A comment on the weak NN scattering length of HAL QCD Collaboration

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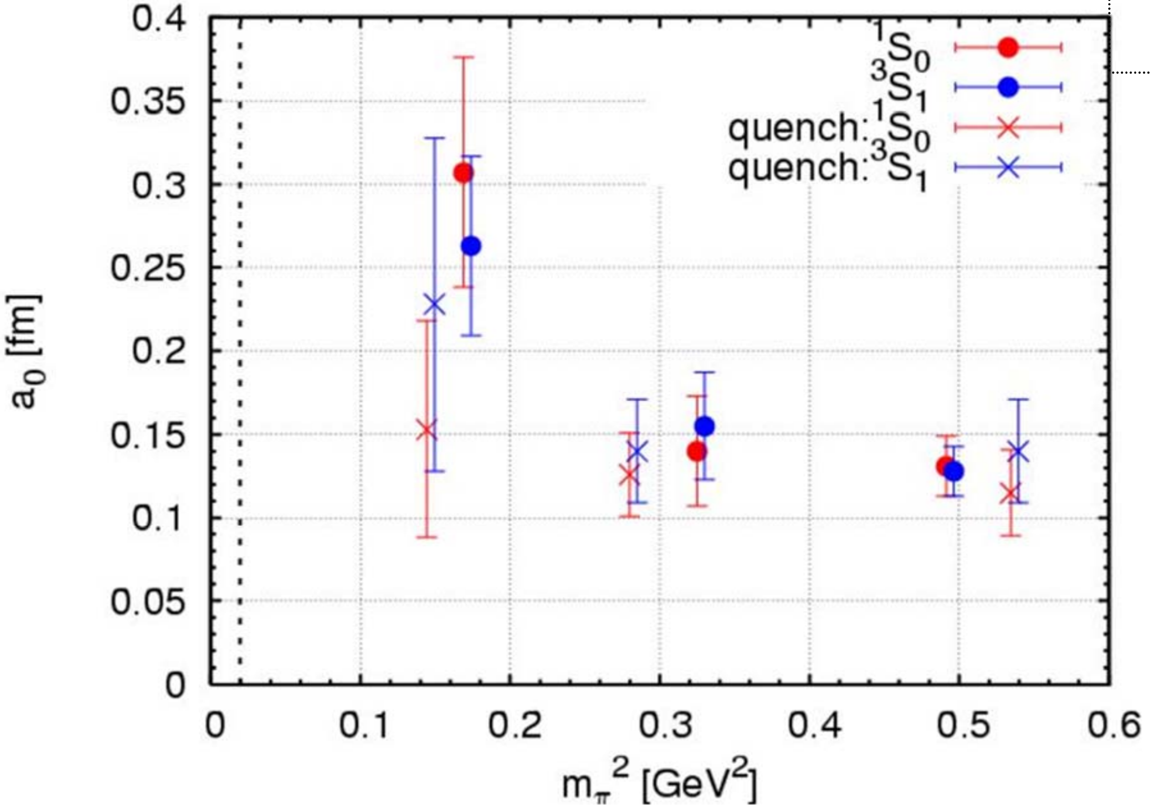
Considerably weak NN scattering length



BS wave →k²→ Luscher's formula

Attractive scattering length
It increases with $m_\pi \to 0$

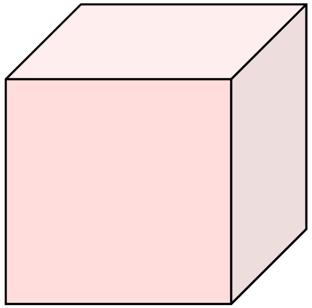
particle physics convention
a > 0 ←→ attractive

But, these values are considerably weaker than exp. values.

$$a_0^{(exp)}(^1S_0) \sim 20 \text{ fm}, \qquad a_0^{(exp)}(^3S_1) \sim -5 \text{ fm}$$

Background

Luescher's method

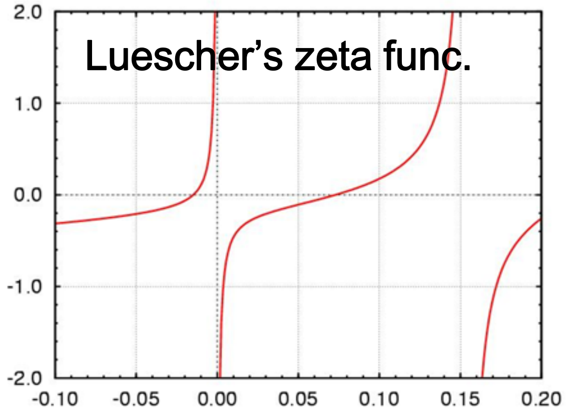


two particle system
in a finite box

Asymptotic momentum |k|

scattering phase

$$k\cot\delta(k) = \frac{2}{\sqrt{\pi}L} Z\left(1; \left(\frac{kL}{2\pi}\right)^2\right)$$



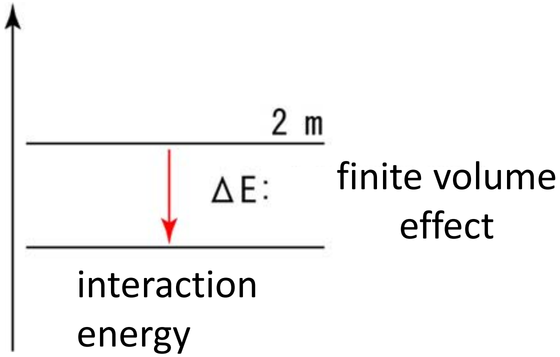
➢ From temporal correlation

(from energy spectrum of two-particle system)

$$R(t) \equiv C_{NN}(t) / \left(C_N(t)\right)^2$$

$$\sim A \cdot \exp\left(-\Delta E t\right)$$

$$\Delta E(\vec{k}) \equiv 2\left(\sqrt{m^2 + \vec{k}^2} - m\right)$$



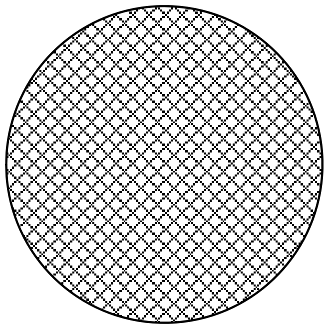
➢ From spatial correlation

(from long distance behavior of BS wave function)

