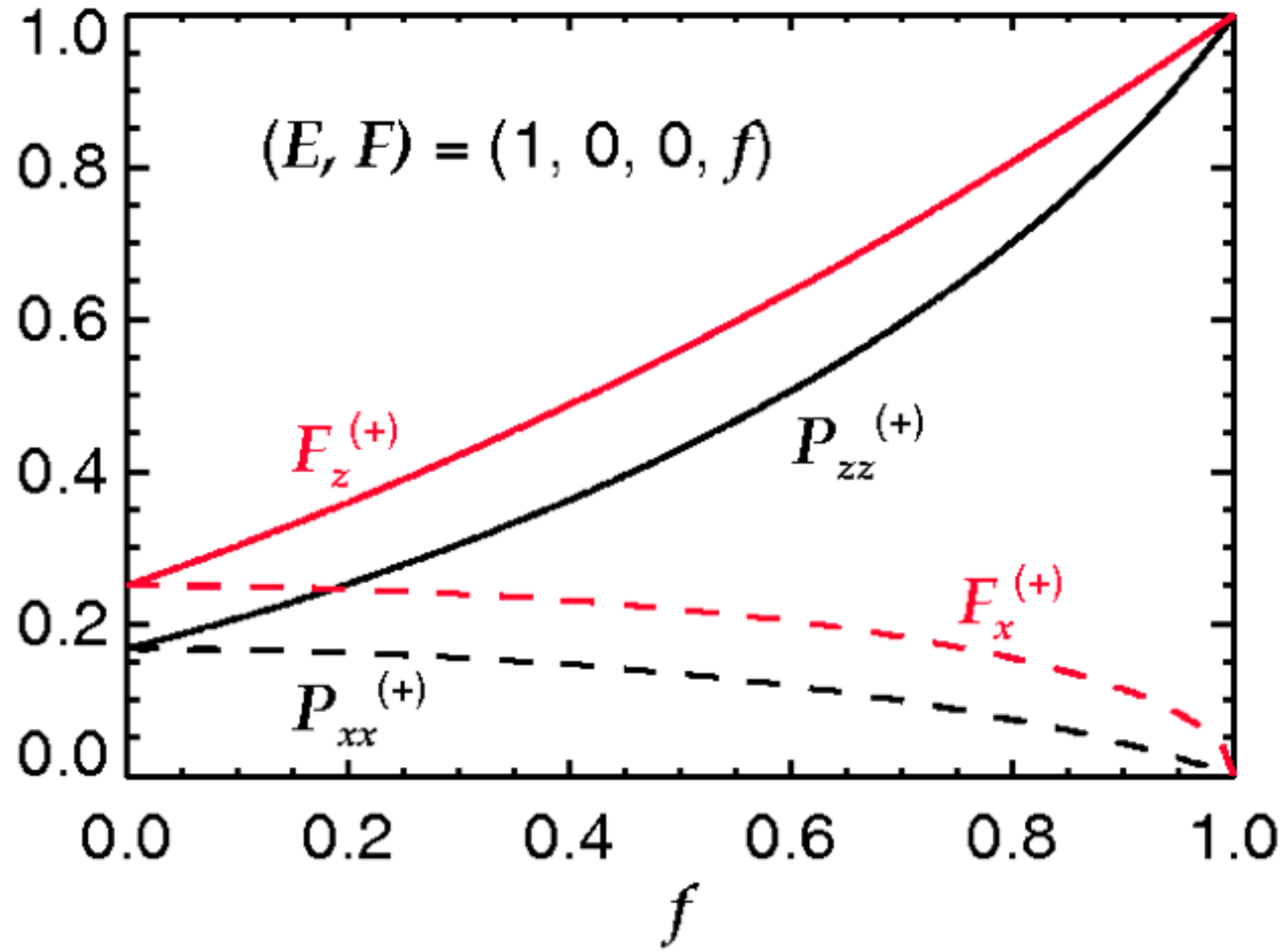
Numerical Fluxes Given by Reconstruction

$$
\begin{aligned}
F_z^+ &= \left(3q^4 + 6\beta^2 c^2 q^2 - c^4\beta^4 + 8\beta c q^3\right)\left\{8\left(3 + \beta^2\right) q^3\right\}^{-1} \\
P_{zz}^+ &= \frac{1}{2\left(3 + \beta^2\right)}\left[\frac{\beta^3 c^3}{q} + 3\beta c q + 4\beta^2 c^2 + 1 - \beta^2\right] \\
P_{xz}^+ &= \beta\,\frac{f_x}{f}\,F_z \\
P_{yz}^+ &= \beta\,\frac{f_y}{f}\,F_z \\
c &= \cos\psi \equiv \frac{f_z}{f} \\
q &= \sqrt{1 - \beta^2 s^2} = \sqrt{1 - \beta^2 + \beta^2 c^2}
\end{aligned}
$$

$$
\beta = \frac{3f}{2 + \sqrt{4 - 3f^2}}
$$

Fluxes from Each Cube Face





Numerical Flux Evaluated by Reconstruction

Numerical Flux Modified by Absorption and Emission

emission

$$
\begin{aligned}
F_{\nu,x,i+1/2,j,k} &= F'^{(+)}_{\nu,x,i+1/2,j,k} + F'^{(-)}_{\nu,x,i+1/2,j,k} \\
F'^{(+)}_{\nu,x,i+1/2,j,k} &= e^{-\Delta\tau_i/2} F^{(+)}_{\nu,x,i+1/2,j,k} + \left(1 - e^{-\Delta\tau_i/2}\right)\frac{S_\nu}{4} \\
F'^{(+)}_{\nu,x,i+1/2,j,k} &= e^{-\Delta\tau_{i+1}/2} F^{(+)}_{\nu,x,i+1/2,j,k} - \left(1 - e^{-\Delta\tau_{i+1}/2}\right)\frac{S_\nu}{4} \\
P_{\nu,xx,i+1/2,j,k} &= P'^{(+)}_{\nu,xx,i+1/2,j,k} + P'^{(-)}_{\nu,xx,i+1/2,j,k} \\
P'^{(+)}_{\nu,x,i+1/2,j,k} &= e^{-\Delta\tau_i/2} P^{(+)}_{\nu,x,i+1/2,j,k} + \left(1 - e^{-\Delta\tau_i/2}\right)\frac{S_\nu}{6} \\
P'^{(-)}_{\nu,x,i+1/2,j,k} &= e^{-\Delta\tau_{i+1}/2} P^{(-)}_{\nu,x,i+1/2,j,k} + \left(1 - e^{-\Delta\tau_i/2}\right)\frac{S_\nu}{6} \\
P_{\nu,xy,i+1/2,j,k} &= P'^{(+)}_{\nu,xy,i+1/2,j,k} + P'^{(-)}_{\nu,xy,i+1/2,j,k} \\
P'^{(+)}_{\nu,x,i+1/2,j,k} &= e^{-\Delta\tau_i/2} P^{(+)}_{\nu,x,i+1/2,j,k} \\
P'^{(-)}_{\nu,x,i+1/2,j,k} &= e^{-\Delta\tau_{i+1}/2} P^{(-)}_{\nu,x,i+1/2,j,k}
\end{aligned}
$$

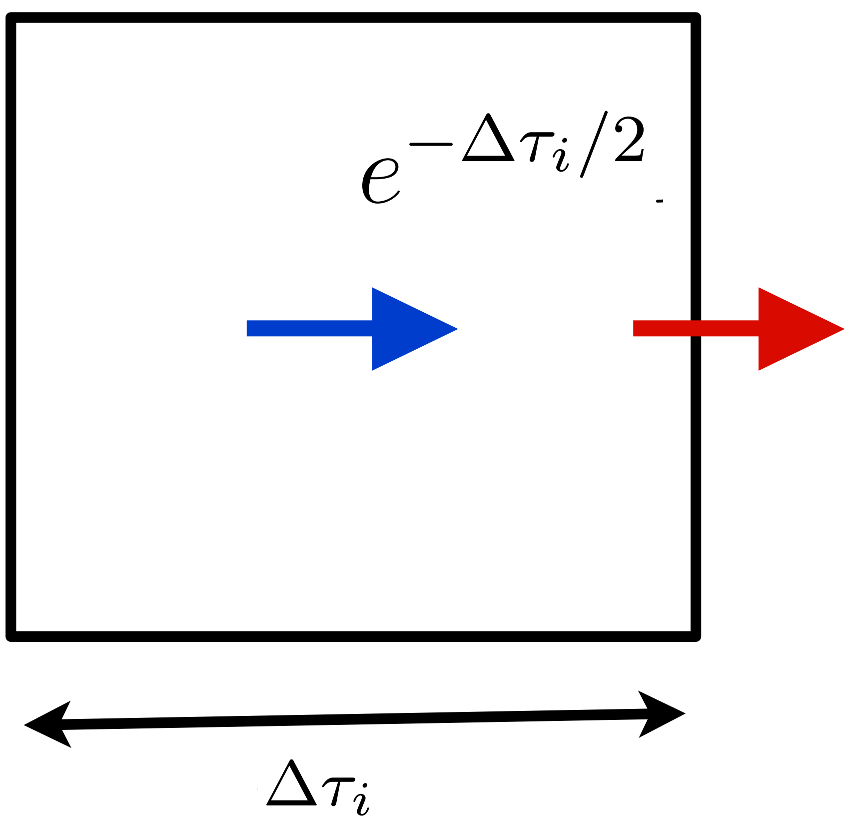
absorption

Modification due to Emission and Absorption

Effects of Absorption Evaluated by the Formal Solution

$$F'^{(+)}_{\nu,x,i+1/2,j,k} = e^{-\Delta\tau_i/2} F^{(+)}_{\nu,x,i+1/2,j,k} + \left(1 - e^{-\Delta\tau_i/2}\right) \frac{S_\nu}{4}$$

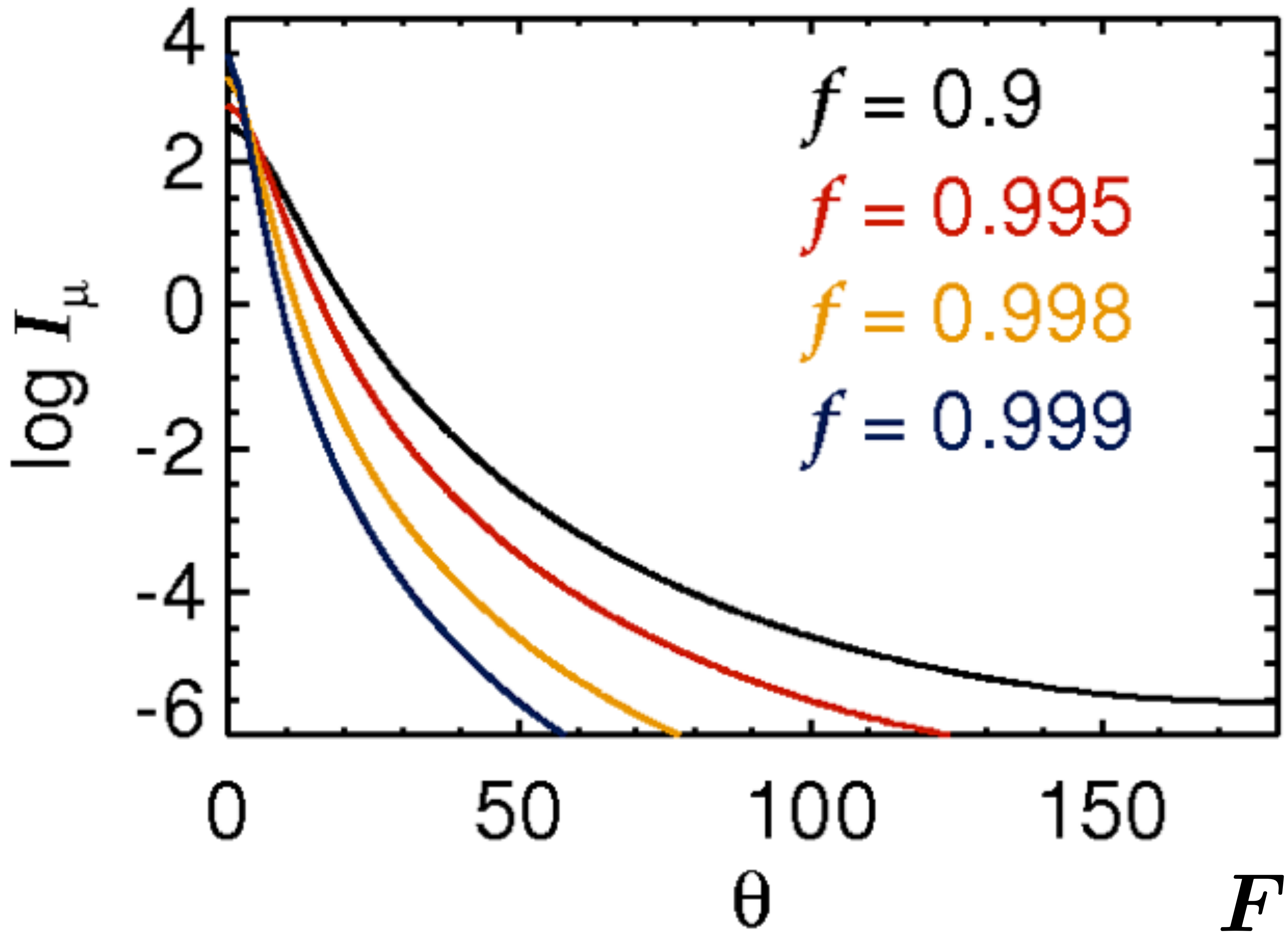
Flux at Boundary   Absorption   Flux at Center   Emissivity



