# *Lattice QCD in and out of the epsilon regime*

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for JLQCD Collaboration

JLQCD collaboration, arXiv:1111.0417

S. Aoki (Tsukuba) & HF, PRD 84, 014501 (2011) [arXiv:1105.1606]

[P.H. Damgaard (NBI,NBIA) & HF, JHEP0901:052, 2009]

新学術領域

「素核宇宙融合による計算科学に基づいた重層的物質構造の解明」

公募研究　23105710

「カイラル対称性の破れにともなう物理の定量評価」

# 1. Introduction

Q.

What is the “epsilon regime” ?

A.

Quantum Chromo Dynamics (QCD) in (vicinity of) the $m_{\mathrm{quark}} = 0$ limit in a finite volume.

# 1. Introduction

[Nambu, 1961 ] (2008 Nobel Prize)

Chiral symmetry is important near $m_{\text{quark}} = 0$

- Chiral symmetry breaking and constituent mass

$$\langle \bar{q}q \rangle \neq 0$$

$\rightarrow$ "effective" quark action $\quad \mathcal{L} \rightarrow \bar{q}(D+m)q + C\bar{q}q\bar{q}q + \cdots$

acquires constituent mass $\quad \rightarrow \bar{q}(D+m+2C\langle\bar{q}q\rangle)q + \cdots$

$\sim \Lambda_{QCD} \sim 300\text{MeV}$

$\rightarrow$ hadron masses are ~ O(1) GeV.

- Pion effective theory

(pseudo) Nambu-Goldstone boson = pion

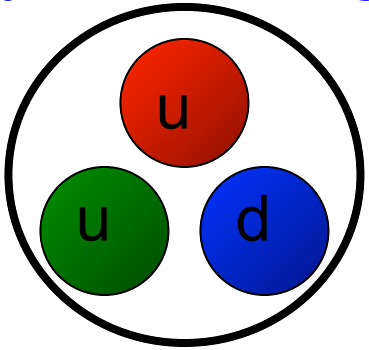
described by Chiral perturbation theory (ChPT)

[Weinberg 1979]

# 1. Introduction

Origin of mass = chiral symmetry breaking.

- Hadrons consist of quarks.




- But

  proton mass >> quark mass ×3

  (1GeV) (3-6MeV)

- Chiral symmetry breaking generate ~90% of mass.

# 1. Introduction

JLQCD (+ TWQCD) collaboration

have been simulating lattice QCD

with exact chiral symmetry

using overlap fermion action.

# 1. Introduction

## JLQCD (+ TWQCD) collaboration

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News@KEK プレス・リリース ～ 07-03 ～ For immediate release：2007年04月24日

**量子色力学における自発的対称性の破れを厳密に実証**

大学共同利用機関法人高エネルギー加速器研究機構
国立大学法人京都大学

大学共同利用機関法人高エネルギー加速器研究機構（KEK）、国立大学法人京都大学などからなる研究チーム（研究責任者　橋本省二（高エネルギー加速器研究機構・准教授））は、物質の質量の起源となる量子色力学における自発的対称性の破れの現象を厳密な計算機シミュレーションにより世界で初めて実証しました。

# 1. Introduction

## Chiral symmetry is expensive.

differenct action = different errors and cost.

| Fermion action | Chiral symmetry | Discretization error | Numerical cost | Lattice size |
|---|---|---|---|---|
| **Overlap** | **Exact** | **O(a²)** | **Very expensive** | **< 2.4 fm** |
| Domain-wall | Weakly broken | O(a²) | Expensive | ~4fm |
| Wilson | Broken | O(a) | Marginal | ~5fm |
| Staggered | Broken (U(1) remains) | O(a²) | Cheap | ~6fm |

JLQCD = Small volume QCD…

# 1. Introduction

Finite volume = Pion physics.

Correlation length (1/M) of QCD particles

Pions(~140MeV) ~ 1.4fm

Kaons(~500MeV)~ 0.4fm

Rho (~800MeV)~0.26fm

Proton (~1GeV) ~0.2fm

At $E \sim p \lesssim 200 \text{ MeV}$, QCD = pion (+ kaon) theory.

Finite volume correction in QCD =

chiral perturbation theory (ChPT)

weakly coupled = analytically calculable.

# 1. Introduction

## Our strategy

Numerical calculation

Small lattice QCD
with exact chiral symmetry

Analytic calculation

Finite volume correction by pion theory

QCD at V=∞

Chiral symmetry is
important but expensive.

# Contents

✓ 1. Introduction
2. Analytic calculation of finite V effects (on the pions)
3. Numerical lattice QCD results
4. Summary

# 2. Analytic calculation of finite V effects

## Pion correlator at V= $\infty$

(In Euclidean space-time,)

$$\int d^3x \langle P(x)P(0)\rangle = A\delta^{ab}\int \frac{dp_4}{2\pi}\frac{e^{ip_4 t}}{p_4^2+M_\pi^2}$$

$$\propto \exp(-M_\pi t)$$

# 2. Analytic calculation of finite V effects

Pion correlator at finite V ( in the p-expansion )

(periodic boundary for t-direction)

$$\int d^3x \langle P^a(x) P^b(0) \rangle = B\delta^{ab} \frac{1}{T} \sum_{p_4} \frac{e^{ip_4 t}}{p_4^2 + M_\pi^2}$$

$$p_4 = 2\pi n_t / T \quad (n_t : \text{integer})$$

$$= B\delta^{ab} \int dp \sum_n \delta(p - 2\pi n/T) \frac{1}{T} \frac{e^{ipt}}{p^2 + M_\pi^2} \qquad \left( \sum_k \delta(p - 2\pi k/T) = \sum_n \frac{T e^{ipnT}}{2\pi} \right)$$

$$\propto \frac{\cosh(M_\pi(t - T/2))}{\sinh(M_\pi T/2)}$$

# 2. Analytic calculation of finite V effects

BUT… In the limit $M_\pi \to 0$ ,

$$V = \infty : \quad \exp(-M_\pi t) \to 1$$

$$V \neq \infty : \quad \frac{\cosh(M_\pi(t - T/2))}{\sinh(M_\pi T/2)} \to \frac{2}{M_\pi T} \to \infty$$

Infra-red divergence due to finite V ???

despite we have IR cut-off $1/V^{1/4}$?