[CP-PACS Coll., PRD71,094504(2005)]

$$\psi_{\vec{k}}(\vec{r}) \equiv \left\langle 0 \middle| N(\vec{x}) N(\vec{y}) \middle| N(\vec{k}) N(-\vec{k}), in \right\rangle$$

asymptotic momentum
|k|



temporal correlation vs spatial correlation

$$E(\vec{q}) = \sqrt{m^2 + \vec{q}^2}$$

BS wave func for E=2E(q)

$$\psi_{\vec{q}}(\vec{x}) \equiv \langle 0 | N(\vec{x}) N(\vec{0}) | N(\vec{q}) N(-\vec{q}), in \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3 2E(\vec{p})} \langle 0 | N(\vec{x}) | N(\vec{p}) \rangle \cdot \langle N(\vec{p}) | N(\vec{0}) | N(\vec{q}) N(-\vec{q}), in \rangle + I(\vec{x})$$

This is related to T-matrix via reduction formula

$$\simeq Z \left(e^{i\vec{q}\cdot\vec{x}} + \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2E(\vec{p})} \frac{T(\vec{p},\vec{q})}{4E(\vec{q}) \cdot (E(\vec{q}) - E(\vec{q}) - i\epsilon)} e^{i\vec{p}\cdot\vec{x}} \right)$$

$$\simeq Z \left(e^{i\vec{q}\cdot\vec{x}} + \frac{1}{2i} \left(e^{2i\delta_0(s)} - 1 \right) \frac{e^{iqr}}{qr} \right) + \cdots \text{ as } |\vec{x}| \to \text{large}$$

cf) C.-J.D.Lin et al.,NPB619,467(2001).
CP-PACS Coll., PRD71,094504(2005).

q controls the long distance behavior

⇒

Energy of the state
[E: temporal correlation]

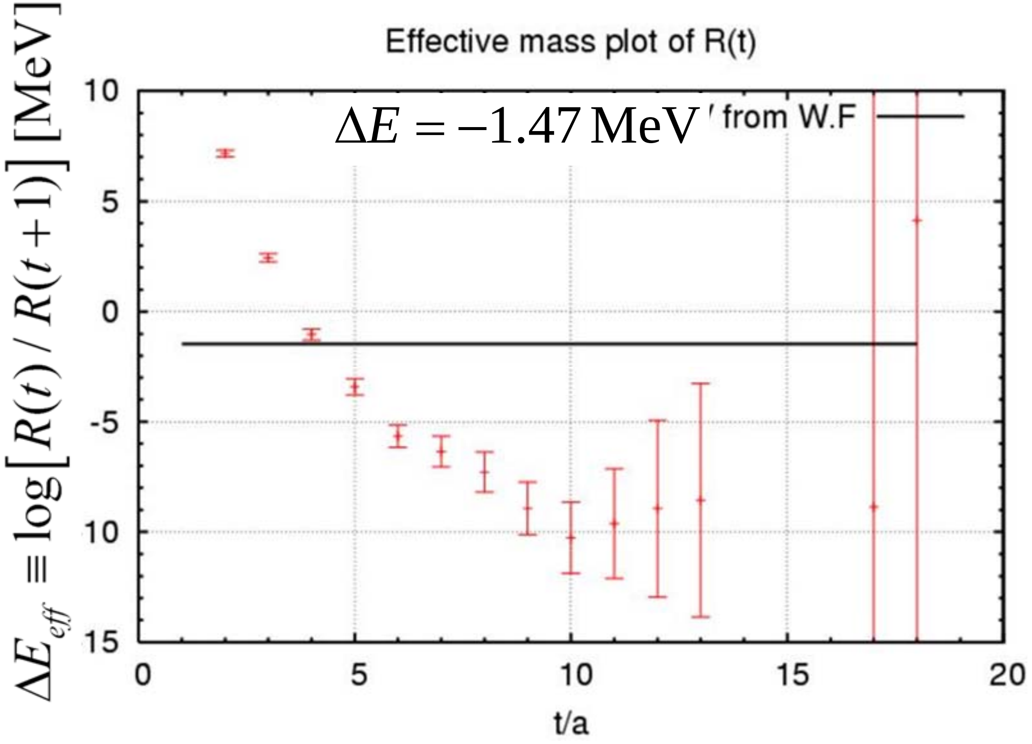
⇔

Asymptotic form of BS wave
[q: spatial correlation]

These two should be compatible with each other.

Actual calculation

$m_\pi \simeq 700\,\mathrm{MeV}$



$$R(t) \equiv \frac{\sum_{\vec{x}} G_{NN}(\vec{x},t)}{G_N(\vec{x},t)^2} \simeq Ae^{\Delta E \cdot t}$$ for large t.

Scattering length:

$a_0 \sim 4.8(5)$ fm [from spatial]

$a_0 \sim 0.131(18)$ fm [from temporal]

Possible source of the problem

ground state saturation

$$C_{NN}(\vec{x}-\vec{y},t) \equiv \left\langle 0 \middle| T\left[N(\vec{x},t)N(\vec{y},t)\cdot\overline{NN}(t=0)\right] \middle| 0 \right\rangle$$
$$= \sum_n \psi_n(\vec{x}-\vec{y})\cdot a_n \exp\left(-E_n t\right)$$