Good Approximation at a large $\Delta\tau_i$

$$P'^{(+)}_{\nu,x,i+1/2,j,k} = e^{-\Delta\tau_i/2} P^{(+)}_{\nu,x,i+1/2,j,k} + \left(1 - e^{-\Delta\tau_i/2}\right) \frac{S_\nu}{6}$$

Aboid Extremely Sharp Beam

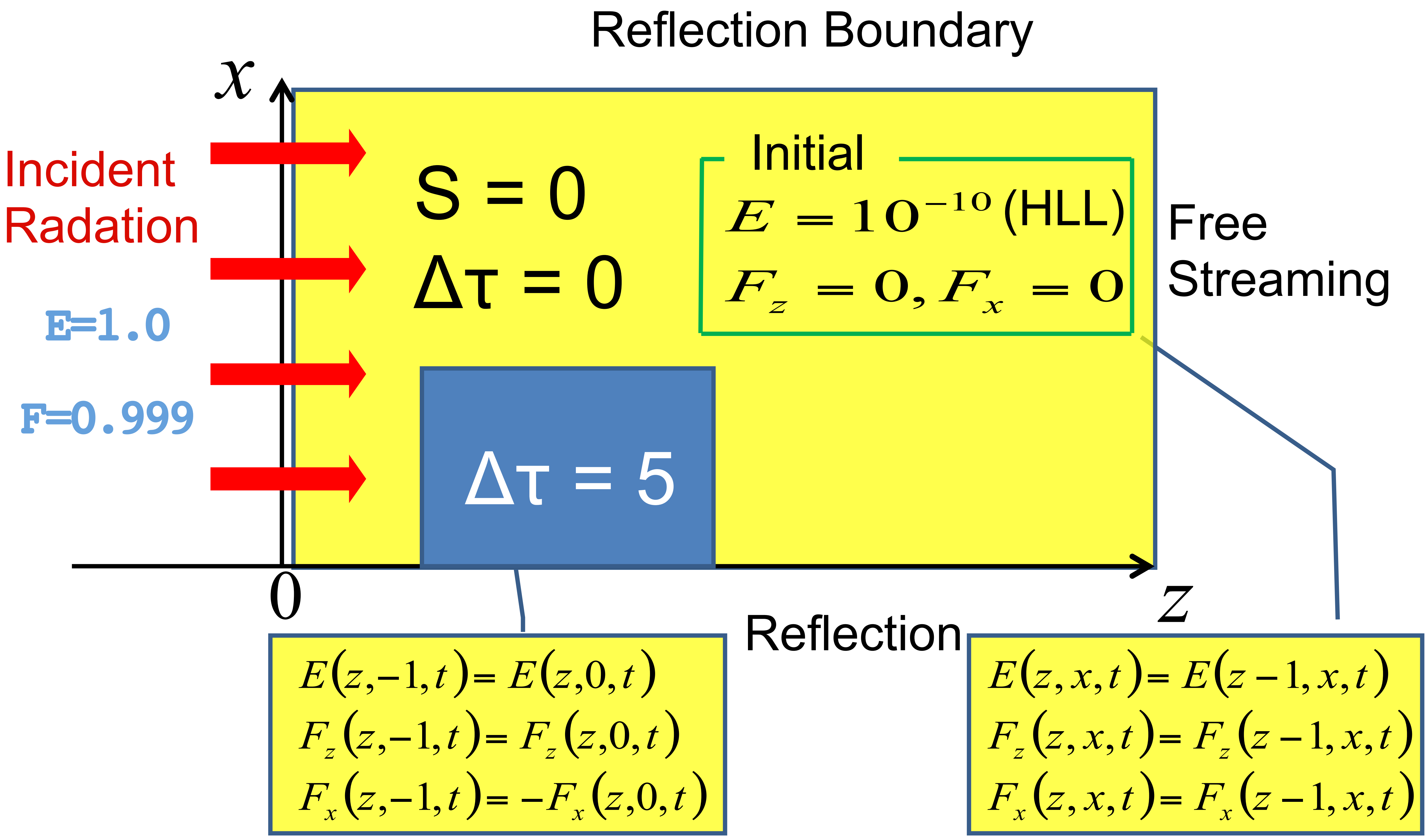


If

$$|\boldsymbol{F}| > f_{\rm lim} E$$

$$\boldsymbol{F} \rightarrow \frac{f_{\rm lim}\, E\, \boldsymbol{F}}{|\boldsymbol{F}|}$$

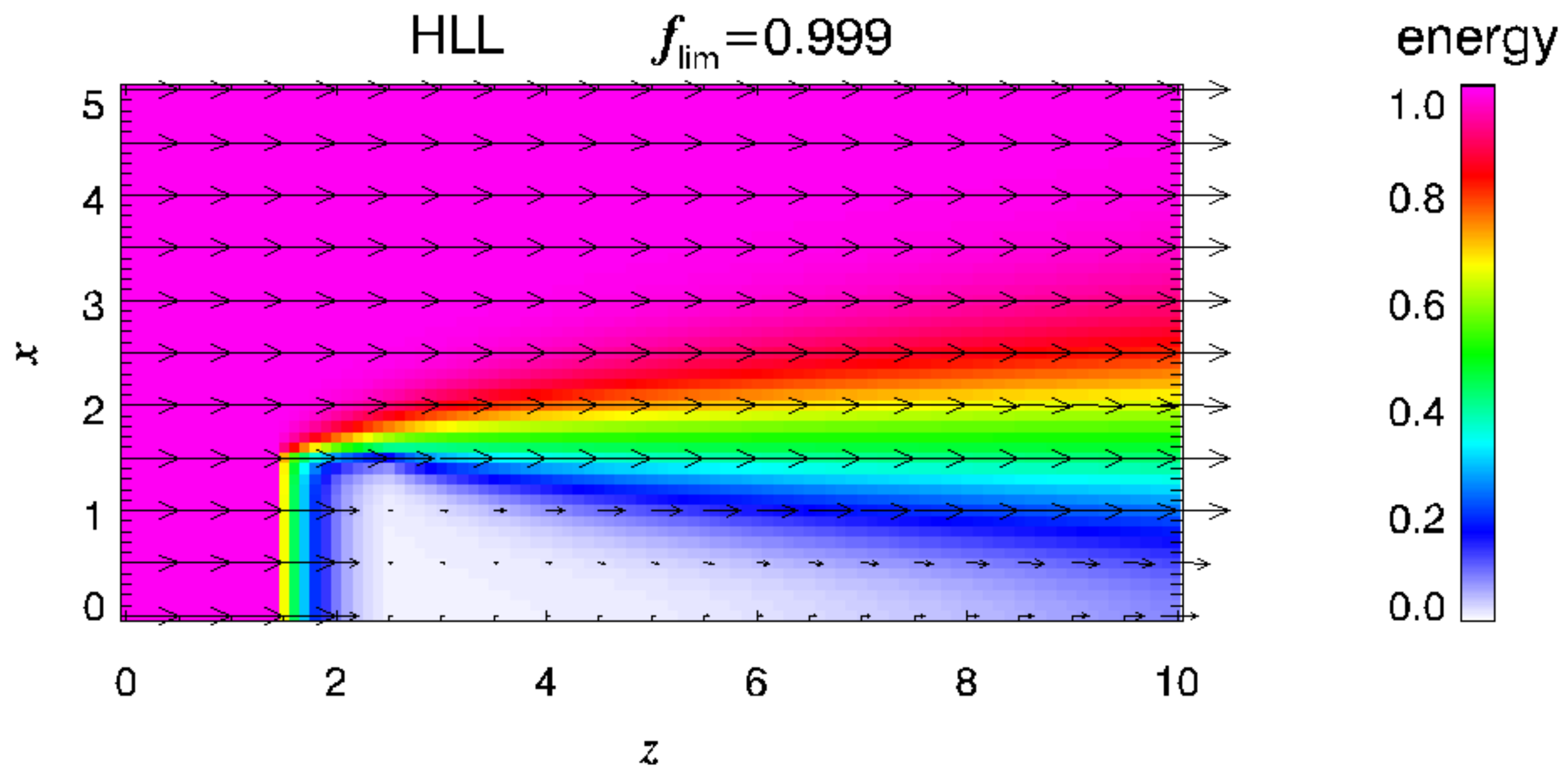
Beam Test



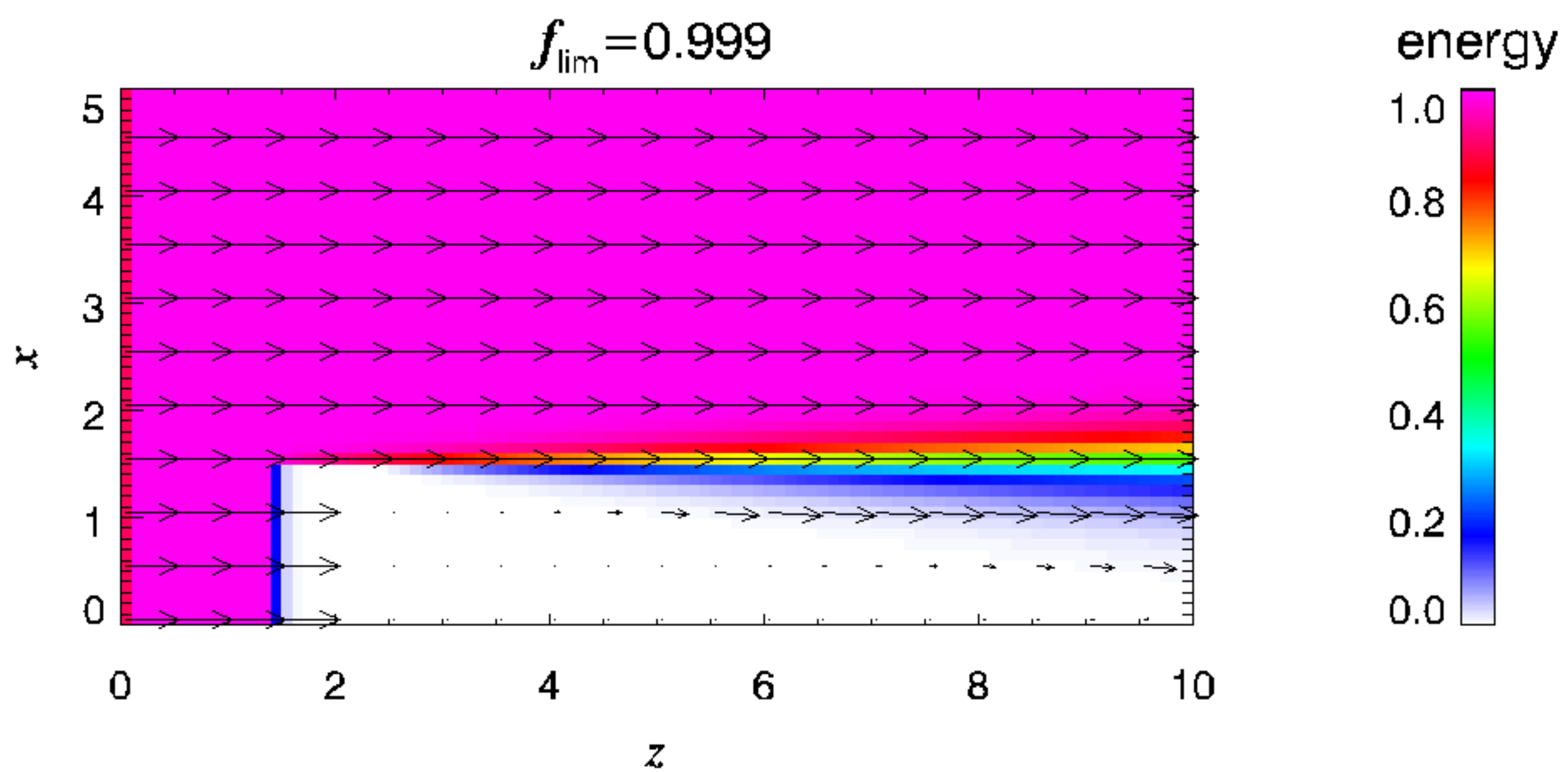
Reflection Boundary
Incident
Radation
E=1.0
F=0.999
S = 0
Δτ = 0
Initial
$E = 10^{-10}$ (HLL)
$F_z = 0, F_x = 0$
Free
Streaming
Δτ = 5
x
z
0
Reflection
$E(z,-1,t) = E(z,0,t)$
$F_z(z,-1,t) = F_z(z,0,t)$
$F_x(z,-1,t) = -F_x(z,0,t)$
$E(z,x,t) = E(z-1,x,t)$
$F_z(z,x,t) = F_z(z-1,x,t)$
$F_x(z,x,t) = F_x(z-1,x,t)$