Something wrong ! Exp->Cosh is not enough !

# 2. Analytic calculation of finite V effects

## Many vacua contributes at finite V

This fake IR divergence is due to a fixed vacuum:

$$U(x) = 1 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right) \in SU(N_f)$$

but at finite V, the vacuum is not uniquely determined: vacuum= moduli = dynamical variable

$$U(x) = U_0 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right),$$

$U_0$ should be non-perturbatively treated.

$\rightarrow$ ε-expansion (is needed for $M_\pi V^{1/4} \ll 1.$)

# 2. Analytic calculation of finite V effects

## ε and p expansions are really useful ?

ε expansion

$$M_\pi L \ll 1.$$

p expansion

$$M_\pi L \gg 1.$$

| Group | Nf | Action | a(fm) | L | Mπ (MeV) |
|---|---|---|---|---|---|
| ETMC | 2 | Twisted mass | 0.05-0.100 | ~3fm | 280 |
| MILC | 2+1 | Staggered | 0.045-0.12 | 3~6 fm | 250 |
| RBC/UKQCD | 2+1 | Domain wall | 0.085-0.11 | 3~4fm | 290 |
| JLQCD | 2+1 | Overlap | 0.11 | 1.8fm | 310 |
| PACS-CS | 2+1 | Wilson | 0.09 | ~3fm | 140 |
| BMW | 2+1 | Wilson | 0.065-0.125 | 3~5fm | 190 |
| ALV | 2+1 | DW on MILC | 0.06-0.12 | 3~4fm | 250 |
| HPQCD | 2+1 | HISQ | 0.045-0.15 | 3~4fm | 360 |

But on the lattice,

$$M_\pi L = 2 \sim 5.$$

No requirement in original theory ( if $E, p \ll m_\rho$ ):
both <u>expansions are bad.</u> $\rightarrow$ Better way of expansion ?

# 2. Analytic calculation of finite V effects

## New expansion

p-expansion

$$U(x) = 1 \exp\left( i \frac{\sqrt{2}\pi(x)}{F} \right), \qquad M_\pi \sim \mathrm{LO}$$

ε-expansion

$$U(x) = U_0 \exp\left( i \frac{\sqrt{2}\pi(x)}{F} \right), \qquad M_\pi \sim \mathrm{NLO}$$

# 2. Analytic calculation of finite V effects

## New expansion

p-expansion

$$U(x) = 1 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right), \qquad M_\pi \sim \mathrm{LO}$$

ε-expansion

$$U(x) = U_0 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right), \qquad M_\pi \sim \mathrm{NLO}$$

New *i* (interpolating)- expansion

$$U(x) = U_0 \exp\left(i\frac{\sqrt{2}\pi(x)}{F}\right), \qquad M_\pi \sim \mathrm{LO}$$

[Damgaard & HF, 2008]

# 2. Analytic calculation of finite V effects

## The pion (chiral) Lagrangian at finite V

$$\mathcal{L} = -\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}\right]$$

Zero-mode = SU(N) matrix model

$$+\frac{1}{2}\mathrm{Tr}(\partial_\mu \xi)^2 + \frac{1}{2}\sum_a M_a^2 (\xi^a)^2$$

Non-zero-mode = massive bosons

$$+\frac{\Sigma}{2F^2}\mathrm{Tr}[\mathcal{M}^\dagger (U_0 - 1)\xi^2 + \xi^2 (U_0 - 1)^\dagger \mathcal{M}]$$

(perturbative) interactions

$$+\cdots$$

= a hybrid system of matrix model and bosonic fields

# 2. Analytic calculation of finite V effects

## Pion correlator at 1-loop

[Aoki & HF, 2011]

Because of the mixing of zero and non-zero modes, the calculation is fairly tedious :

$$\begin{aligned}
\langle P(x)P(0)\rangle = & -\frac{\Sigma^2}{4}(Z_M^{12}Z_F^{12})^4\mathcal{C}^{0a} + \frac{\Sigma^2}{\mu_1+\mu_2}\left(\frac{\Sigma_{\mathrm{eff}}}{\Sigma} - (Z_M^{12}Z_F^{12})^2\right)\mathcal{C}^{0b} \\
& + \frac{\Sigma^2}{2}(\Delta Z_{11}^{\Sigma} - \Delta Z_{22}^{\Sigma})\mathcal{C}^{0c} + \frac{\Sigma^2}{2F^2}\Big[(Z_F^{12}(Z_M^{12})^2)^2\mathcal{C}^1\bar{\Delta}(x, M_{12}'^2) \\
& + \mathcal{C}^2\left(\frac{\Sigma}{F^2}\partial_{M^2}\right)\bar{\Delta}(x, M^2)\Big|_{M^2=M_{12}^2} \\
& + \mathcal{C}_{12}^4\left(\bar{\Delta}(x, M_{11}^2) - \bar{\Delta}(x, M_{12}^2)\right) + \mathcal{C}_{21}^4\left(\bar{\Delta}(x, M_{22}^2) - \bar{\Delta}(x, M_{12}^2)\right) \\
& + \sum_{j\neq 1}\mathcal{C}_{1j}^5\left(\bar{\Delta}(x, M_{2j}^2) - \bar{\Delta}(x, M_{12}^2)\right) + \sum_{i\neq 2}\mathcal{C}_{2i}^5\left(\bar{\Delta}(x, M_{1i}^2) - \bar{\Delta}(x, M_{12}^2)\right) \\
& + \mathcal{C}^6\bar{G}(x, M_{11}^2, M_{22}^2) \\
& + \mathcal{C}_{12}^7\left(\bar{G}(x, M_{11}^2, M_{22}^2) - \bar{G}(x, M_{11}^2, M_{11}^2)\right) \\
& + \mathcal{C}_{21}^7\left(\bar{G}(x, M_{11}^2, M_{22}^2) - \bar{G}(x, M_{22}^2, M_{22}^2)\right)\Big],
\end{aligned}$$