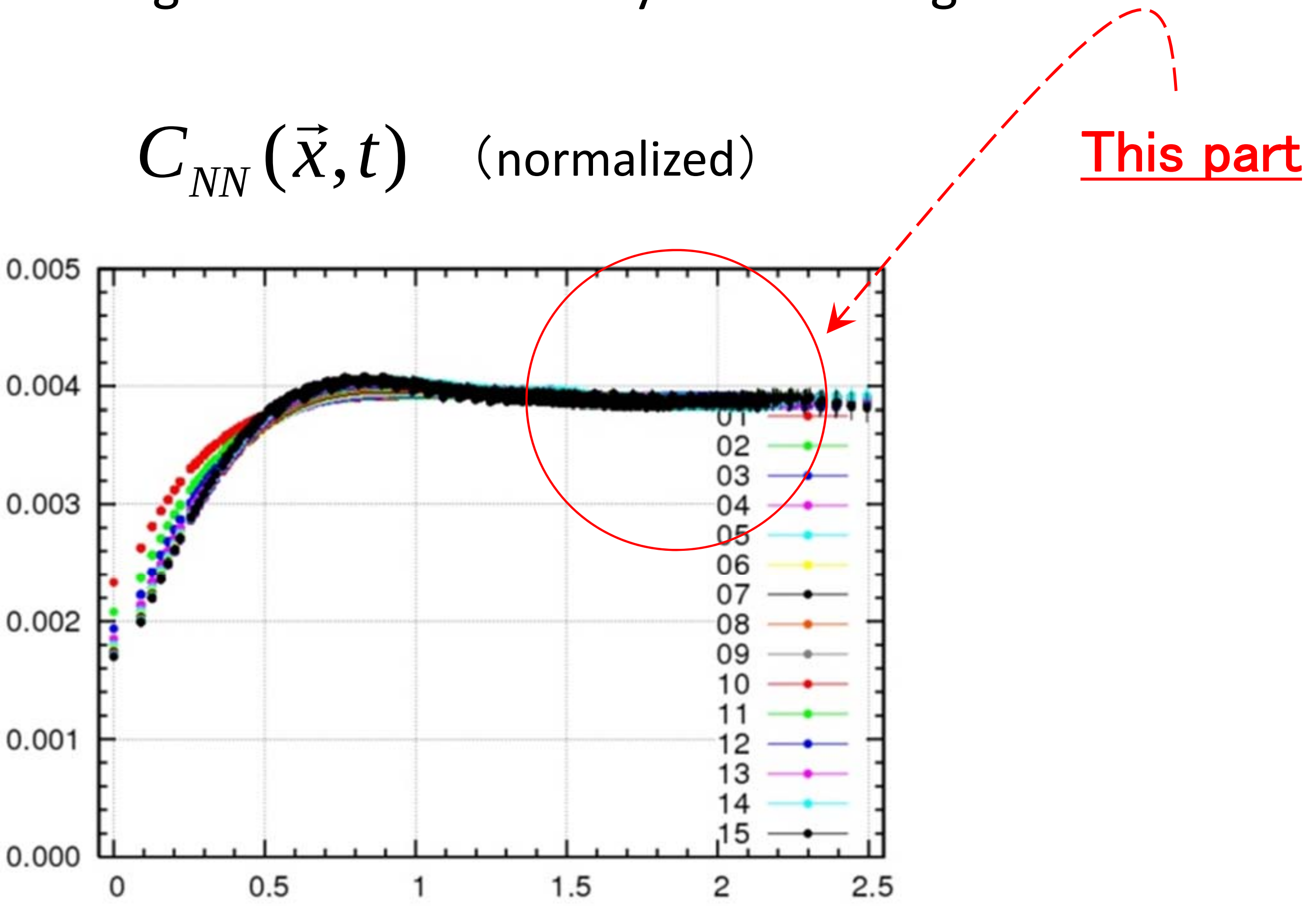
In principle, it can be solved by $t \to \infty$

For two nucleon system, we cannot go to such a large t region in practice.

Time evolution of “BS wave function”

- Convergence is considerably slow at long distance

$C_{NN}(\vec{x},t)$ （normalized）

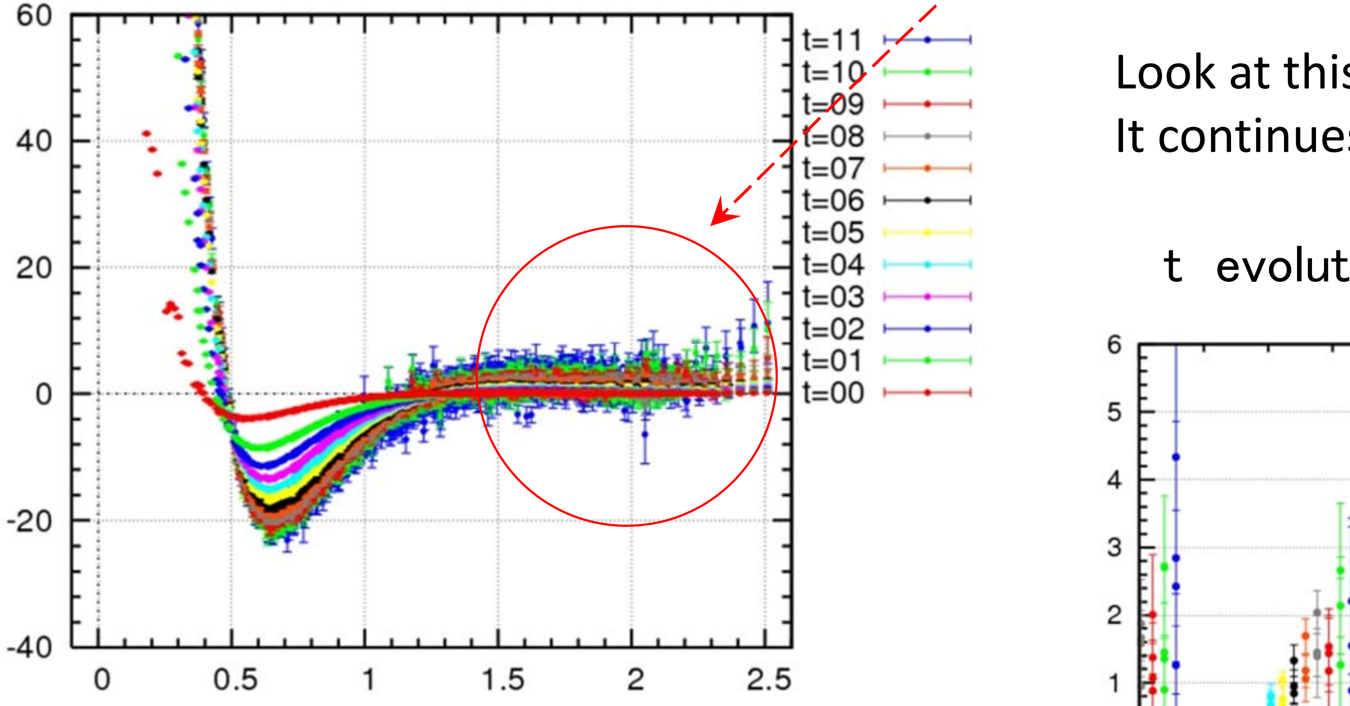


（It is difficult to see in this form）

It is easier to see in the form of "potential"