Shadow

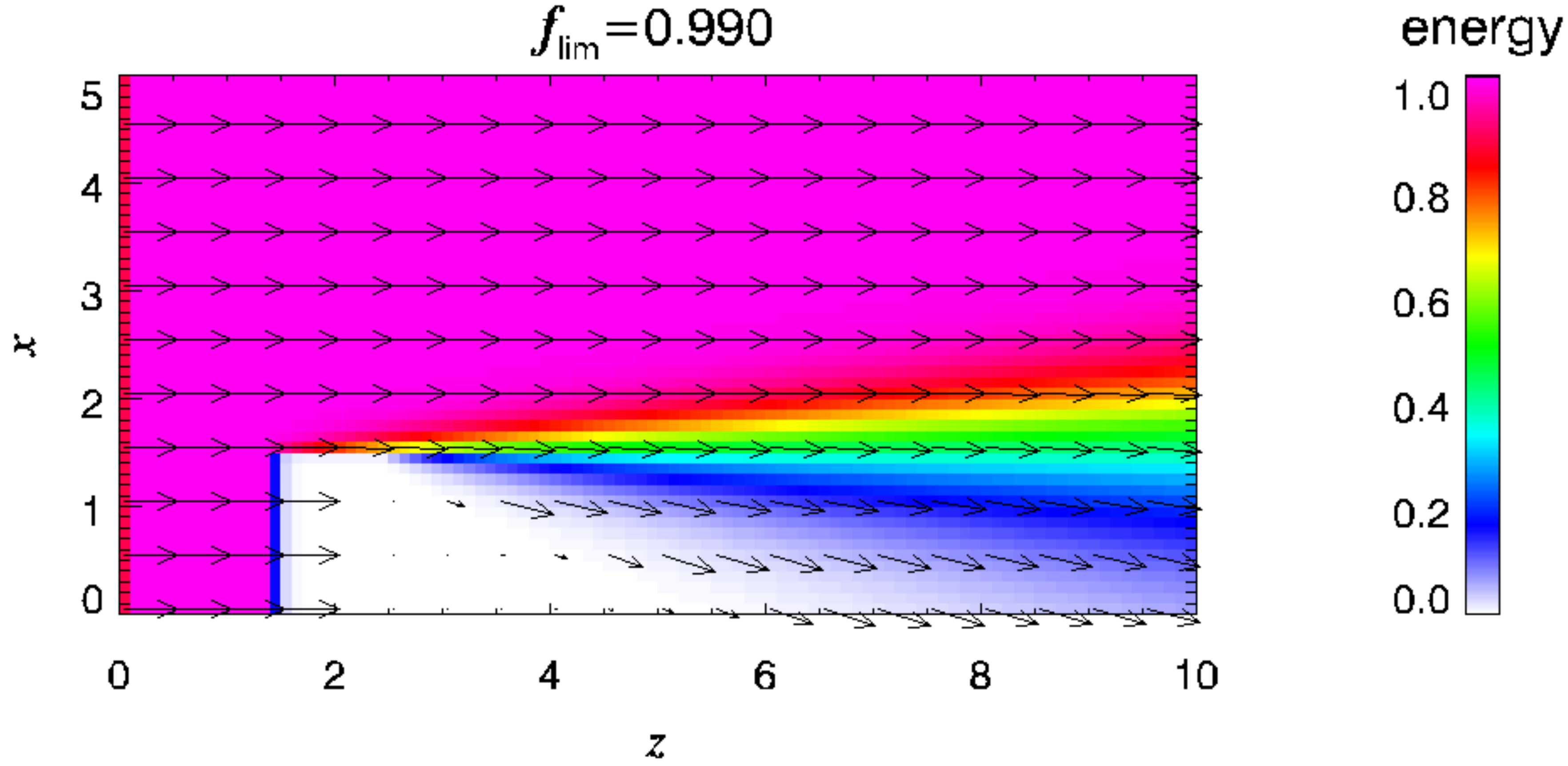
simple HLL

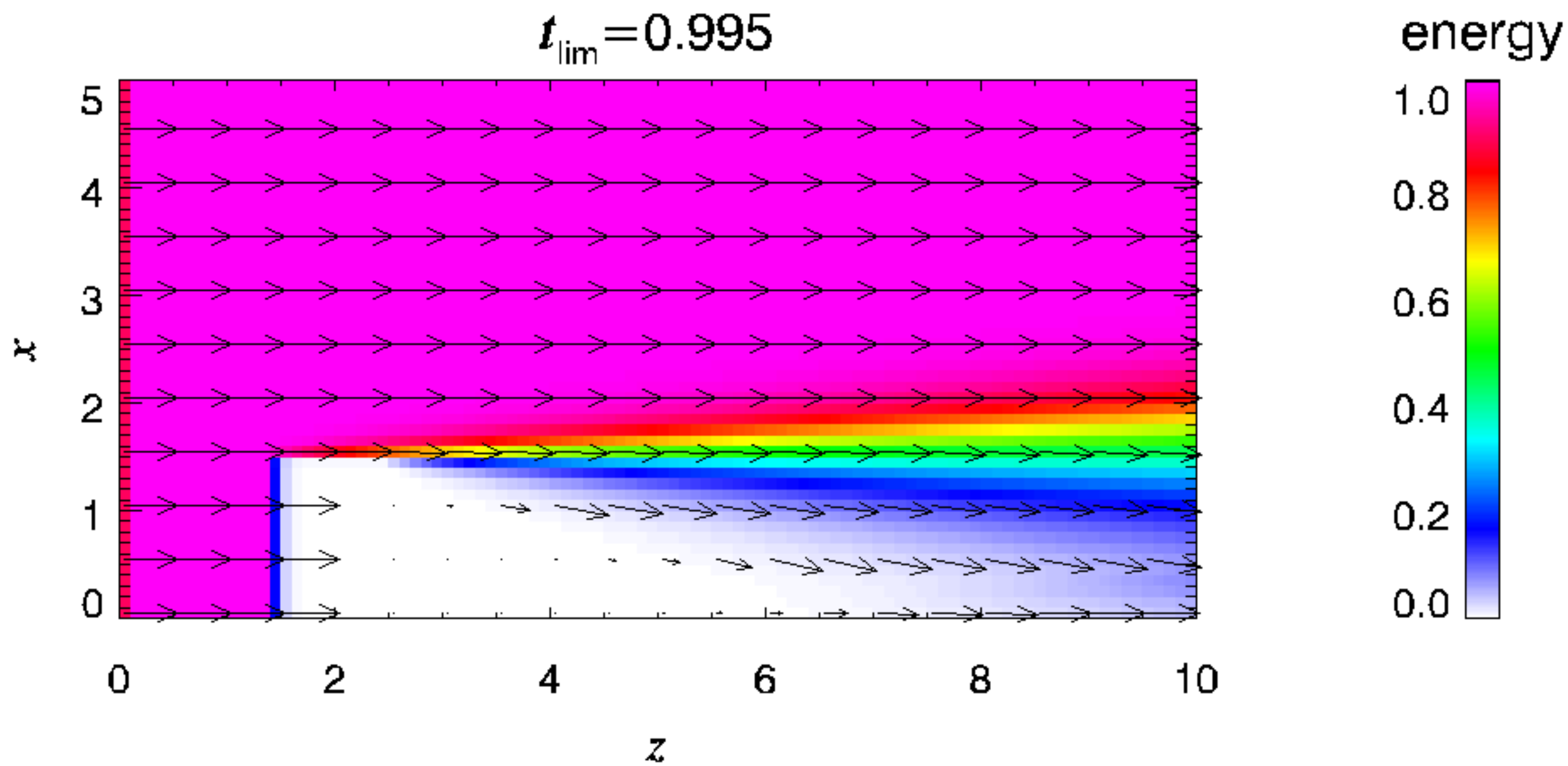


this work



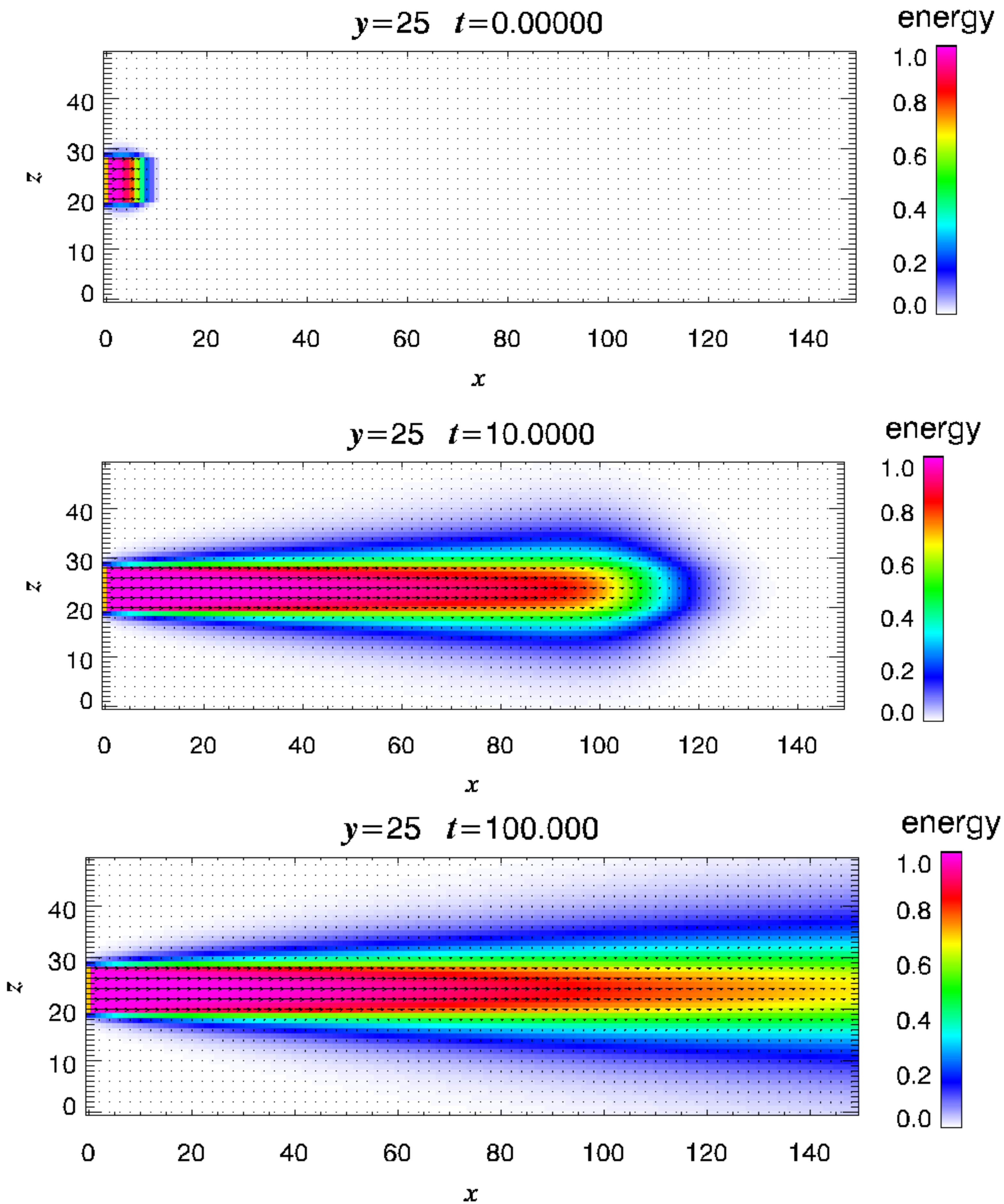
Shadow Test (2)