# 2. Analytic calculation of finite V effects

[Aoki & HF, 2011]

where

$$\mathcal{C}^{0a} \equiv \left\langle ([U_0]_{12} - [U_0^\dagger]_{21})([U_0]_{21} - [U_0^\dagger]_{12}) + \frac{1}{2}([U_0]_{12} - [U_0^\dagger]_{21})^2 + \frac{1}{2}([U_0]_{21} - [U_0^\dagger]_{12})^2 \right\rangle_{U_0},$$

$$\boxed{\begin{array}{rcl} U_0 & \in & SU(N) \text{ in } \theta = 0 \text{ vacuum}, \\ U_0 & \in & U(N) \text{ in a fixed } Q \text{ sector} \end{array}}$$

$$\mathcal{C}^{0b} \equiv \left\langle \frac{[U_0 + U_0^\dagger]_{11}}{2} + \frac{[U_0 + U_0^\dagger]_{22}}{2} \right\rangle_{U_0},$$

$$\mathcal{C}^{0c} \equiv \frac{1}{4}\langle([U_0]_{12} - [U_0^\dagger]_{21})^2 - ([U_0]_{21} - [U_0^\dagger]_{12})^2\rangle_{U_0},$$

$$\mathcal{C}^1 \equiv \left\langle ([U_0]_{11} + [U_0^\dagger]_{22})([U_0]_{22} + [U_0^\dagger]_{11}) + \sum_{j\neq 1}^{N_f}[U_0]_{1j}[U_0^\dagger]_{j1} + \sum_{i\neq 2}^{N_f}[U_0]_{2i}[U_0^\dagger]_{i2} \right.$$

$$\mathcal{C}^2 \equiv \left\langle 2([\mathcal{R}]_{11} + [\mathcal{R}]_{22}) - \sum_{j\neq 1}\frac{[\mathcal{R}]_{1j}[\mathcal{R}]_{j1}}{m_j - m_1} - \sum_{i\neq 2}\frac{[\mathcal{R}]_{2i}[\mathcal{R}]_{i2}}{m_i - m_2} \right\rangle_{U_0},$$

$$\mathcal{C}^3_{ij} \equiv \frac{1}{2}\langle([U_0]_{ji})^2 + ([U_0^\dagger]_{ij})^2\rangle_{U_0} + \frac{\langle[\mathcal{R}]_{ij}[U_0^\dagger]_{ij} + [U_0]_{ji}[\mathcal{R}]_{ji}\rangle_{U_0}}{m_i - m_j} + \frac{\langle([\mathcal{R}]_{ij})^2 + ([\mathcal{R}]_{ji})^2\rangle_{U_0}}{2(m_i - m_j)^2},$$

$$\mathcal{C}^4_{ij} \equiv \langle[U_0]_{ij}[U_0^\dagger]_{ji}\rangle_{U_0} + \frac{\langle[\mathcal{R}]_{ji}[U_0]_{ij} + [\mathcal{R}]_{ij}[U_0^\dagger]_{ji}\rangle_{U_0}}{m_j - m_i} + \frac{\langle[\mathcal{R}]_{ij}[\mathcal{R}]_{ji}\rangle_{U_0}}{(m_j - m_i)^2},$$

$$\mathcal{C}^5 \equiv -\left\langle ([U_0]_{12} + [U_0^\dagger]_{21})([U_0]_{21} + [U_0^\dagger]_{12}) + \frac{1}{2}([U_0]_{12} + [U_0^\dagger]_{21})^2 + \frac{1}{2}([U_0]_{21} + [U_0^\dagger]_{12})^2 \right\rangle_{U_0},$$

$$\mathcal{C}^6_{ij} \equiv \frac{1}{2}\langle([U_0]_{ji} + [U_0^\dagger]_{ij})^2\rangle_{U_0} + \frac{\langle([\mathcal{R}]_{ij} + [\mathcal{R}]_{ji})([U_0]_{ji} + [U_0^\dagger]_{ij})\rangle_{U_0}}{m_i - m_j}$$

# 2. Analytic calculation of finite V effects

Pion correlator at 1-loop [Aoki & HF, 2011]

$$\boxed{\int d^3x \langle P(x)P(0)\rangle = C_{PP}\frac{\cosh(M_\pi^{NLO}(t-T/2))}{\sinh(M_\pi^{NLO}T/2)} + D_{PP}}$$

$D_{PP}$ cancels IR divergence: $D_{PP} \underset{M\to 0}{\sim} -2\frac{C_{PP}}{M_\pi^{NLO}T} + E + \cdots ,$

disappears in the p-regime: $\lim_{M_\pi \to \text{large}} D_{PP} \sim \exp(-m\Sigma V) \to 0.$

( $C_{PP}, D_{PP}, M_\pi^{NLO}$ : functions of $\Sigma$ and $F_\pi$ )

$$\boxed{\to_{V\to\infty} \exp(-M_\pi t)}$$

# 3. Numerical lattice QCD results (preliminary)

QCD simulation with exact chiral symmetry

[JLQCD & TWQCD collaborations, 2006-2011]

2+1-flavor overlap Dirac fermions [Neuberger 98]

Iwasaki gauge action, β=2.3, 1/a ~ 1.759 GeV.

Lattice size : L=16 [1.8 fm], T=48.