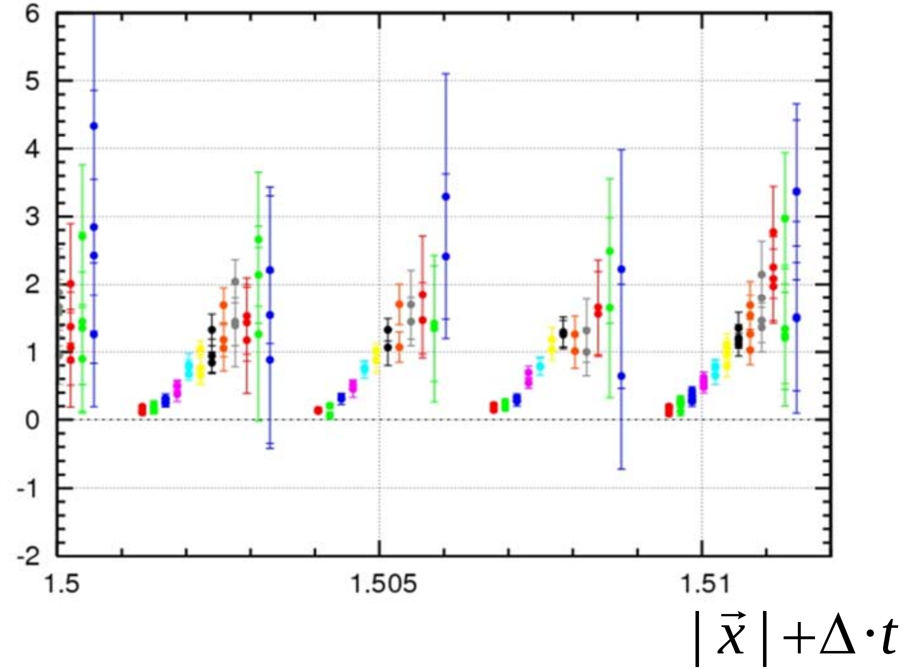
- Convergence is considerably slow at long distance

$$-\frac{H_0 C_{NN}(\vec{x},t)}{C_{NN}(\vec{x},t)}$$



Look at this part very carefully !
It continues to grow quite slowly.

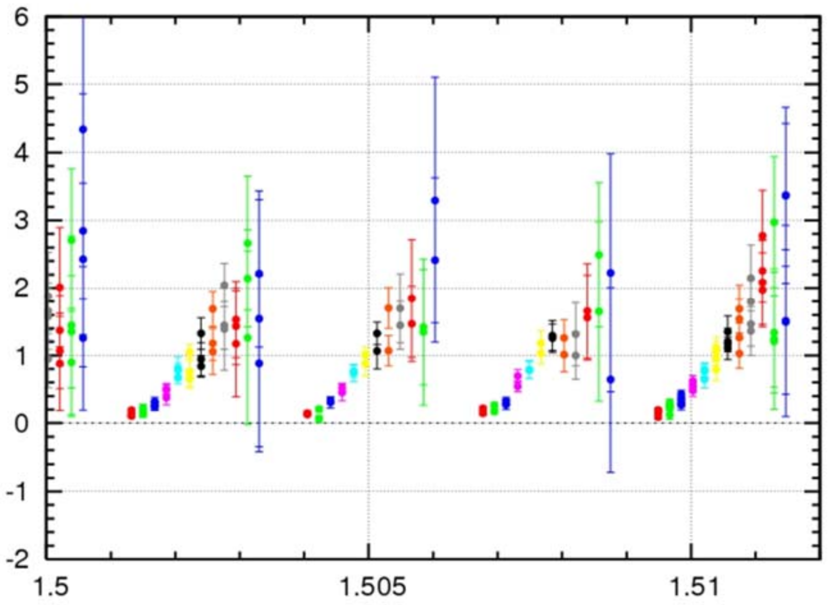
t evolution of $-\frac{H_0 C_{NN}(\vec{x},t)}{C_{NN}(\vec{x},t)}$



This uncertainty leads to the considerably weak scat. length

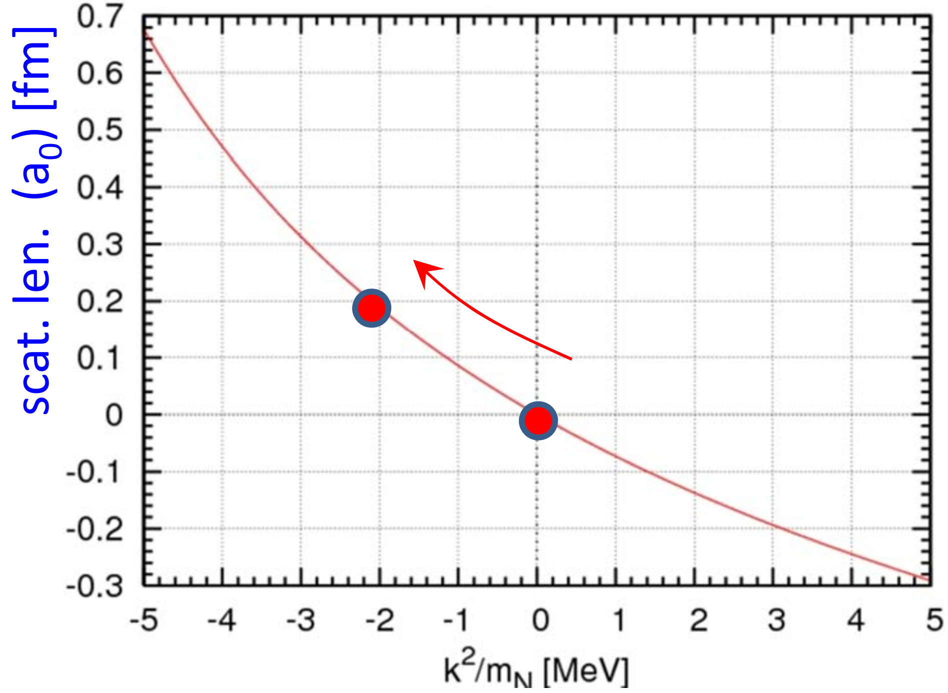
$$-\frac{H_0 C_{NN}(\vec{x},t)}{C_{NN}(\vec{x},t)} \rightarrow -\frac{H_0 \psi_{\vec{k}_0}(\vec{x})}{\psi_{\vec{k}_0}(\vec{x})} \quad \left(= V(\vec{x}) - \frac{\vec{k}_0^2}{m_N}\right)$$

$$\frac{1}{\frac{4\pi}{L^3}\sum_{\vec{n}\in\mathbb{Z}^3}\frac{1}{(2\pi/L)^2\vec{n}^2 - k^2}} \quad (\sim a_0)$$



$$|\vec{x}| + \Delta \cdot t$$

$$-\frac{\vec{k}_0^2}{m_N}$$



Candidate of $-\mathrm{k}^2/m_N$ at each t

Before the ground state saturation is achieved

➔ We get a smaller $-k^2/m_N$ than its real value.