Beam Test (1) Propagation

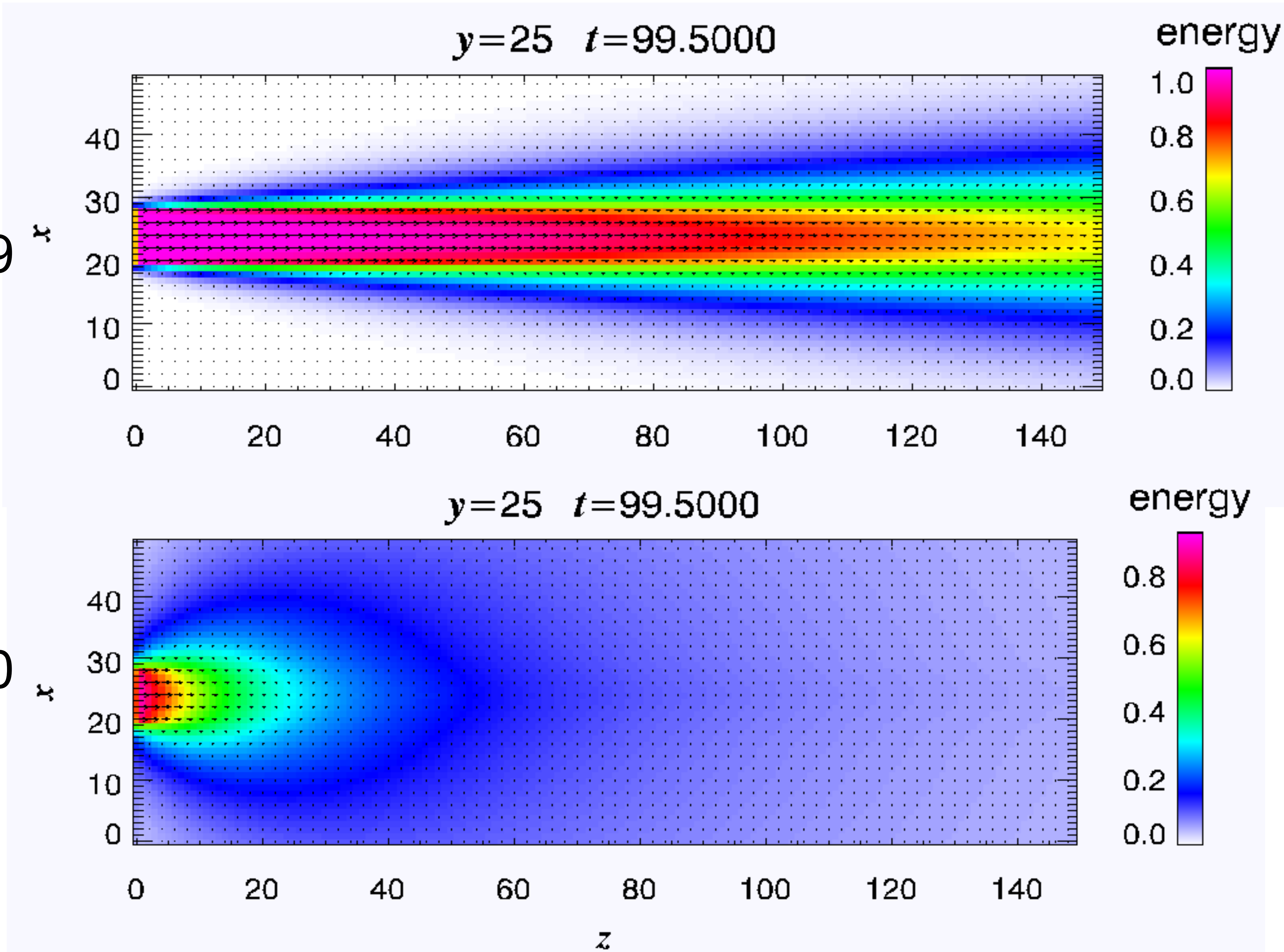
First Order Accurate
$\Delta x = 0.1$



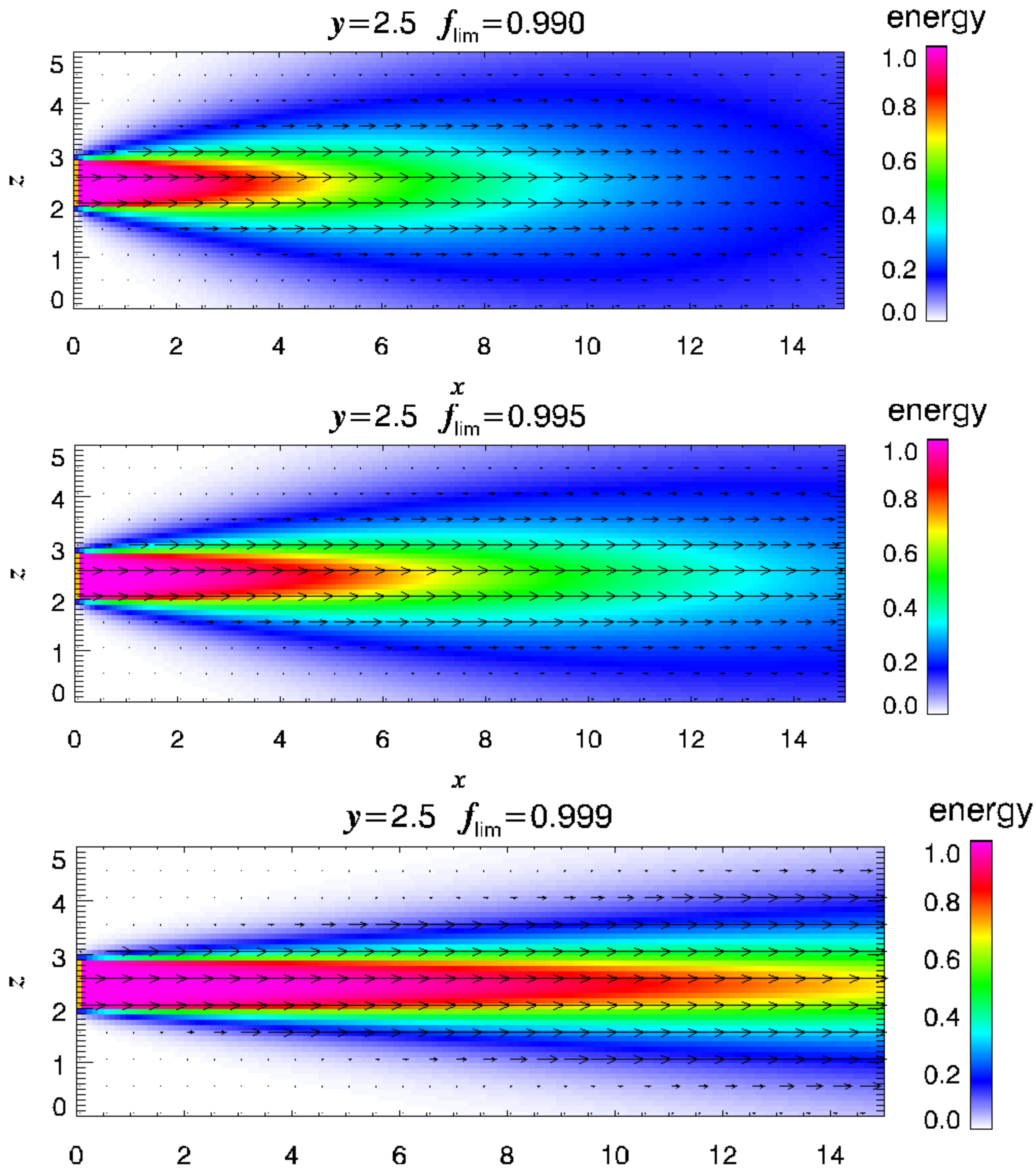
Beam Test(2)

$f = 0.999$

$f = 0.700$

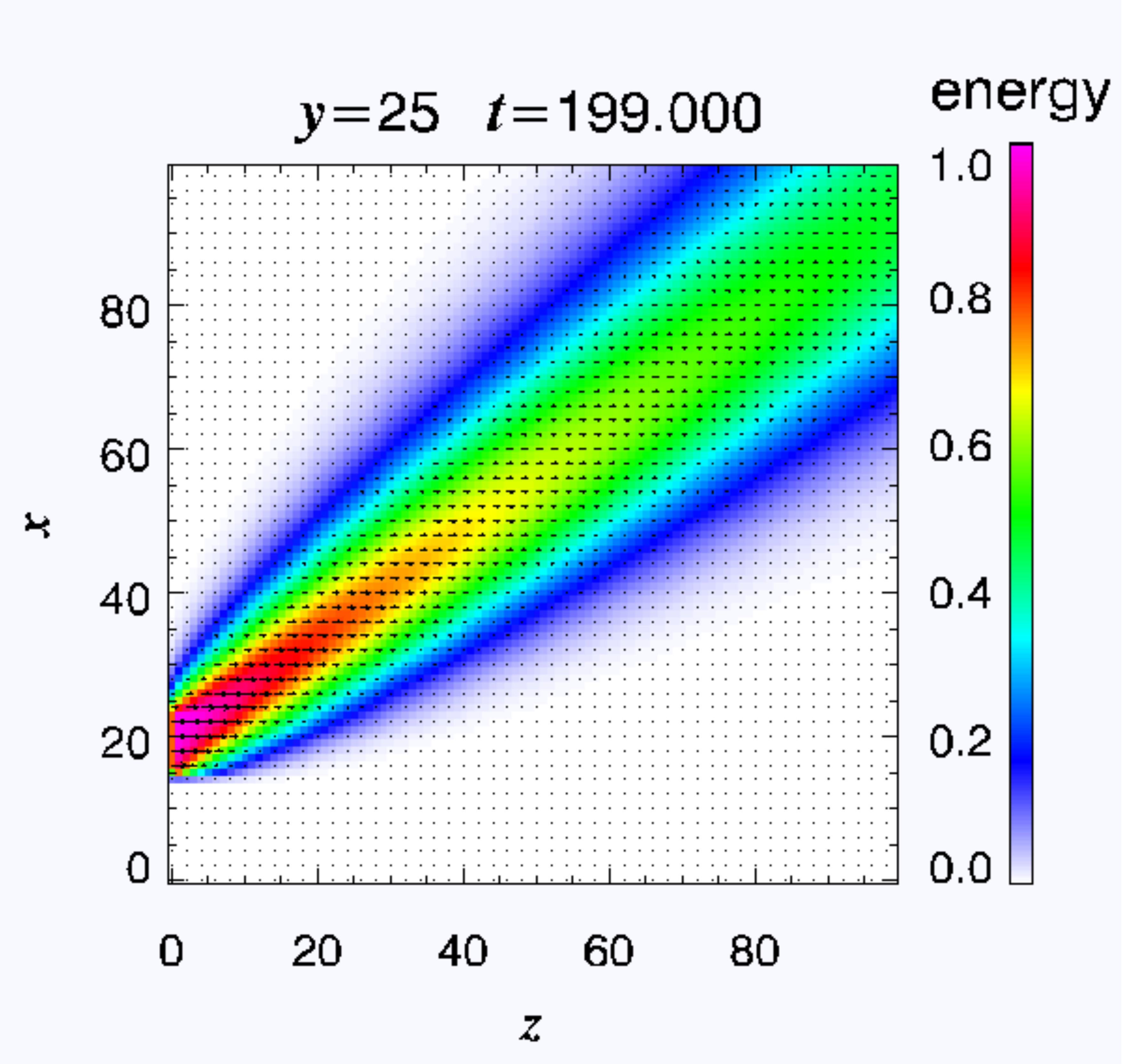


Beam Test (3)

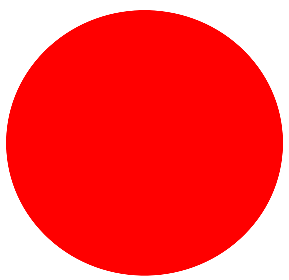


y=2.5 f_lim=0.990
energy
y=2.5 f_lim=0.995
energy
y=2.5 f_lim=0.999
energy
x
z

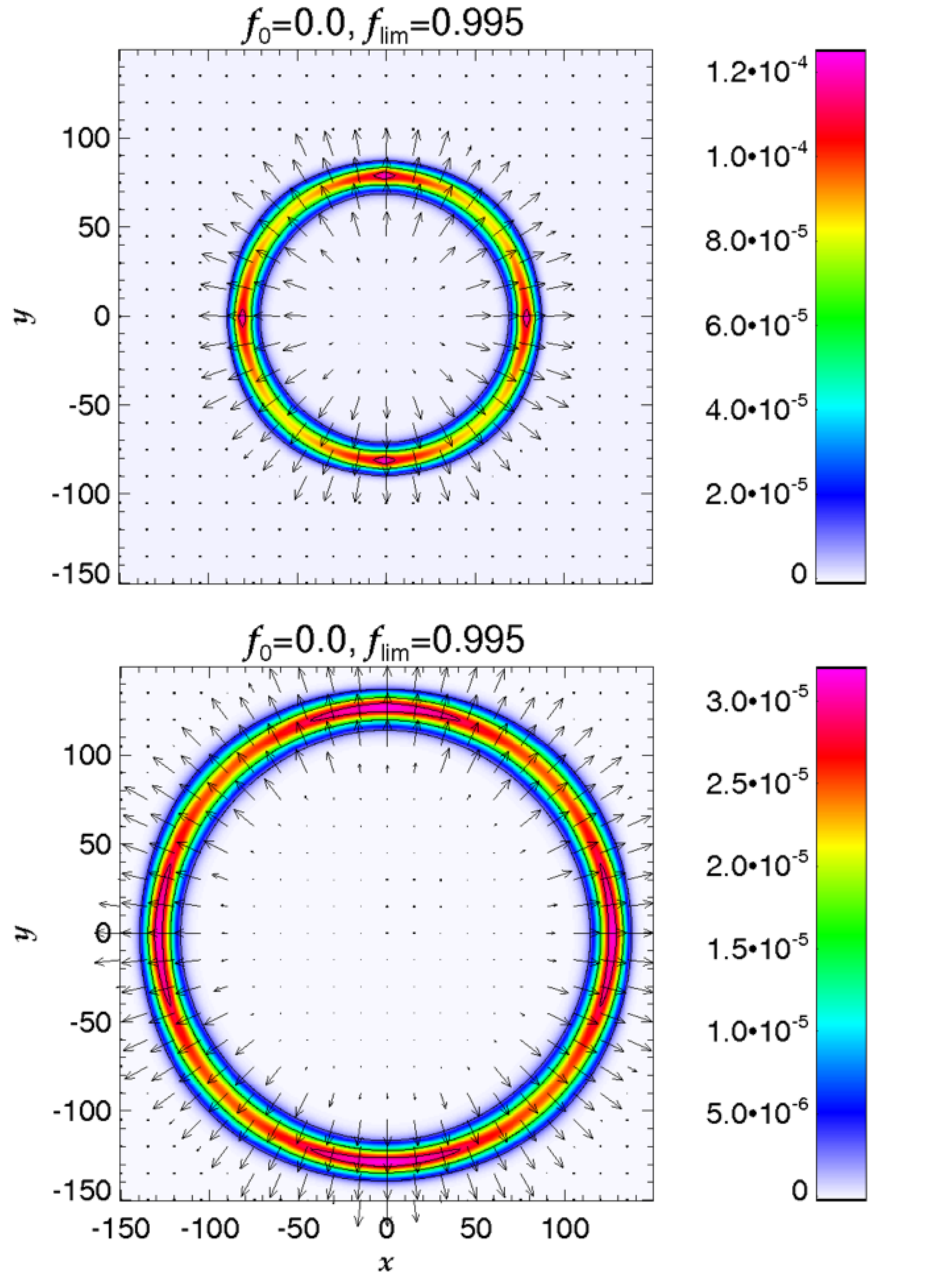
Beam Test (5)