Topology fixed: Q=0 (or 1)

Quark masses : $m_s$=0.08, 0.100,

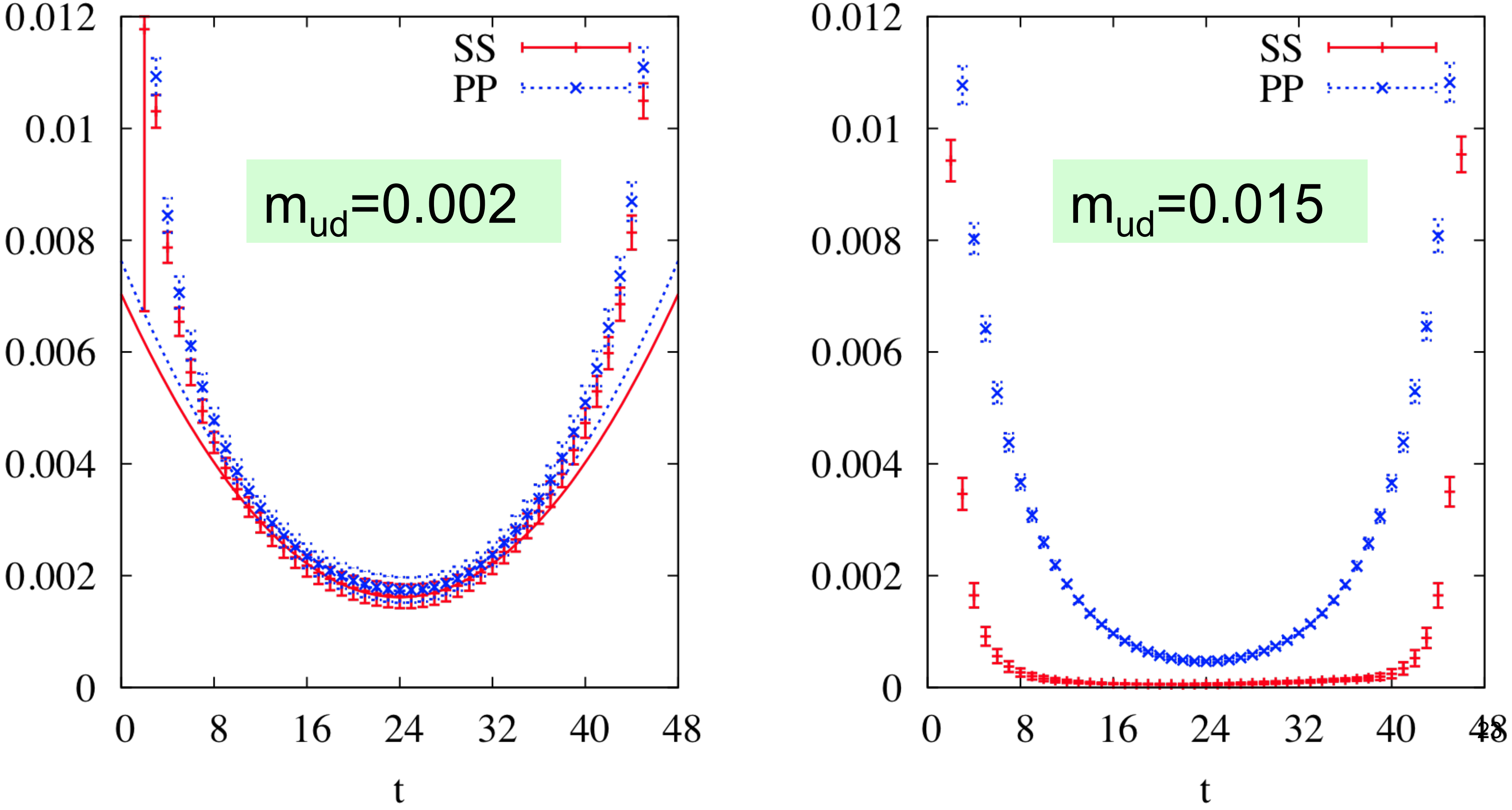
$m_{ud}$= 0.002, 0.015,0.025,0.035,0.050,0.080,0.100

(~3 MeV) (30MeV <)

ε-regime, p-regime

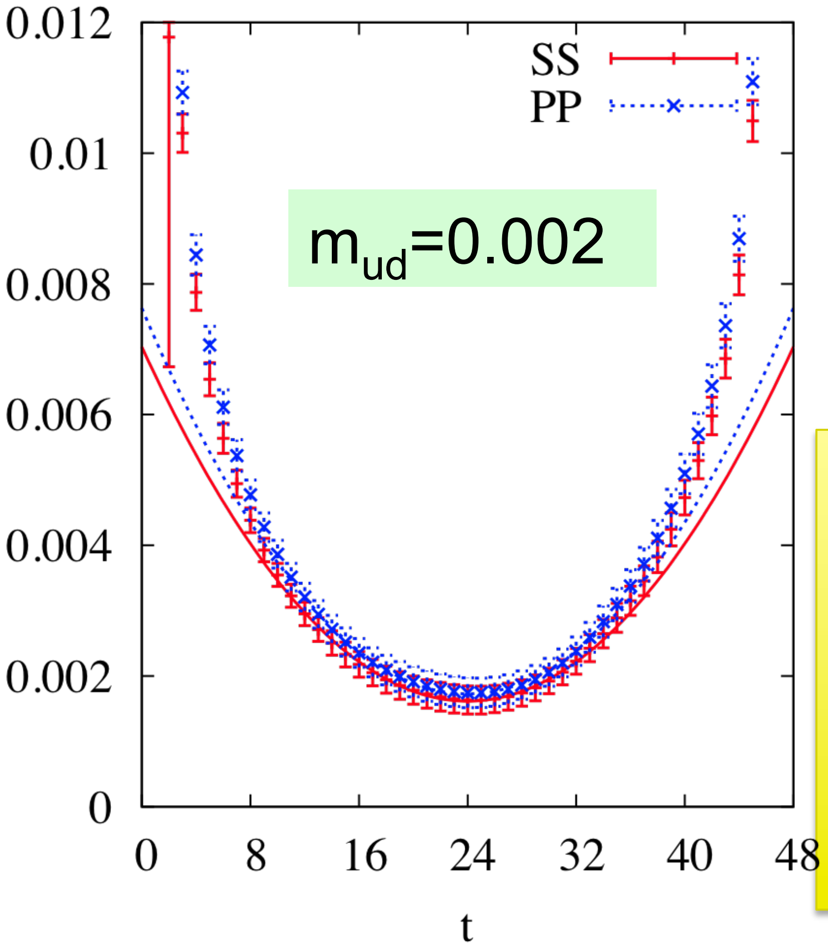
# 3. Numerical lattice QCD results (preliminary)

Really inside the ε-regime?→ Scalar channel knows.