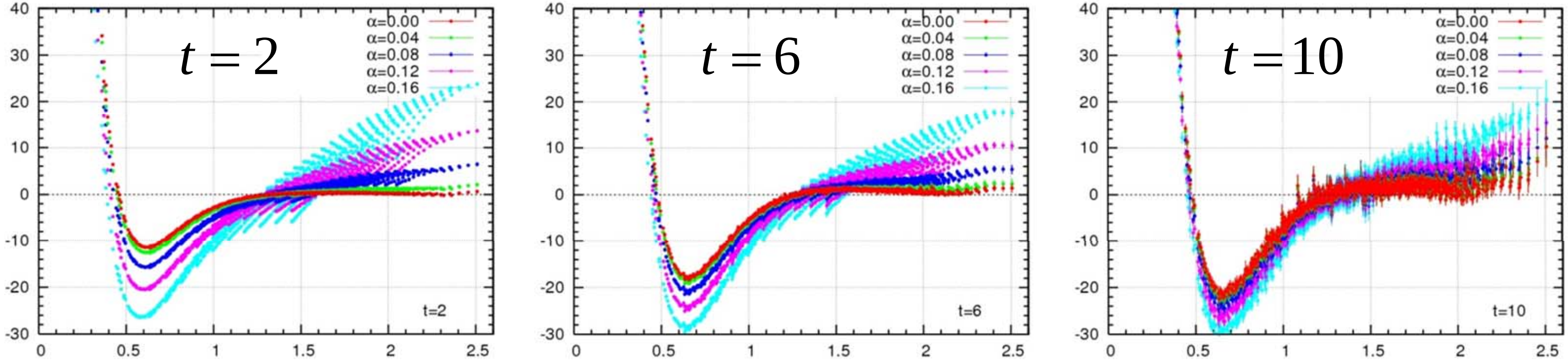
➔ We get a weaker scattering length than its real value.

Attempt to determine an upper bound (by approaching from above)

- α **source** (an extension of the flat wall source)

$$f(x,y,z) = 1 + \alpha\left(\cos(2\pi x/L) + \cos(2\pi y/L) + \cos(2\pi z/L)\right)$$

$$-\frac{H_0 C_{NN}(\vec{x},t)}{C_{NN}(\vec{x},t)} \rightarrow -\frac{H_0 \psi_{\vec{k}_0}(\vec{x})}{\psi_{\vec{k}_0}(\vec{x})} \left(= V(\vec{x}) - \frac{\vec{k}_0^{\,2}}{m_N} \right)$$



Extremely **large t (>> 10)** is needed for a quantitative evaluation

➔

Rather than seeking for a better source,

We will seek for a better procedure.

A New Method

as a modification of HALQCD method

(Time-dependent effective Schrodinger eq.)

Original HAL QCD algorithm

lattice QCD

↓

4-pt correlator

$$C_{NN}(t,\vec{x}) \equiv \langle 0 | T[N(\vec{x},t)N(\vec{0},t)\cdot\bar{J}(t=0)] | 0 \rangle$$

$t \to +\infty$ →

$$C_{NN}(t,\vec{x}) = \sum_n \psi_n(\vec{x}) a_n \exp(-E_n t)$$

BS wave func.

↓ **effective Schrodinger eq.**

$$\left(k^2/m - H_0\right)\psi_{\vec{k}}(\vec{x}) = \int d^3x' U(\vec{x},\vec{x}')\psi_{\vec{k}}(\vec{x}')$$

potential

↓

scattering phase/bound state

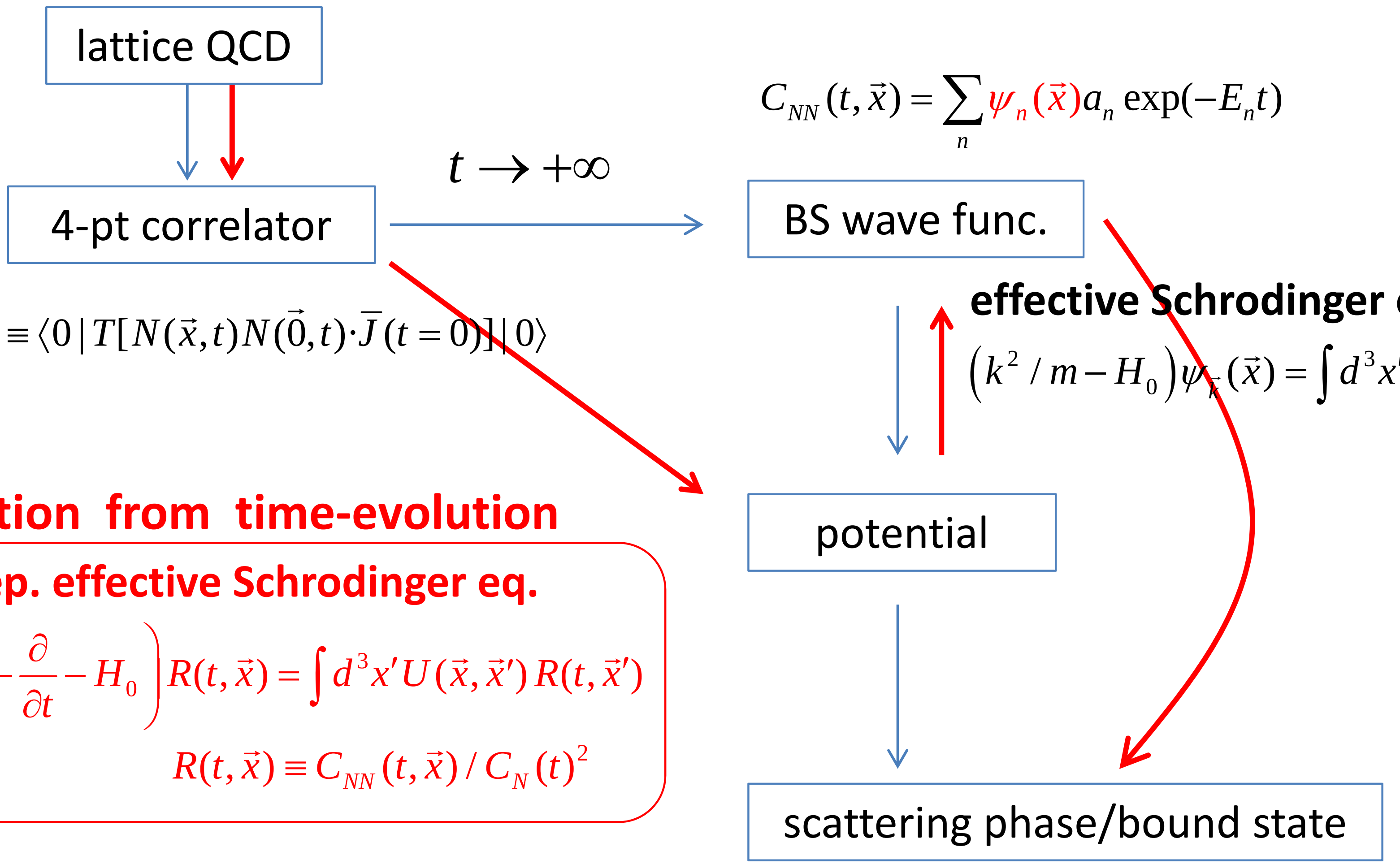
Asymptotic form of BS wave at long distance