Burst Test (1)

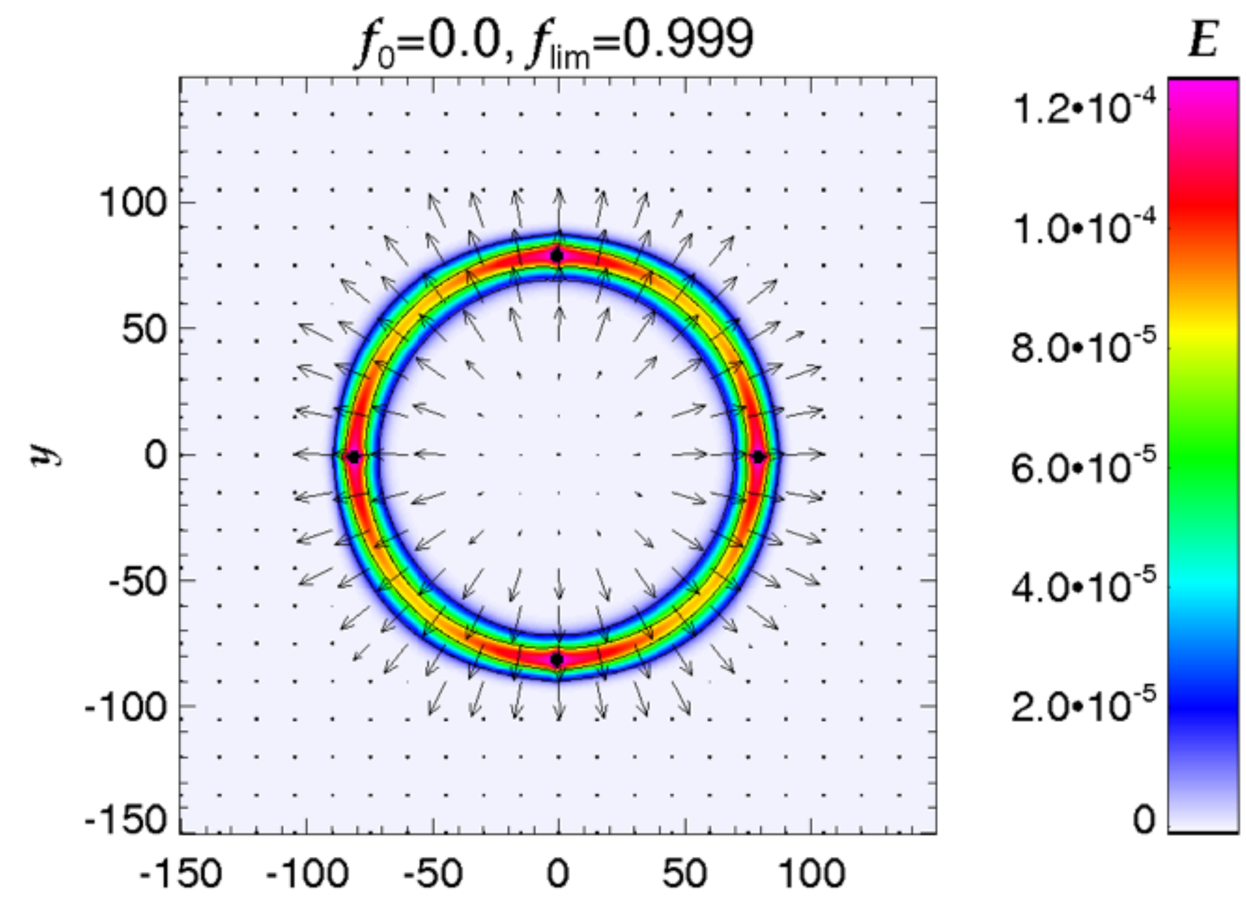


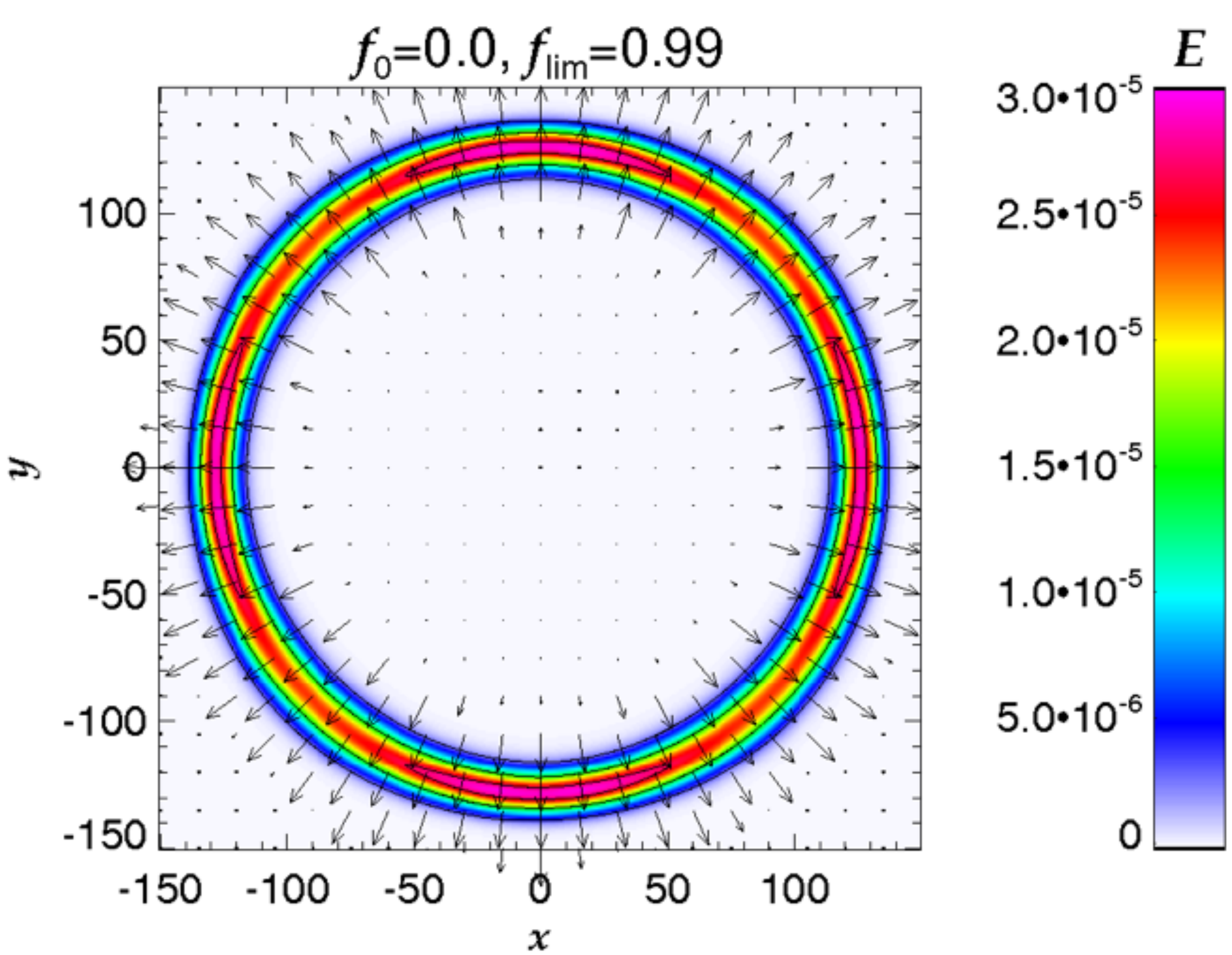
E = const.
F = 0

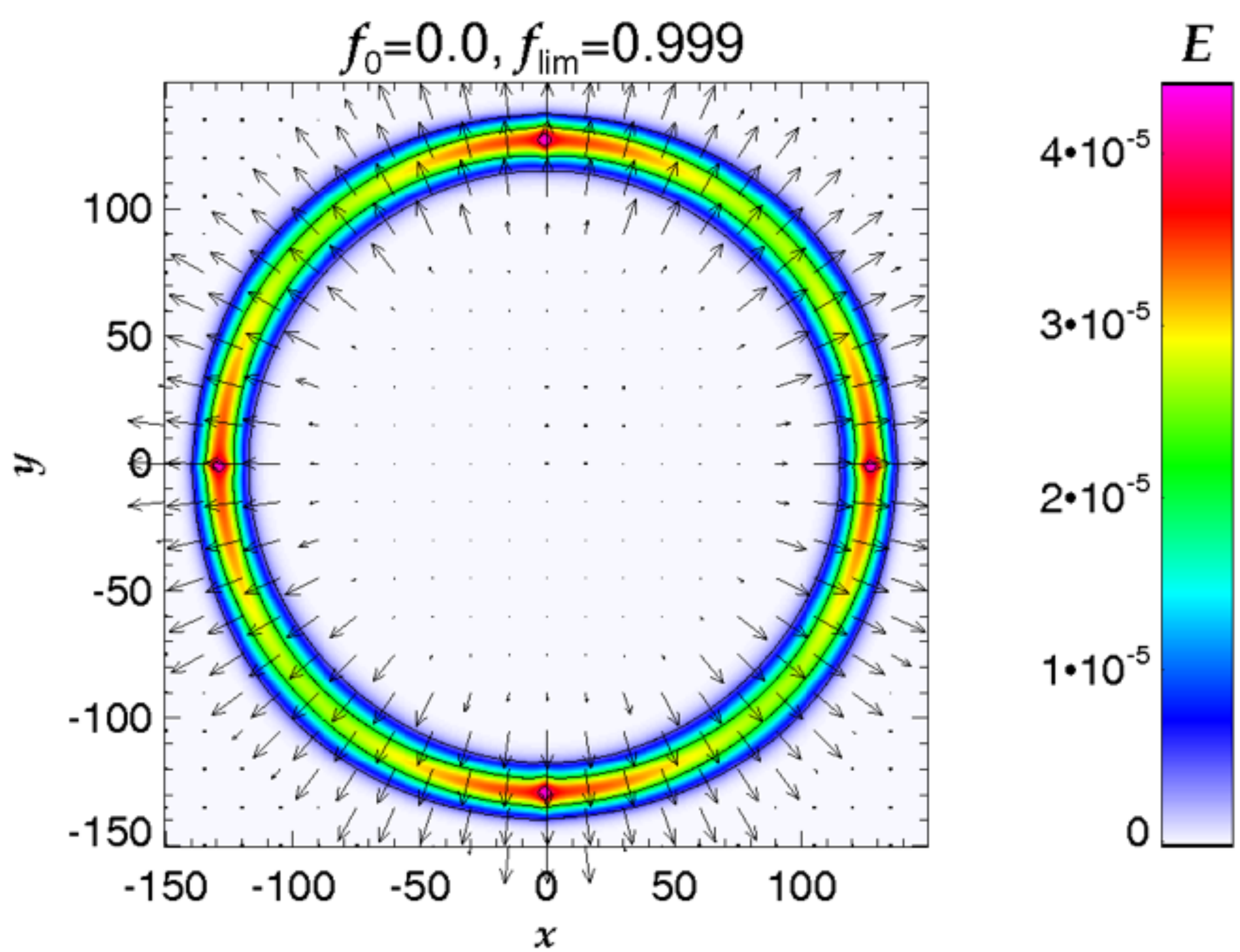


Burst(2)