# 3. Numerical lattice QCD results (preliminary)

Really inside the ε-regime? → Yes.




ε expansion analysis : [Damgaard et al, 2002]

PP → $\Sigma_{\rm eff} = 0.001936(66)$

SS → $\Sigma_{\rm eff} = 0.001938(70)$

consistent with our previous value

$\Sigma_{\rm eff} = 0.002041(70) \sim [241{\rm MeV}]^3$

from Dirac spectrum.

Next, let us extract

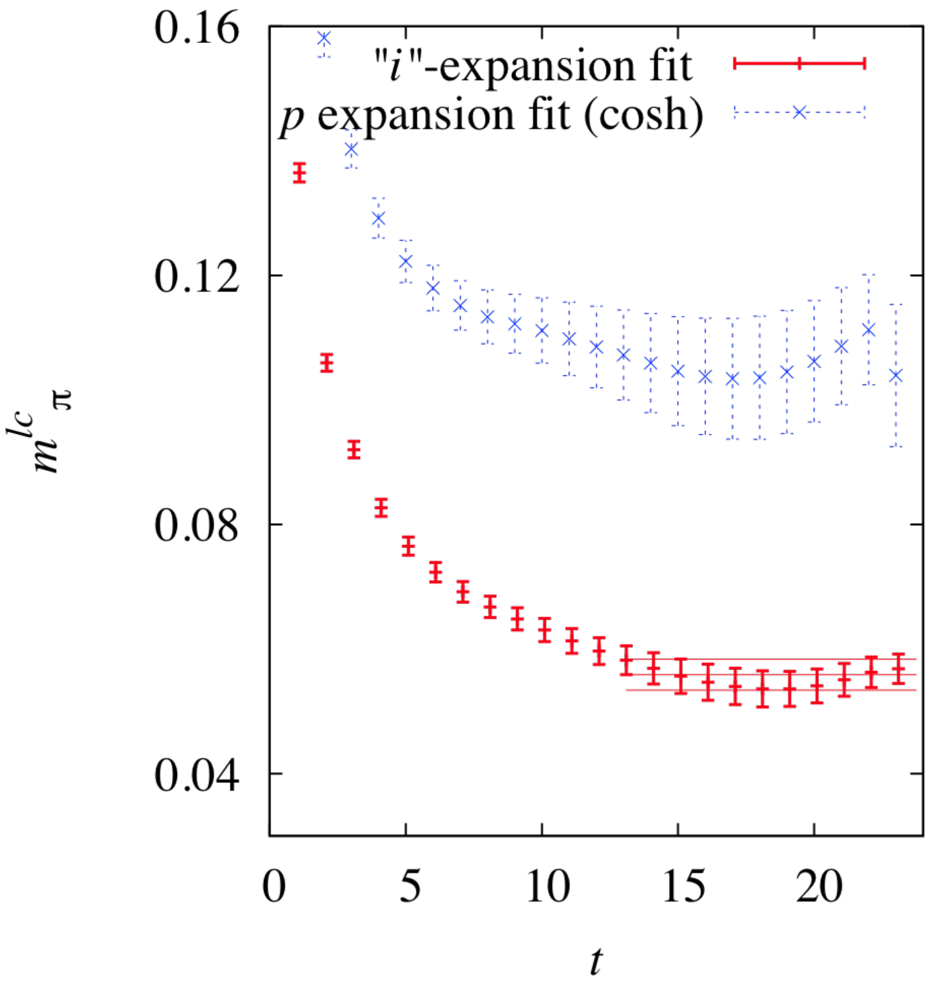
$$M_\pi \quad \text{and} \quad F_\pi .$$

with the "i" expansion formula.

( Naïve GMOR relation suggests $M_\pi \sim 100{\rm MeV}$. )