$$\psi_{\vec{k}}(\vec{x}-\vec{y}) \equiv Z^{-1}\langle 0 | N(\vec{x})N(\vec{y}) | N(\vec{k})N(-\vec{k}), in \rangle$$

$$= e^{i\delta_0(k)} \frac{\sin(k|\vec{x}-\vec{y}| + \delta_0(k))}{k|\vec{x}-\vec{y}|} + \cdots$$

⇒ Potential, which is faithful to scattering phase

New Algorithm



$$C_{NN}(t,\vec{x}) = \sum_n \psi_n(\vec{x}) a_n \exp(-E_n t)$$

$$C_{NN}(t,\vec{x}) \equiv \langle 0 | T[N(\vec{x},t)N(\vec{0},t)\cdot\bar{J}(t=0)] | 0 \rangle$$

effective Schrodinger eq.

$$\left(k^2/m - H_0\right)\psi_{\vec{k}}(\vec{x}) = \int d^3x' U(\vec{x},\vec{x}')\psi_{\vec{k}}(\vec{x}')$$

Interaction from time-evolution

Time-dep. effective Schrodinger eq.

$$\left(\frac{1}{4m}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right) R(t,\vec{x}) = \int d^3x' U(\vec{x},\vec{x}') R(t,\vec{x}')$$

$$R(t,\vec{x}) \equiv C_{NN}(t,\vec{x}) / C_N(t)^2$$

Asymptotic form of BS wave at long distance

$$\psi_{\vec{k}}(\vec{x}-\vec{y}) \equiv Z^{-1}\langle 0 | N(\vec{x})N(\vec{y}) | N(\vec{k})N(-\vec{k}), in \rangle$$

$$= e^{i\delta_0(k)} \frac{\sin(k|\vec{x}-\vec{y}| + \delta_0(k))}{k|\vec{x}-\vec{y}|} + \cdots$$

⇒ Potential, which is faithful to scattering phase

Time-dependent effective Schrodinger equation

◆ Normalized NN correlator (R-correlator)

$$R(t,\vec{x}) \equiv e^{2m_N \cdot t} \cdot C_{NN}(t,\vec{x})$$
$$= \sum_{\vec{k}} a_{\vec{k}} \exp\left(-t\Delta W(\vec{k})\right)\psi_{\vec{k}}(\vec{x})$$

t has to be sufficiently large
(to suppress inelastic contributions in C_{NN})

$$\Delta W(\vec{k}) \equiv 2\sqrt{m_N^2 + \vec{k}^2} - 2m_N$$

◆ Derivation

$$-\frac{\partial}{\partial t} R(t,\vec{x}) = \sum_{\vec{k}} a_{\vec{k}} \Delta W(\vec{k}) \exp\left(-t\Delta W(\vec{k})\right)\psi_{\vec{k}}(\vec{x})$$
$$= \sum_{\vec{k}} a_{\vec{k}} \left\{\frac{\vec{k}^2}{m_N} - \frac{\Delta W(\vec{k})^2}{4m_N}\right\} \exp\left(-t\Delta W(\vec{k})\right)\psi_{\vec{k}}(\vec{x})$$
$$= \sum_{\vec{k}} a_{\vec{k}} \left\{H_0 + U - \frac{1}{4m_N}\frac{\partial^2}{\partial t^2}\right\} \exp\left(-t\Delta W(\vec{k})\right)\psi_{\vec{k}}(\vec{x})$$

an identity

$$\Delta W(\vec{k}) = \frac{\vec{k}^2}{m_N} - \frac{\Delta W(\vec{k})^2}{4m_N}$$

Def. of HAL QCD potential

$$\left(H_0 + U\right)\psi_{\vec{k}}(\vec{x}) = \frac{\vec{k}^2}{m_N}\psi_{\vec{k}}(\vec{x})$$

⇒

Time-dep. effective Schrodinger eq.

$$\left\{\frac{1}{4m_N}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right\} R(t,\vec{x}) = \int d^3x'\, U(\vec{x},\vec{x}')\, R(t,\vec{x}')$$

Numerical Application

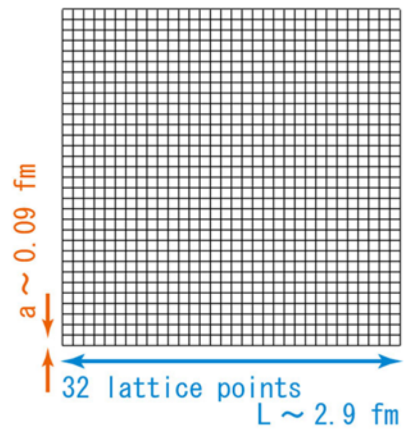
◆ Potential at leading order of derivative expansion

$$U(\vec{x},\vec{x}') = \left(V_C(\vec{x}) + O(\vec{\nabla}^2)\right)\delta(\vec{x}-\vec{x}') \qquad (^1S_0 \text{ channel})$$