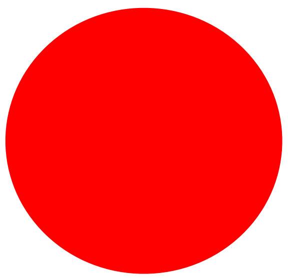
Sphericity and Sharpness
Conflict







Burst (3)

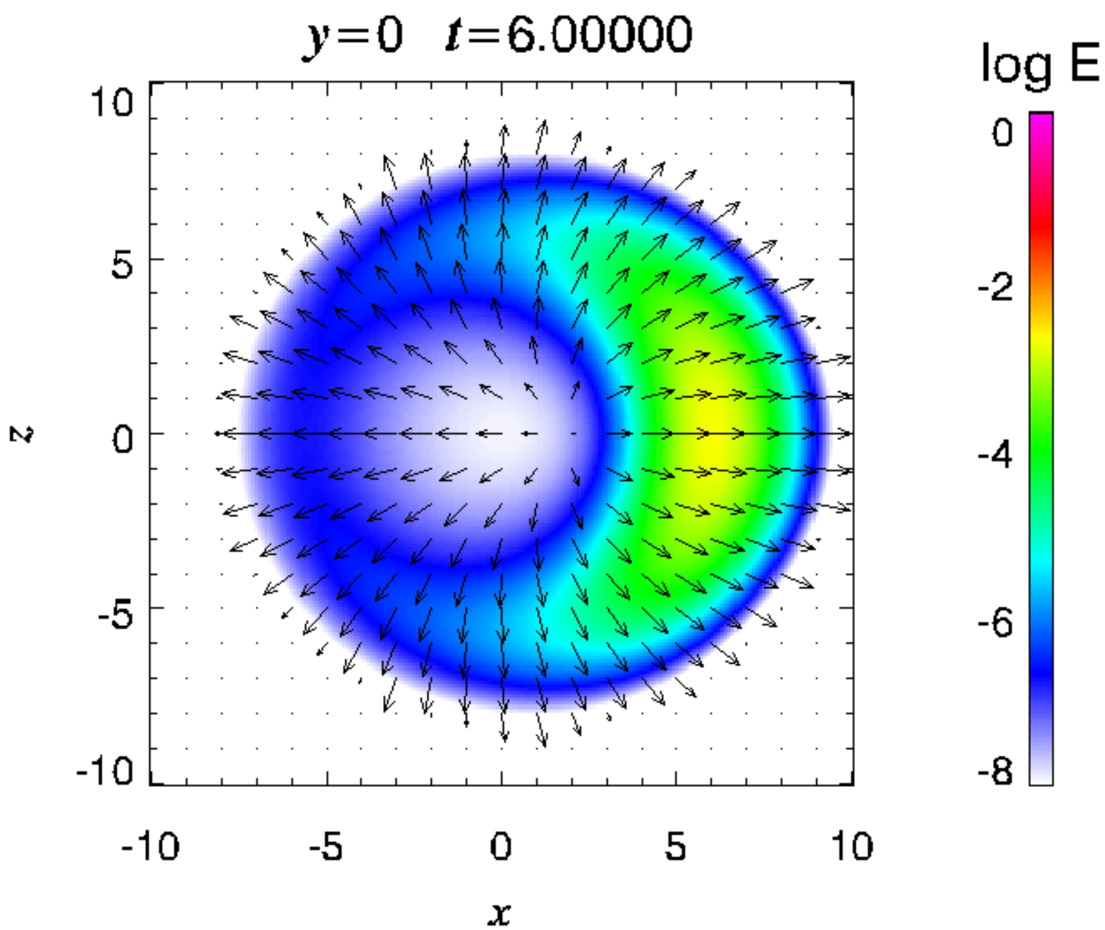


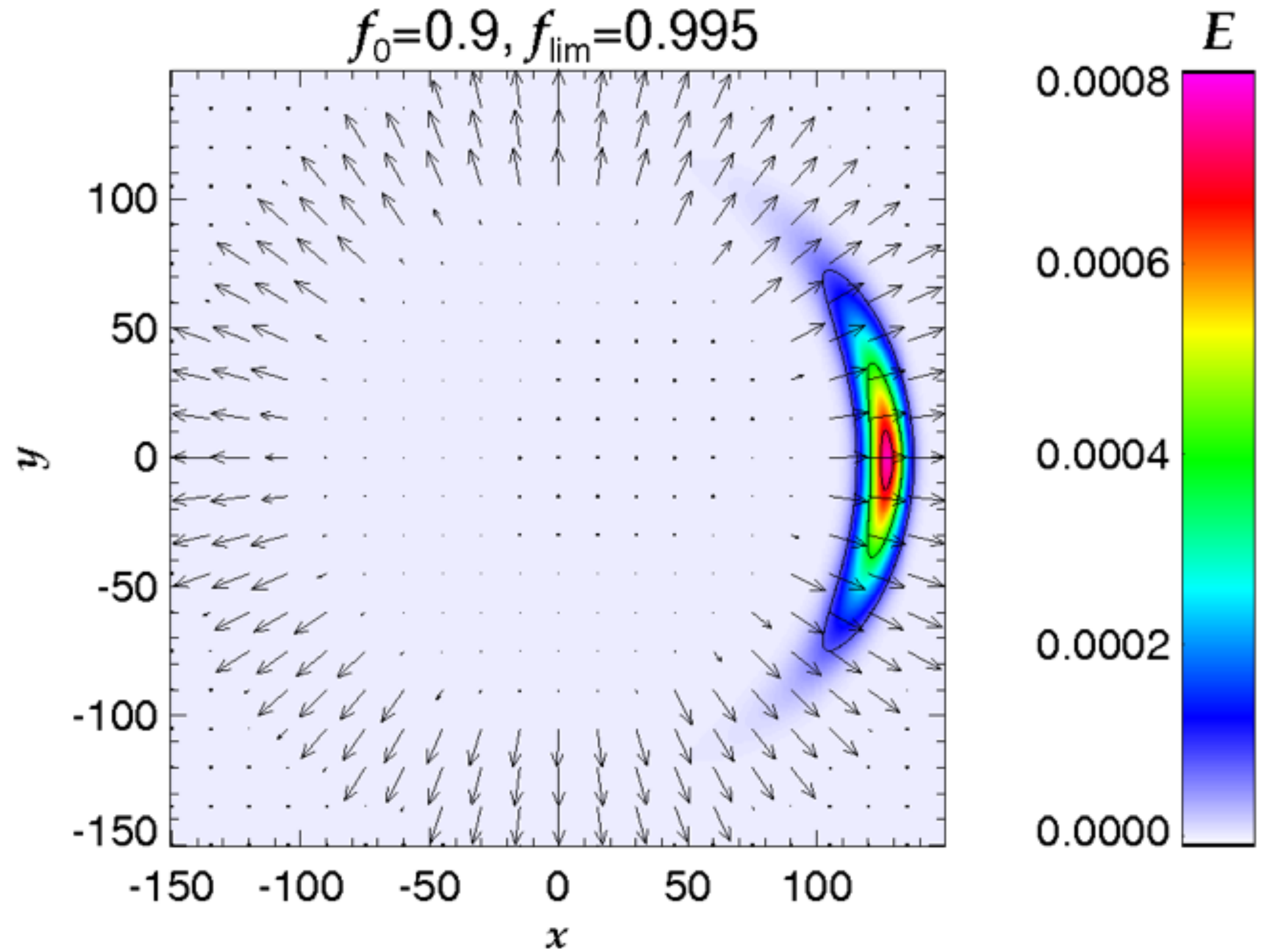
E = const.