# 3. Numerical lattice QCD results (preliminary)

Pion mass and decay constant in the ε regime




Fitting with

$$C_{PP}\frac{\cosh(M_\pi^{NLO}(t-T/2))}{\sinh(M_\pi^{NLO}T/2)}+D_{PP}$$

$$[\Sigma_{\rm eff}=0.002041(70)\ \ \text{input}]$$

after 1-loop (of non-zero modes) volume correction, we obtain

$$M_\pi^{V=\infty}=97.6(4.2)\ \ \text{MeV},$$

$$F_\pi^{V=\infty}=128.6(5.6)\ \ \text{MeV}.$$

$$(1/a=1.759\ \text{GeV})$$

# 3. Numerical lattice QCD results (preliminary)

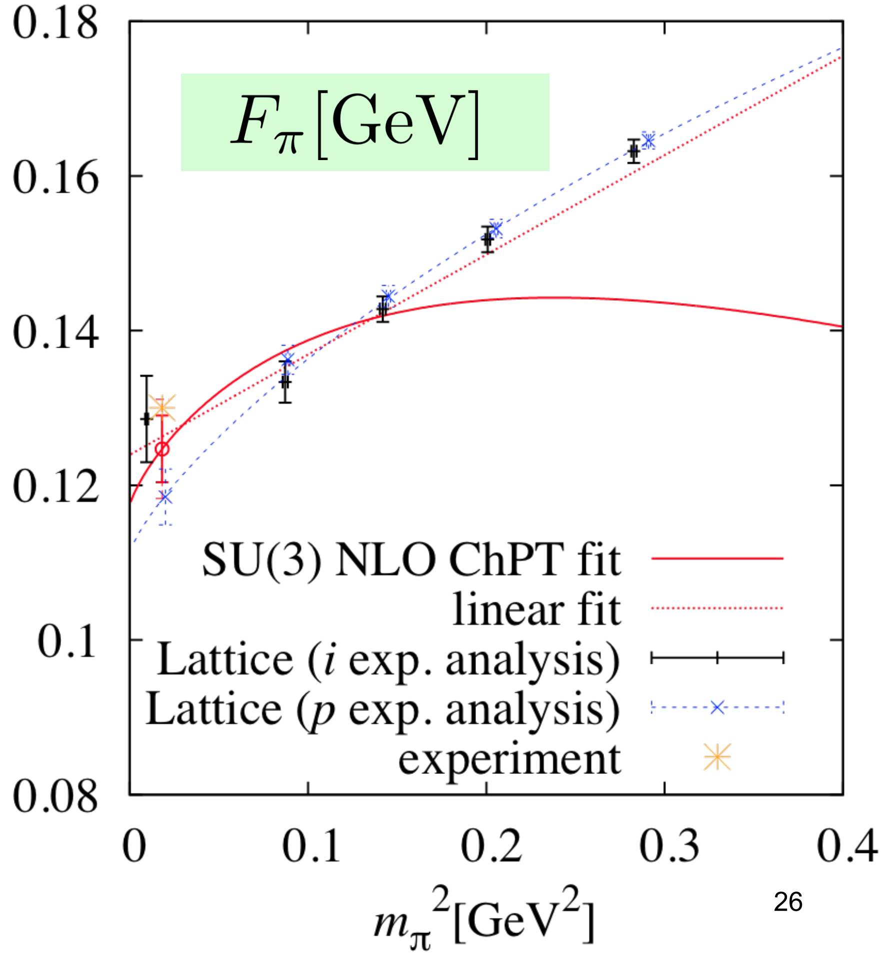
## Chiral "interpolation" for $F_\pi$

$$F_\pi = 125(4)(^{+5}_{-0})\ \ \mathrm{MeV}$$

bigger than our previous analysis using p-regime data only :

$$F_\pi = 119(4)\ \ \mathrm{MeV}$$

Linear fit looks better than NLO ChPT fit, though…



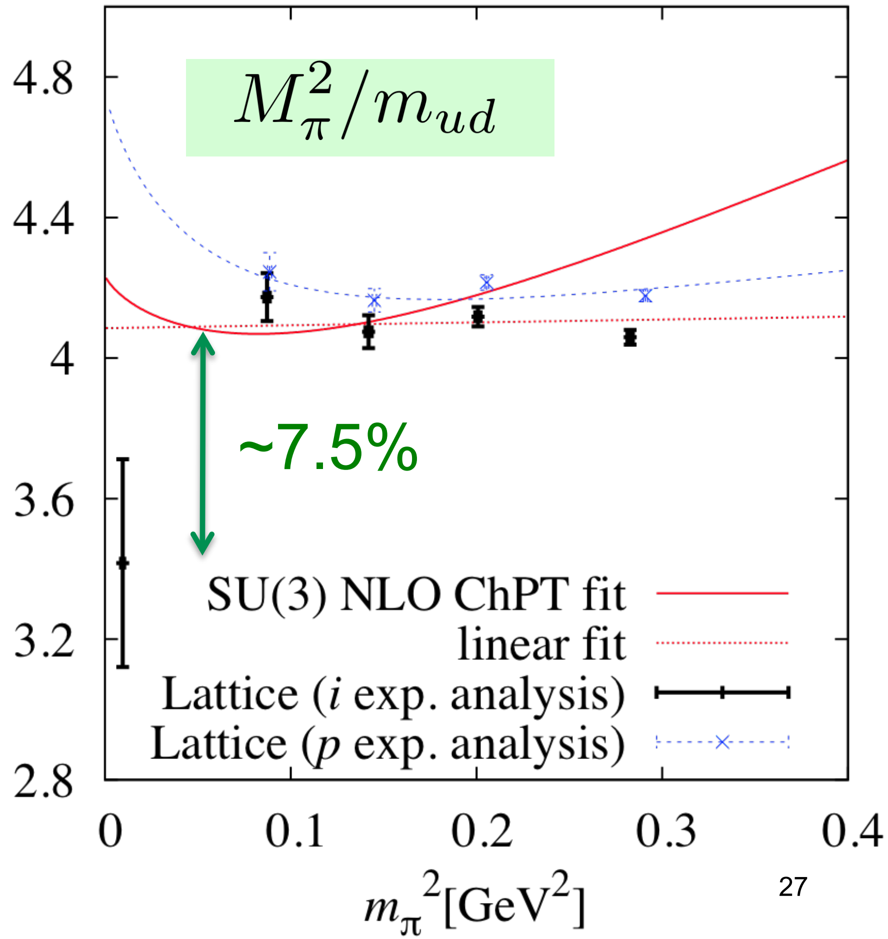
# 3. Numerical lattice QCD results (preliminary)

## Chiral "interpolation" for $M_\pi$

ChPT fit looks bad :

$2.5\sigma\ [7.5\%]$ deviation.

2-loop (1/V) effects from non-zero modes under control ?



# 3. Numerical lattice QCD results (preliminary)

## Bad convergence of ChPT for $M_\pi$ ?

In the limit $M_\pi \to 0$ ,

$$M_\pi^V = \underbrace{M_\pi}_{\to 0} + \underbrace{\delta M_{\mathrm{NLO}}}_{\mathcal{O}(1/L^2)} + \underbrace{\delta M_{\mathrm{NNLO}}}_{\mathcal{O}(1/F^2L^4)} + \cdots$$

Bad.
(big correction/LO)

$$F_\pi^V = \underbrace{F_\pi}_{\text{finite}} + \underbrace{\delta F_{\mathrm{NLO}}}_{\mathcal{O}(1/L^2)} + \underbrace{\delta F_{\mathrm{NNLO}}}_{\mathcal{O}(1/F^2L^4)} + \cdots$$

Good.

# 4. Summary

Small lattice QCD with exact chiral symmetry

+

Finite V correction from pion effective theory (ChPT)

$$\boxed{C_{PP}\frac{\cosh(M_\pi^{NLO}(t-T/2))}{\sinh(M_\pi^{NLO}T/2)}+D_{PP}}$$

can extract physics at V=∞.