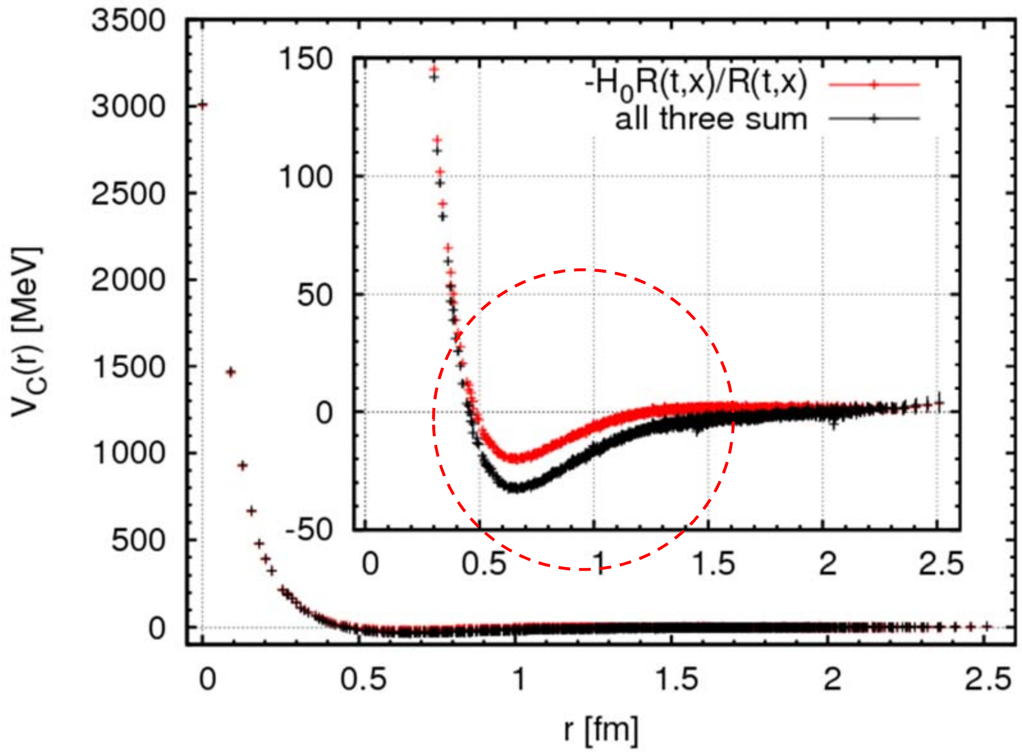
$$\left\{\frac{1}{4m_N}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right\} R(t,\vec{x}) = V_C(\vec{x}) R(t,\vec{x})$$

$$V_C(r) = -\frac{H_0 R(t,\vec{x})}{R(t,\vec{x})} - \frac{(\partial/\partial t) R(t,\vec{x})}{R(t,\vec{x})} + \frac{1}{4m_N}\frac{(\partial/\partial t)^2 R(t,\vec{x})}{R(t,\vec{x})}$$

2+1 flavor gauge config. by PACS-CS Coll.
$m_\pi \sim 700$ MeV,
$m_N \sim 1580$MeV



$(t = 8)$

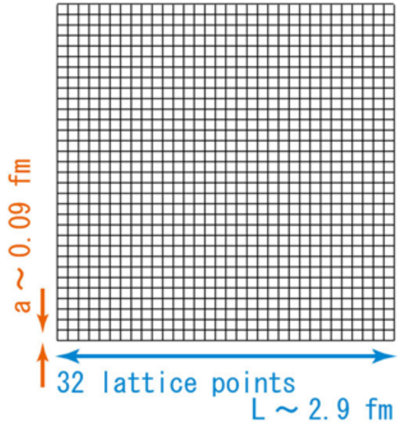


- More attraction
- Range gets longer

(Numerical derivatives are evaluated by 5 point formula.)

Numerical Application

2+1 flavor gauge config.
by PACS-CS Coll.
$m_\pi \sim 700$ MeV,
$m_N \sim 1580$MeV



◆ Potential at leading order of derivative expansion

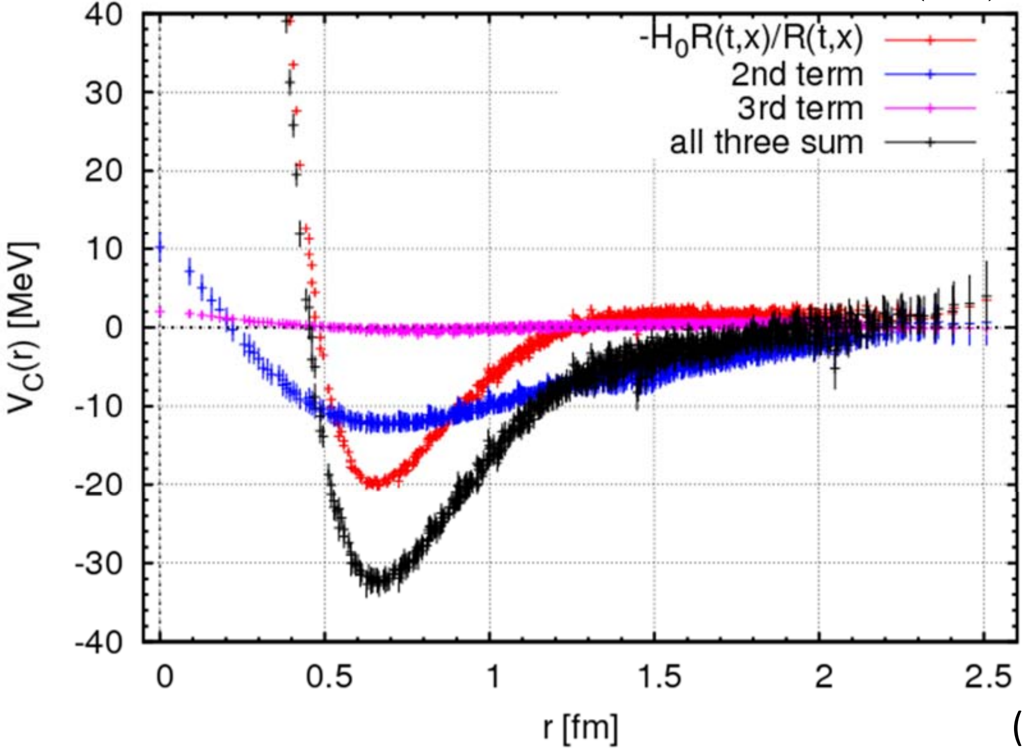
$$U(\vec{x},\vec{x}') = \left(V_C(\vec{x}) + O(\vec{\nabla}^2)\right)\delta(\vec{x}-\vec{x}') \qquad (^1S_0 \text{ channel})$$

$$\left\{\frac{1}{4m_N}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right\} R(t,\vec{x}) = V_C(\vec{x}) R(t,\vec{x})$$