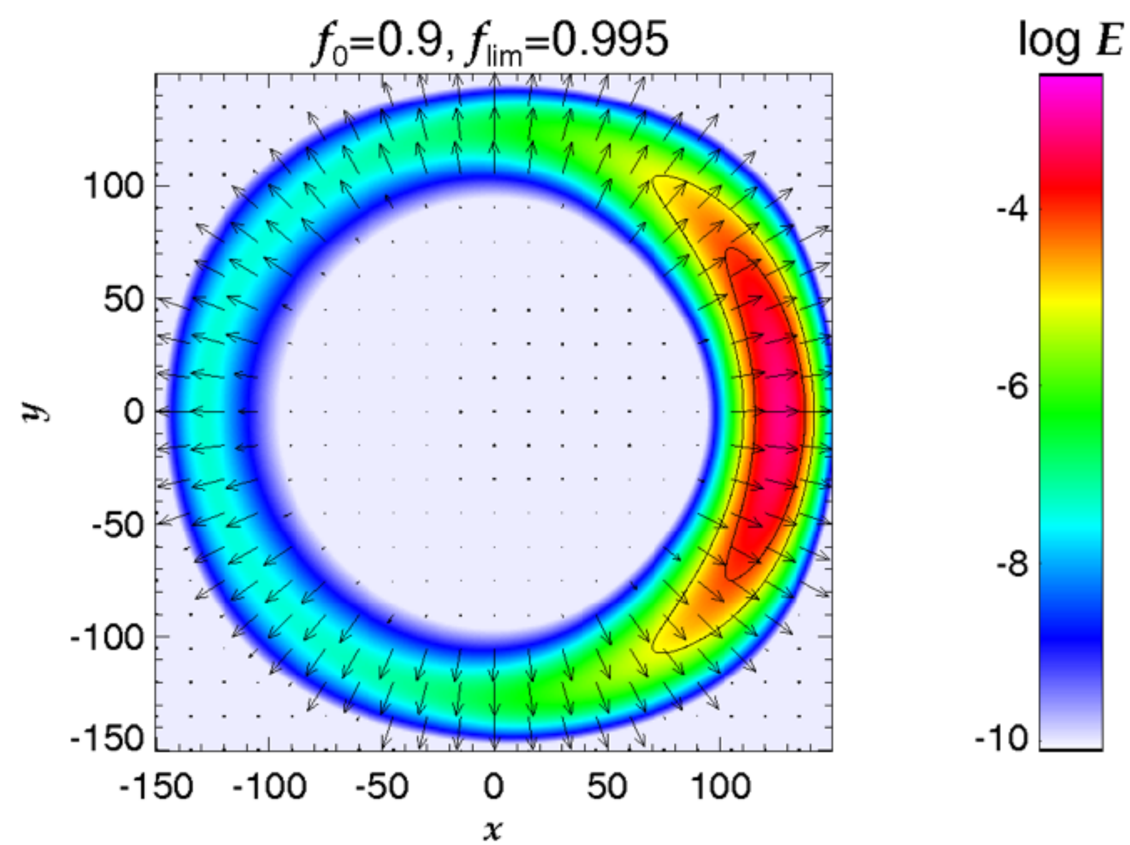
F = (0.9, 0, 0) E

arrows: $\boldsymbol{f} = \boldsymbol{F}/E$

HLL: diffusive, bias



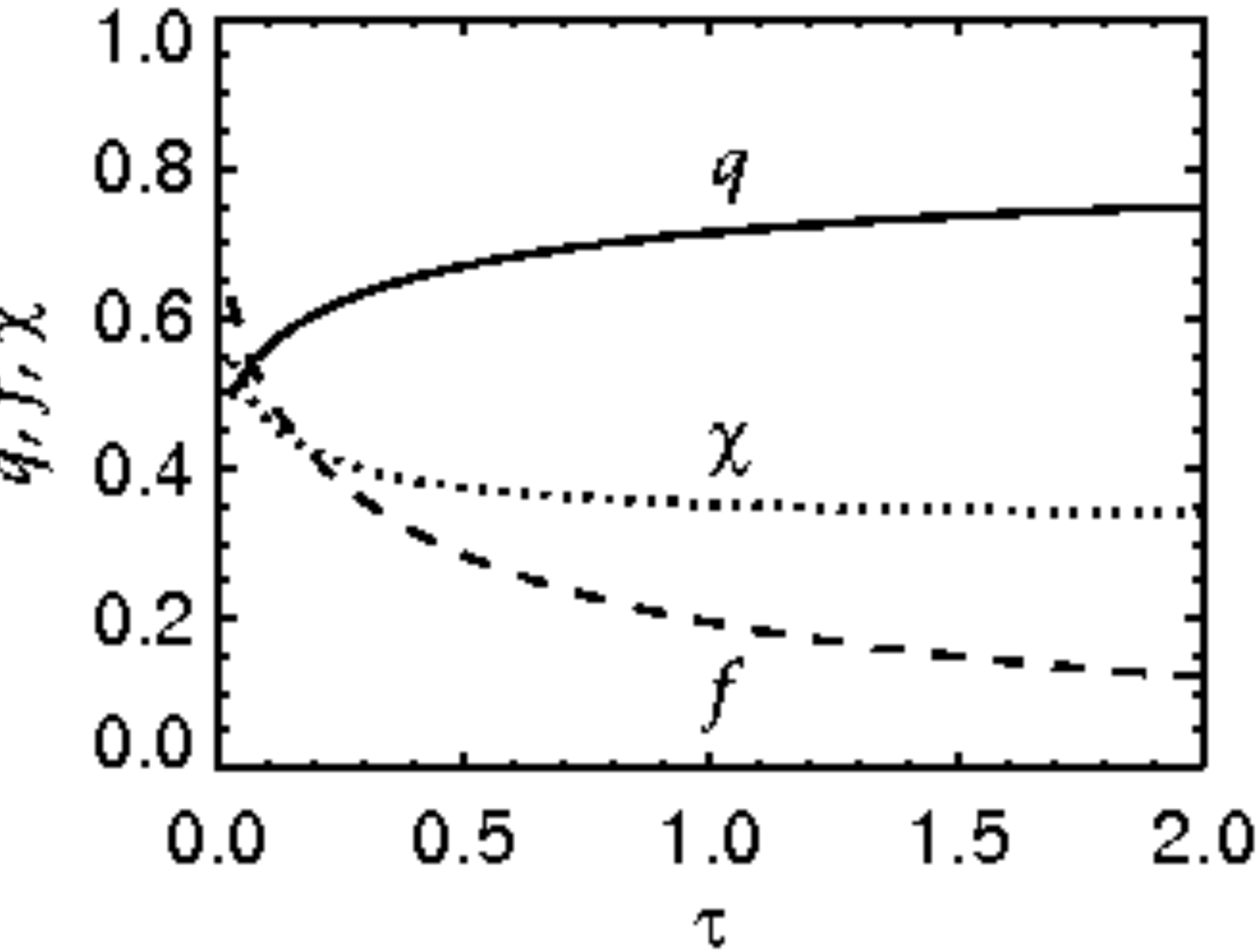




1D Plane Parallel
Closure relation

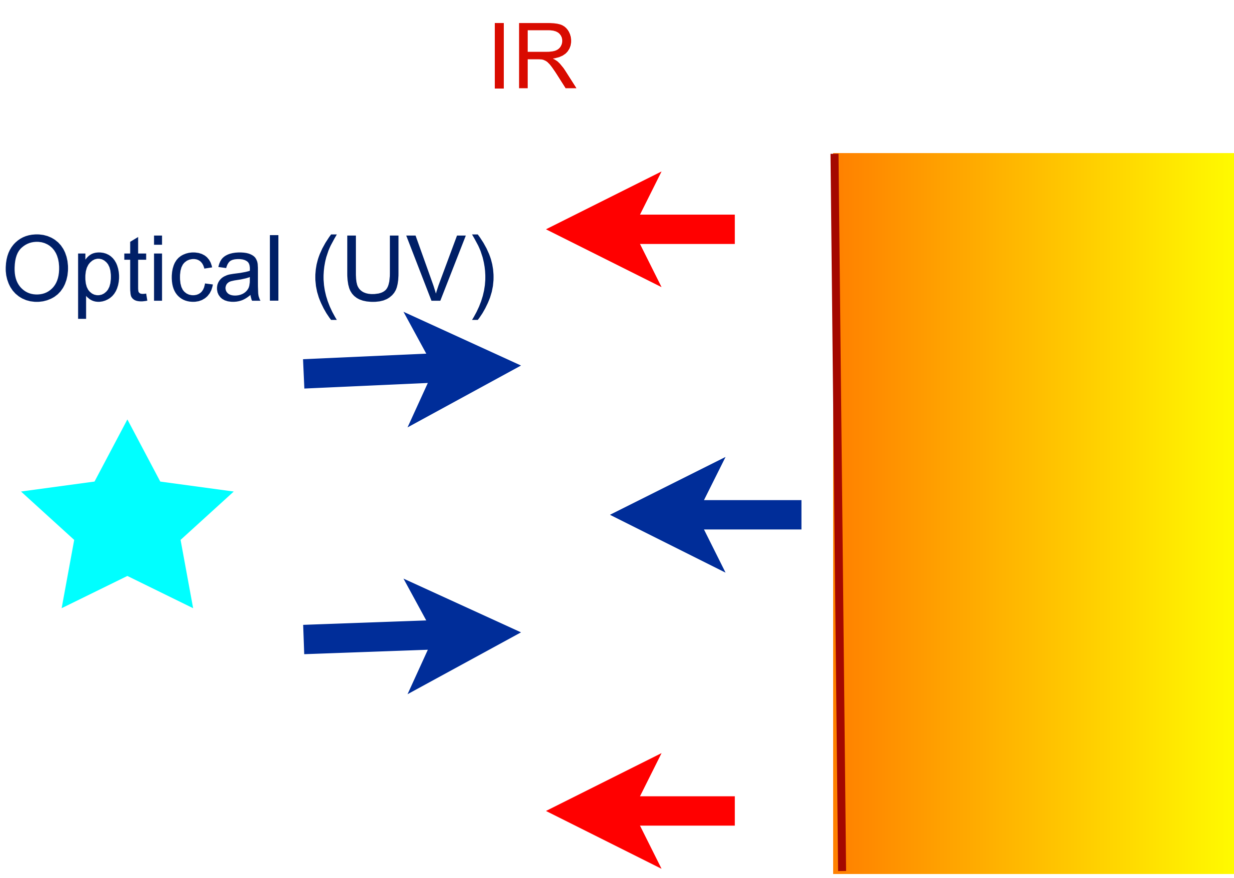
$$\frac{\partial H}{\partial \tau} = J - S = 0$$
$$\frac{\partial K}{\partial \tau} = H$$

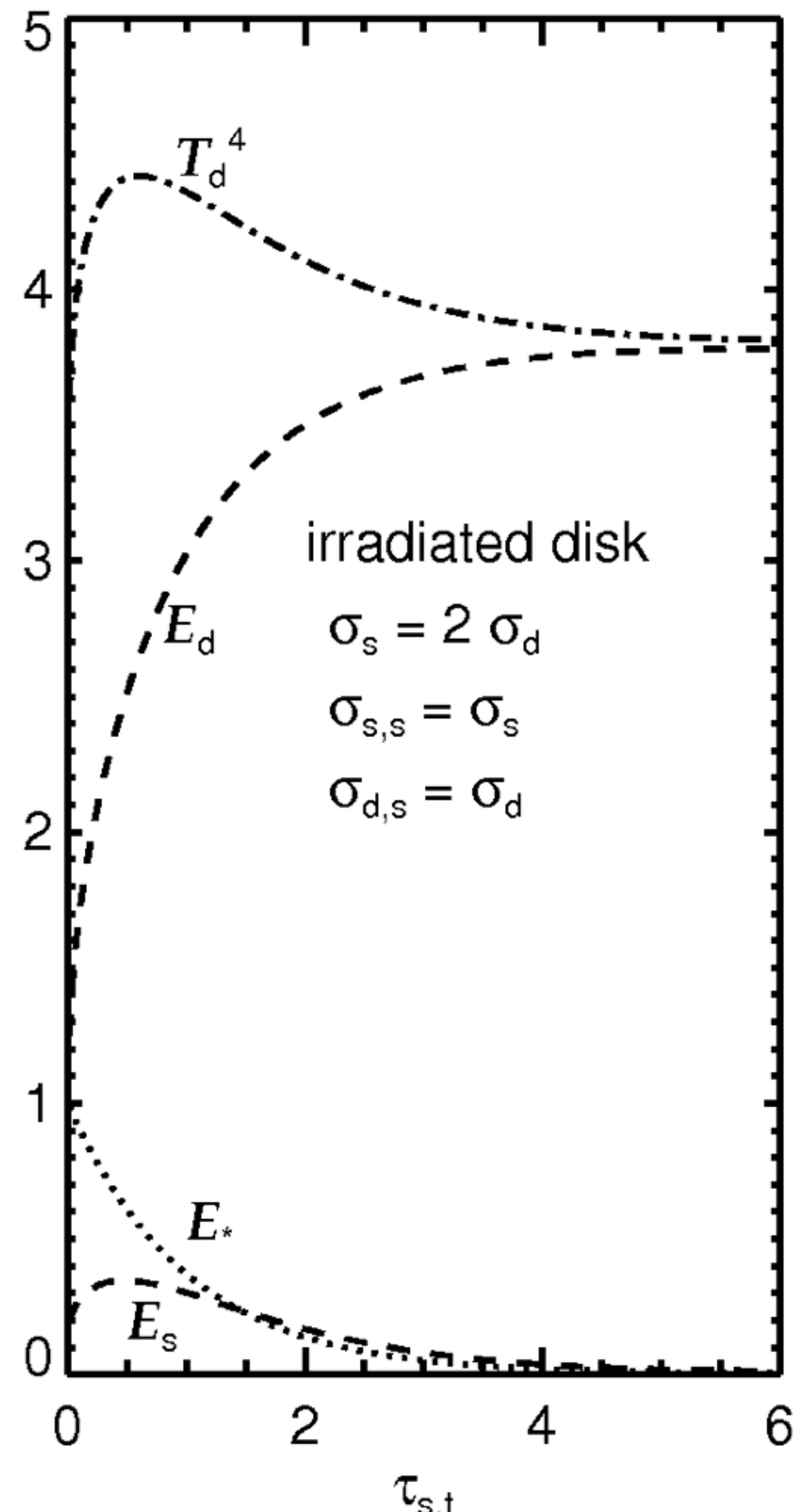
$$J = 3H\left[\tau + q\left(\tau\right)\right]$$
$$q \simeq 2/3$$