# A. A short-cut prescription

Any interpolation is a good interpolation.

[Aoki & HF, 2011]

1. The epsilon-expansion and p-expansion are different expansions of the same theory.
2. They should converge in the limit $V \to \infty$.
3. Any interpolation gives the same results in that limit.
4. No additional information is needed : all the necessary ingredients can be read off from the ε and p expansions.

# A. A short-cut prescription

## A quick recipe

[Aoki & HF, 2011]

Starting from the p-regime result,

1. Remove IR divergent part from the propagator:

$$\frac{1}{V}\sum_p \frac{e^{ipx}}{p^2+M_\pi^2} \rightarrow \frac{1}{V}\sum_p \frac{e^{ipx}}{p^2+M_\pi^2} - \frac{1}{M_\pi^2 V}$$

2. Multiply a factor from zero-mode integrals, which can be read off from the ε-regime results.
3. Add a constant term if it exists in the ε-expansion.

# A. A short-cut prescription

## Results

[Aoki & HF, 2011]

$$\mathcal{PP}(t, m_v, m_v)_Q = C^Q_{PP} \frac{\cosh(M^Q_{vv}(t - T/2))}{M^Q_{vv} \sinh(M^Q_{vv} T/2)} + D^Q_{PP},$$

(The same result as our full calculation !)

$$\mathcal{AP}(t, m_v, m_v)_Q = C^Q_{AP} \frac{\sinh(M^Q_{vv}(t - T/2))}{\sinh(M^Q_{vv} T/2)} + D^Q_{AP} \left( \frac{t}{T} - \frac{1}{2} \right),$$

$$\mathcal{AA}(t, m_v, m_v)_Q = C^Q_{AA} \frac{\cosh(M^Q_{vv}(t - T/2))}{M^Q_{vv} \sinh(M^Q_{vv} T/2)} + D^Q_{AA},$$

Axial-Ward-Takahashi identities also confirmed.

# B. Comparison with Colangelo et al. [2005]

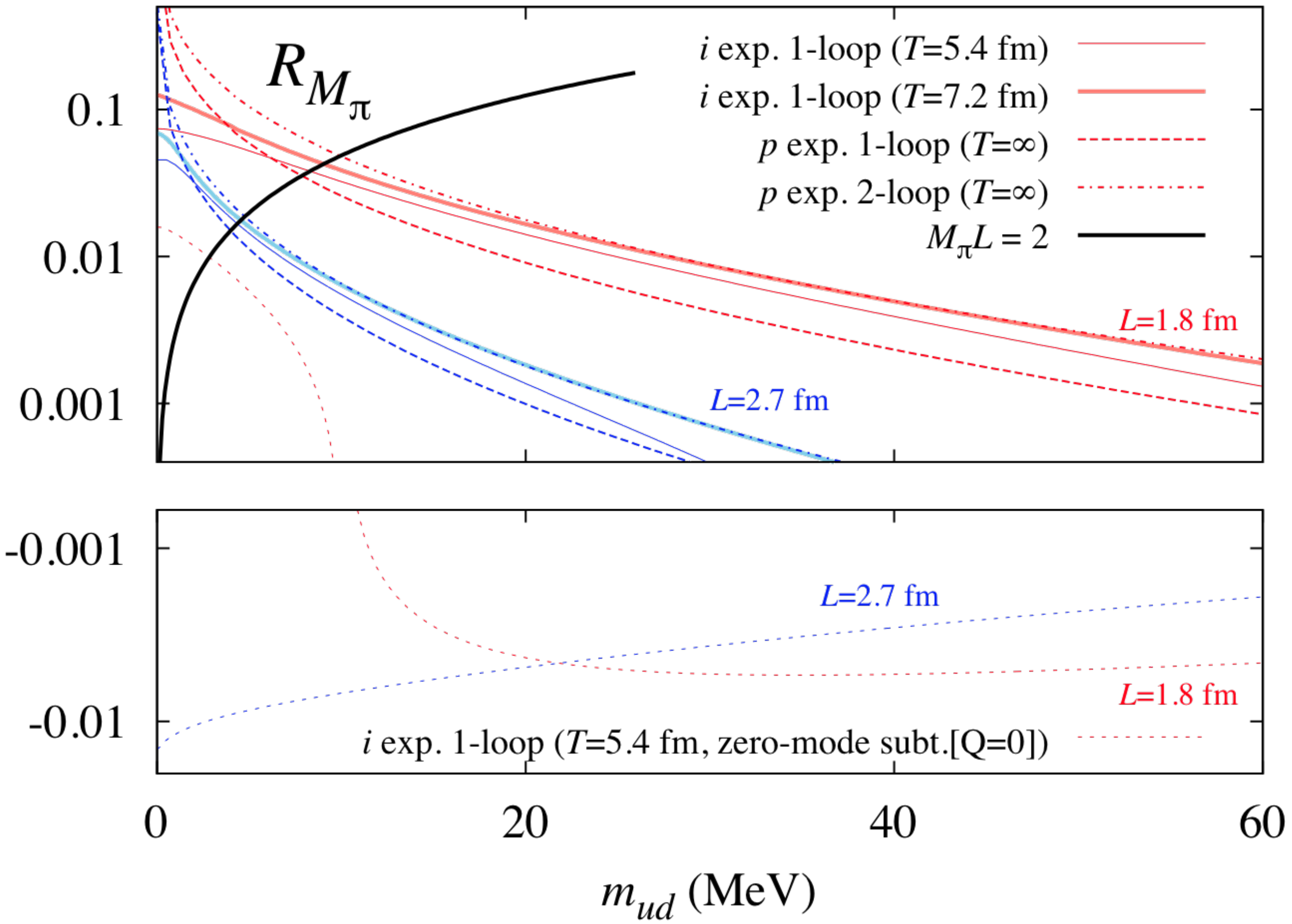
## Differences from p-regime calculation:

"Finite volume effects for meson masses and decay constants," G. Colangelo, S.Durr, C. Haefeli NPB 721 [2005]

| | Loop corrections | Chiral limit | Temporal direction |
|---|---|---|---|
| Colangelo et al. 2005 | 2-loop | fake IR divergence | neglected |
| Our work | 1-loop | finite | finite |

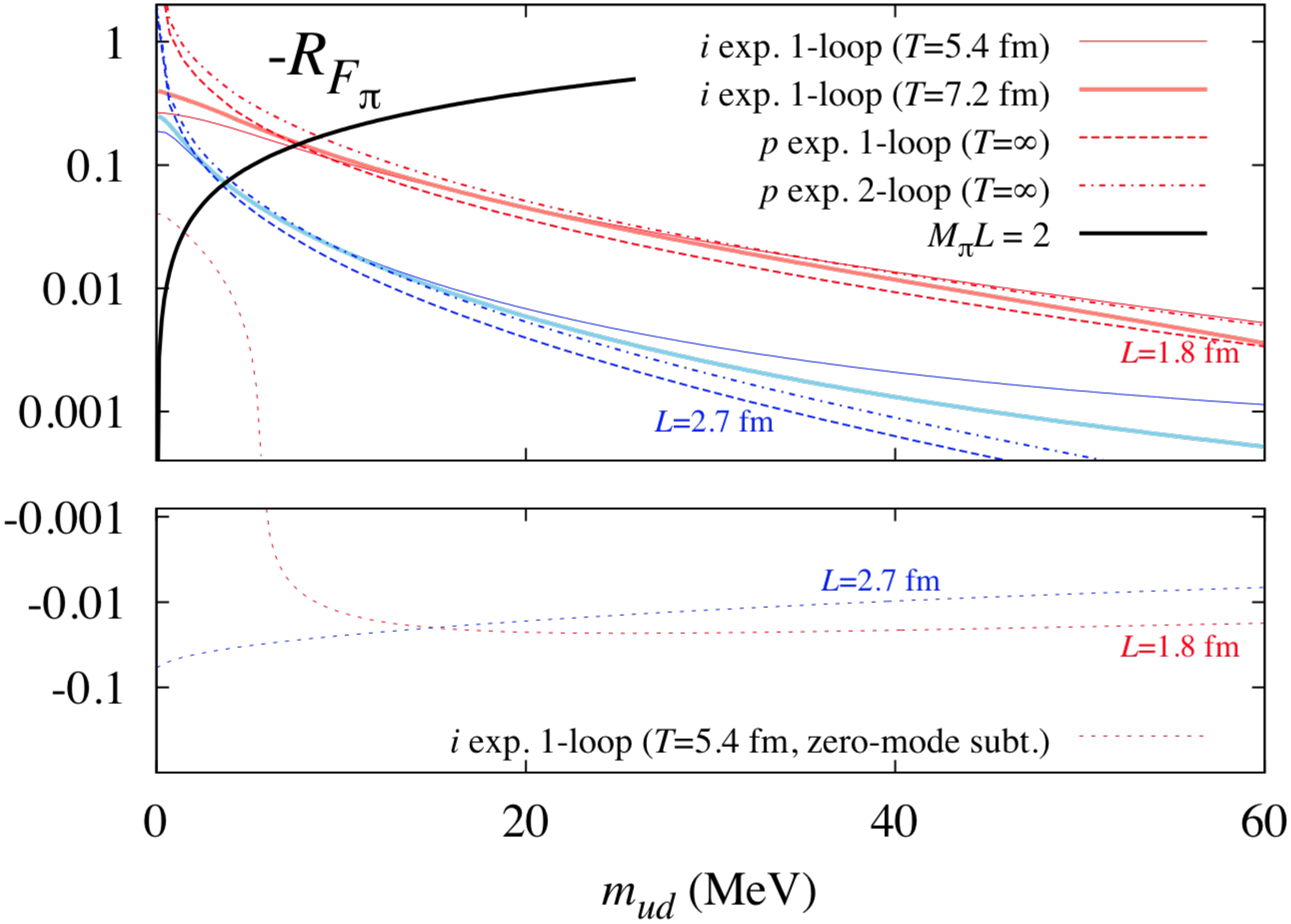
# B. Comparison with Colangelo et al. [2005]

## Pion mass:



# B. Comparison with Colangelo et al. [2005]

## Pion decay constant:



# 3. Numerical lattice QCD results (preliminary)

## How to control convergence of ChPT

p-expansion needs $M_\pi \to 0, \quad \frac{1}{L} \to 0$

keeping $M_\pi L \gg 1.$ $\left(\frac{1}{M_\pi L} e^{-M_\pi L}\right)$

New i-expansion: $M_\pi^2 \ln M_\pi^2, \ \frac{1}{(F_\pi L)^2}, \ \frac{M_\pi^2}{F_\pi^2 (F_\pi L)^2}, \ \frac{1}{(F_\pi L)^4}, \cdots$

We can separately take $M_\pi \to 0, \quad \frac{1}{L} \to 0.$

2-loop corrections -> $\frac{1}{(F_\pi L)^4} \lesssim 0.01 \text{ on } [3.3 \text{ fm}]^4$

Physical point simulation on L~3 fm lattice is interesting on next generation machines.

# 2. New chiral expansion

Mass term plays a key role.

Mass term decomposition into 3 pieces :

$$-\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}^\dagger U(x) + U^\dagger(x)\mathcal{M}\right] = -\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}\right]$$

$$+\frac{1}{2}\sum_i M_{ii}^2[\xi^2]_{ii} \quad \xi(x): \text{generic pion fields (non-zero modes)}$$

$$+\frac{\Sigma}{2F^2}\mathrm{Tr}[\mathcal{M}^\dagger(U_0-1)\xi^2 + \xi^2(U_0^\dagger - 1)\mathcal{M}] + \cdots,$$

Third term is NLO in both of p and ε regimes.

It is natural to assume it is perturbative everywhere:

$$\mathcal{M}^\dagger(U_0 - 1) \sim O(p^3).$$



[Damgaard & HF, 2008]