Boundary:
No incident from outside

IR from Irradiated Protoplanetary Disk



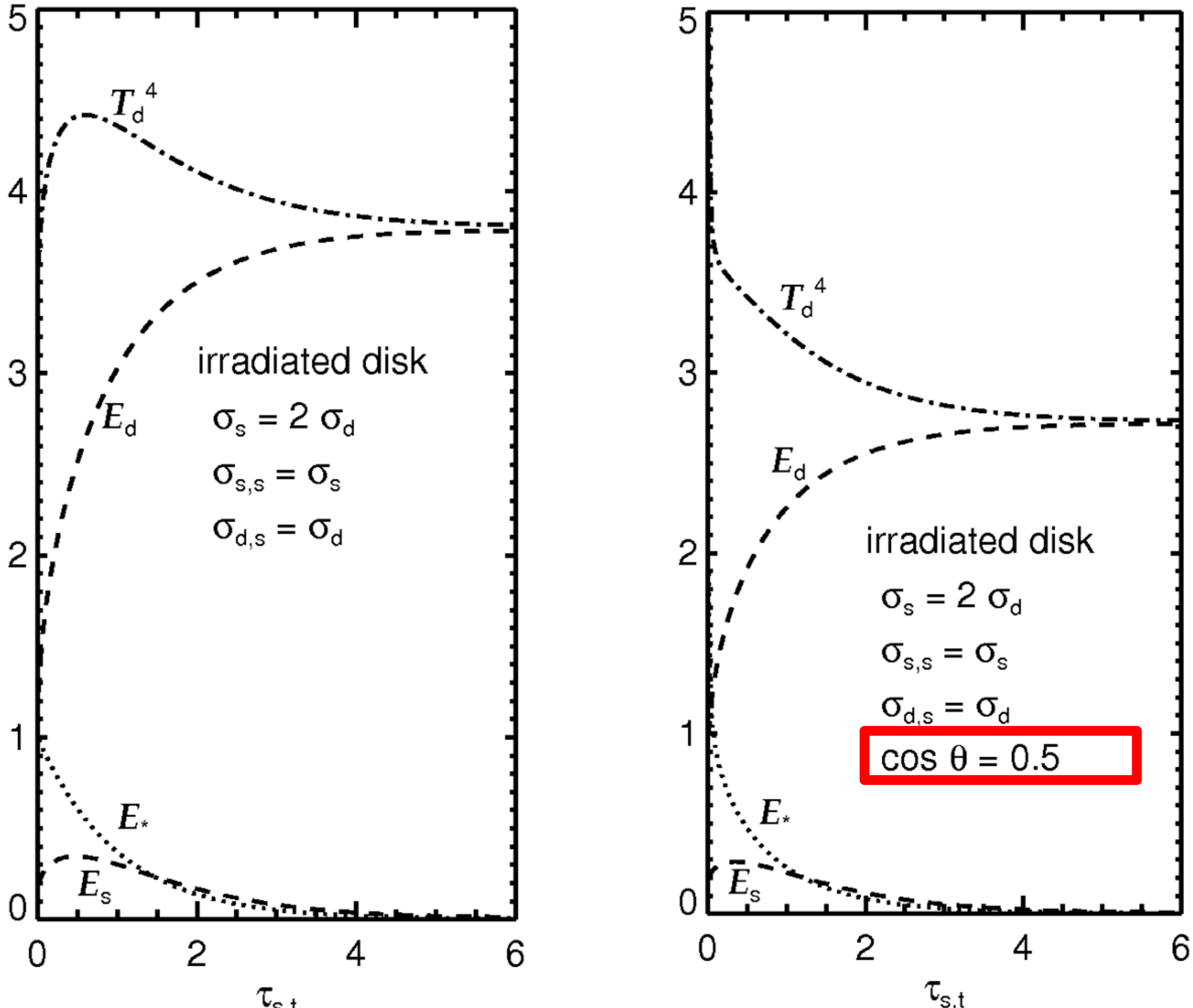


cf. Calvet et al. '91

but

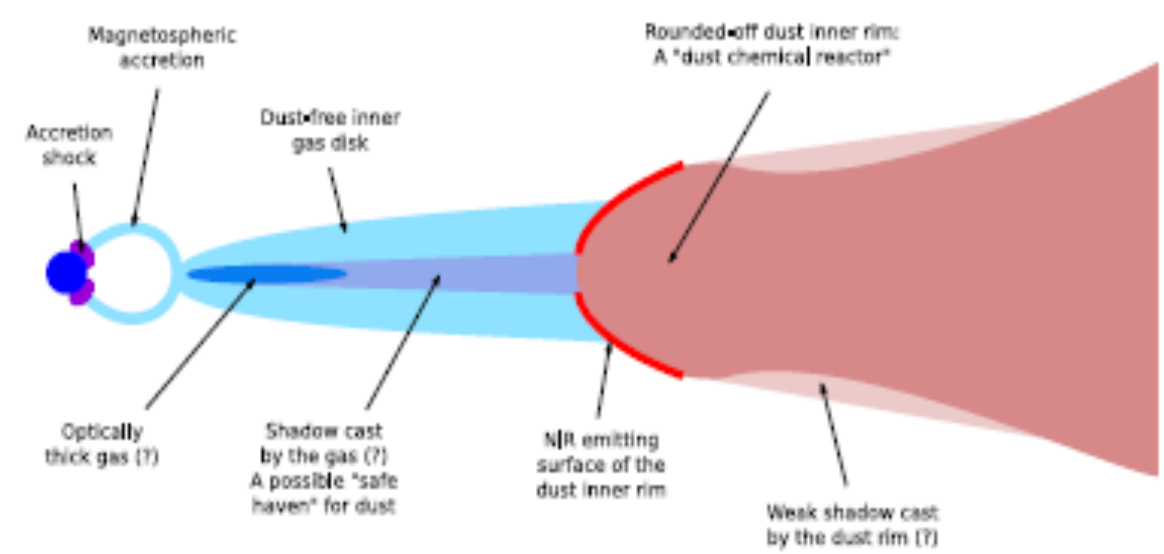
$$\chi = \frac{3 + 4f^2}{5 + 2\sqrt{4 - 3f^2}}$$

Temperature Distribution Depends on the Incident Angle



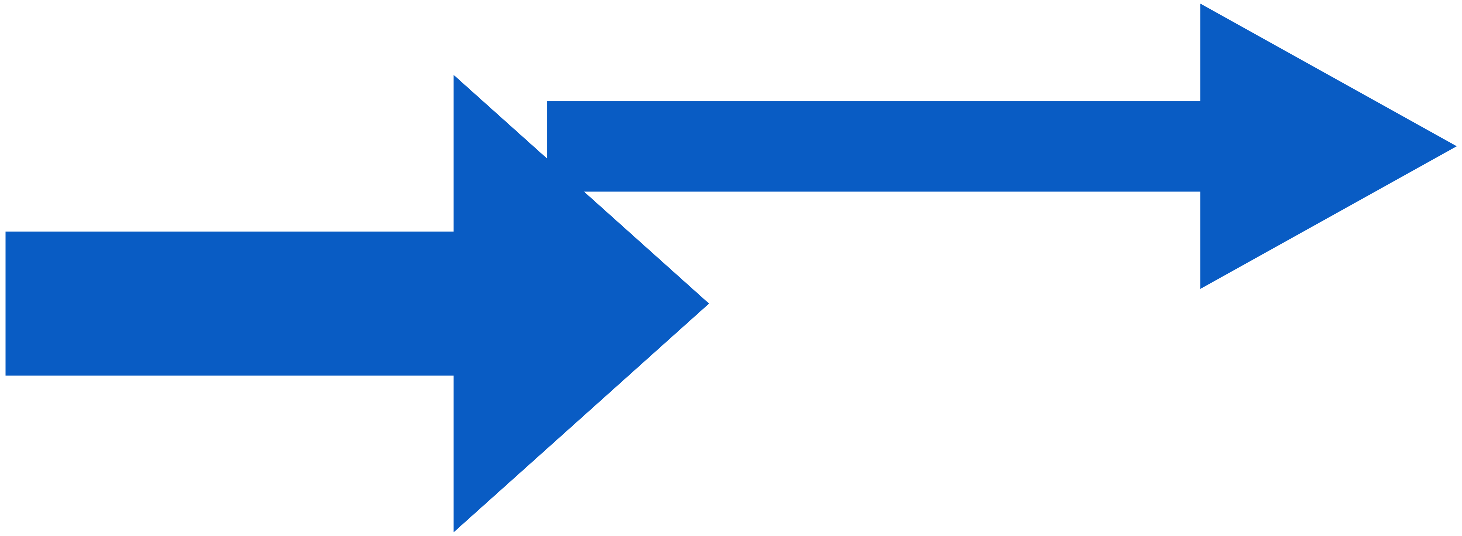
Summary

- M1 model can handle shadow and scattering, both of which have important effects in celestial bodies.
- Reconstruction gives us simple but good numerical fluxes.
  - If $\Delta t < \Delta x / cm$, $|F| < E$
  - Burst Test
  - Light Propagation

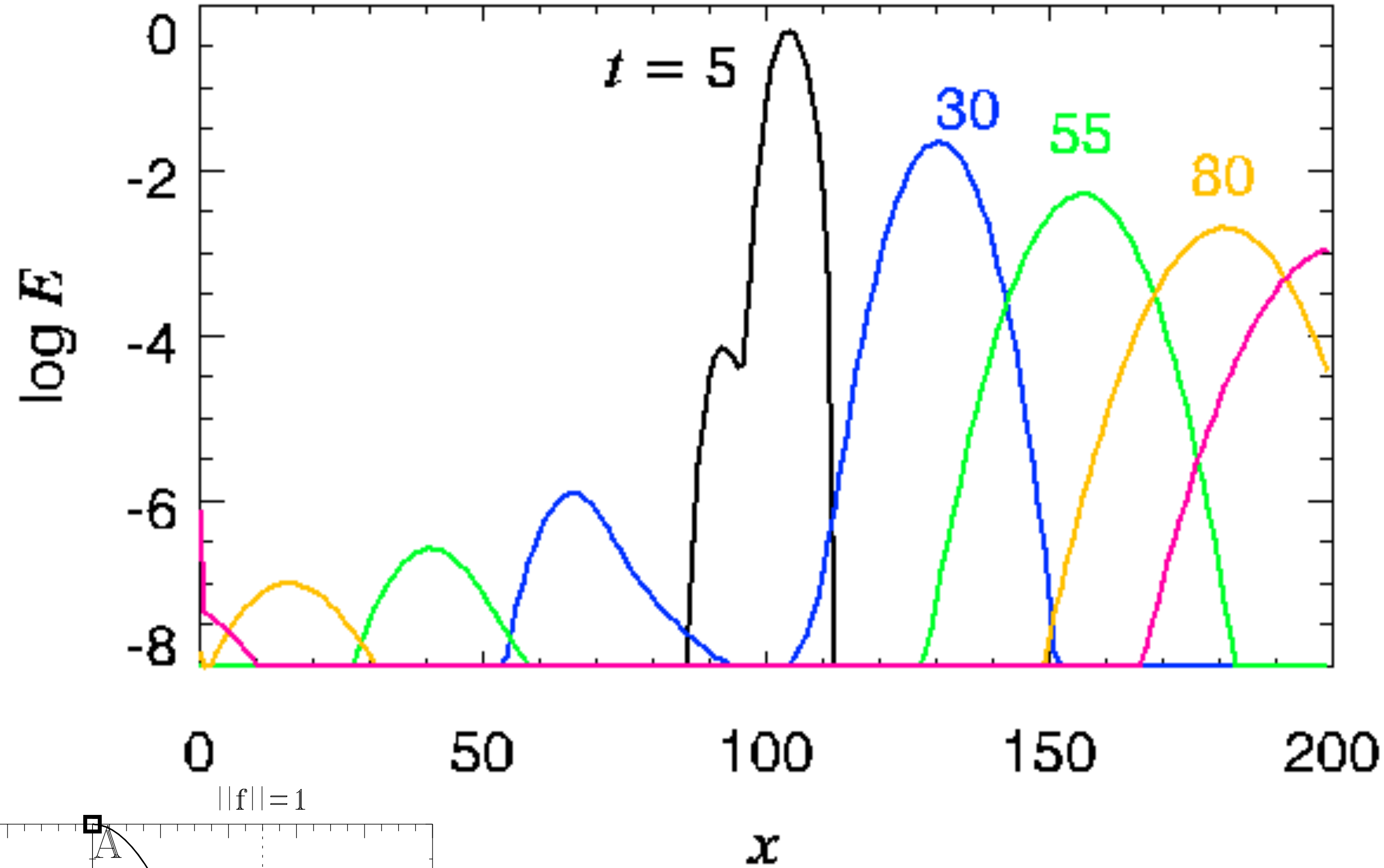


Dullemond & Monnier '10

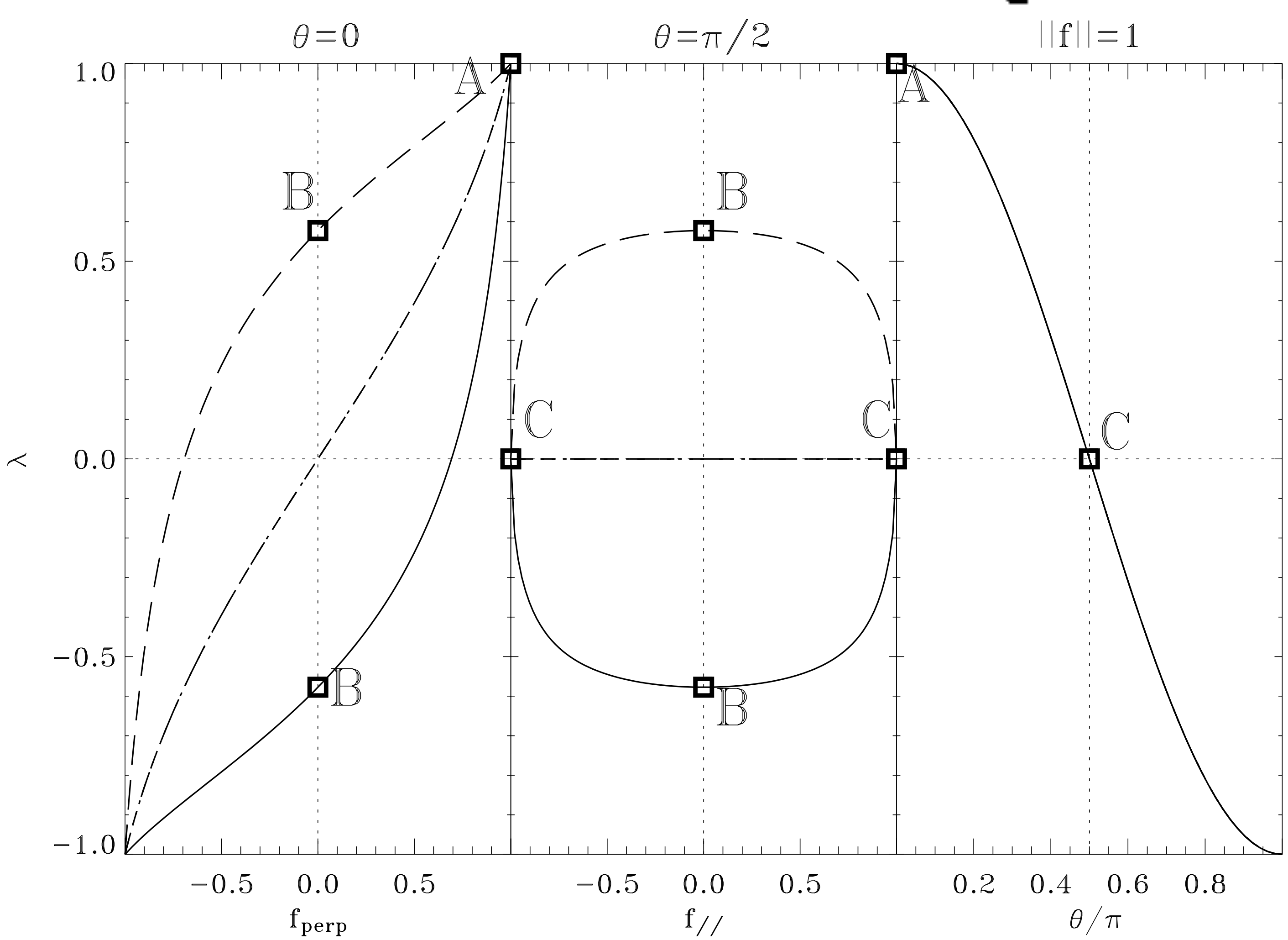
Numerical Instability in 2nd Order Accuarate Flux



Characteristics



Gonzalez+ '07



Characteristics: Average velocity