⇒

$$V_C(r) = -\frac{H_0 R(t,\vec{x})}{R(t,\vec{x})} - \frac{(\partial/\partial t) R(t,\vec{x})}{R(t,\vec{x})} + \frac{1}{4m_N}\frac{(\partial/\partial t)^2 R(t,\vec{x})}{R(t,\vec{x})}$$

$(t=8)$



Thee contributions:

◆ **The 1st term : main contribution**

◆ **The 2nd term: important correction**

$$\frac{(\partial/\partial t) R(t,\vec{x})}{R(t,\vec{x})} = \frac{\partial}{\partial t}\log\left(R(t,\vec{x})\right)$$

point-wise effective mass plot

➔ Its non-constantness implies ground state saturation is not fulfilled

◆ **The 3rd term: negligible**

(Numerical derivatives are evaluated by 5 point formula.)

The source dependence is gone !

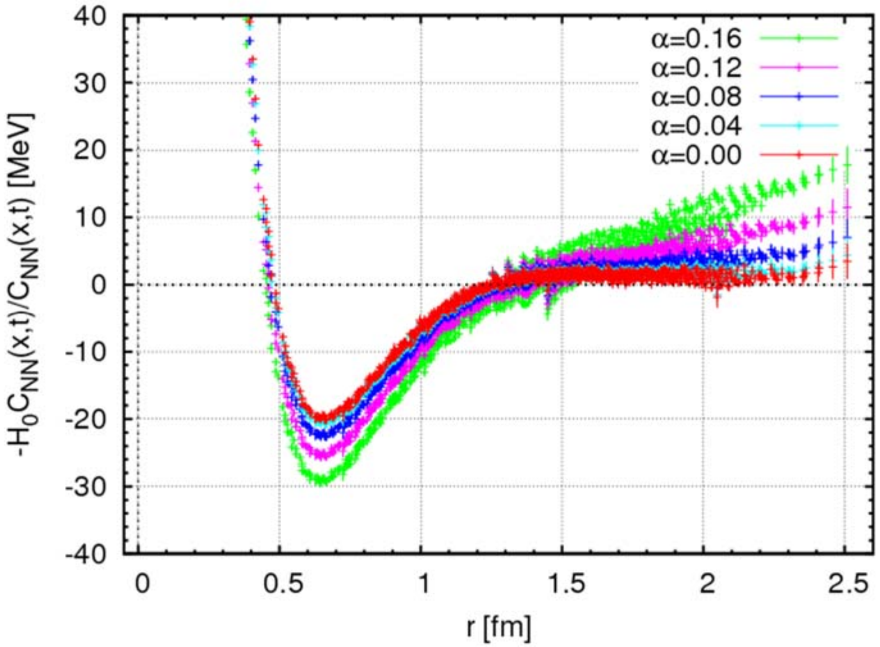
- α source

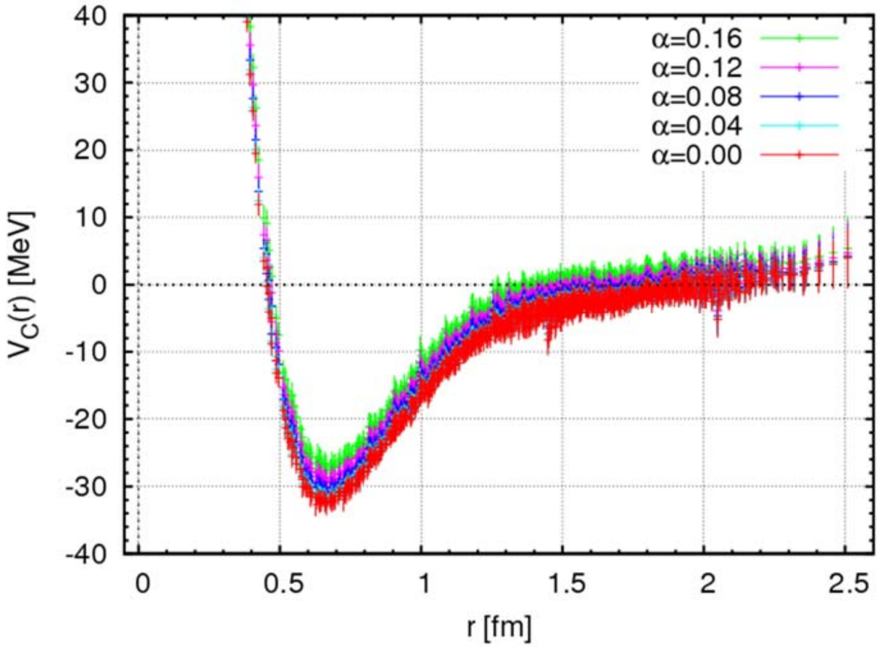
$$f(x,y,z) = 1 + \alpha\left(\cos(2\pi x/L) + \cos(2\pi y/L) + \cos(2\pi z/L)\right)$$

$$H_0 C_{NN}(\vec{x},t)/C_{NN}(\vec{x},t) \to \frac{H_0 \psi_{\vec{k}}(\vec{x})}{\psi_{\vec{k}}(\vec{x})} \left(= V_C(r) - \frac{\vec{k}^2}{m_N} \right)$$

$$(t = 8)$$

$$V_C(r) = -\frac{H_0 R(t,\vec{x})}{R(t,\vec{x})} - \frac{(\partial/\partial t) R(t,\vec{x})}{R(t,\vec{x})} + \frac{1}{4m_N} \frac{(\partial/\partial t)^2 R(t,\vec{x})}{R(t,\vec{x})}$$





Comment: Two-particle system with unequal mass

$$\Delta W(\vec{k}) \equiv \sqrt{m_1^2+\vec{k}^2}+\sqrt{m_2^2+\vec{k}^2}-m_1-m_2$$

reduced mass

$$\mu \equiv \frac{m_1 m_2}{m_1+m_2}$$

No counterpart of the identity holds

$$\Delta W(\vec{k}) = \frac{\vec{k}^2}{2\mu}+\Delta W(\vec{k})^2$$

Non-relativistic approximation

$$\Delta W(\vec{k}) \simeq \frac{\vec{k}^2}{2\mu}$$

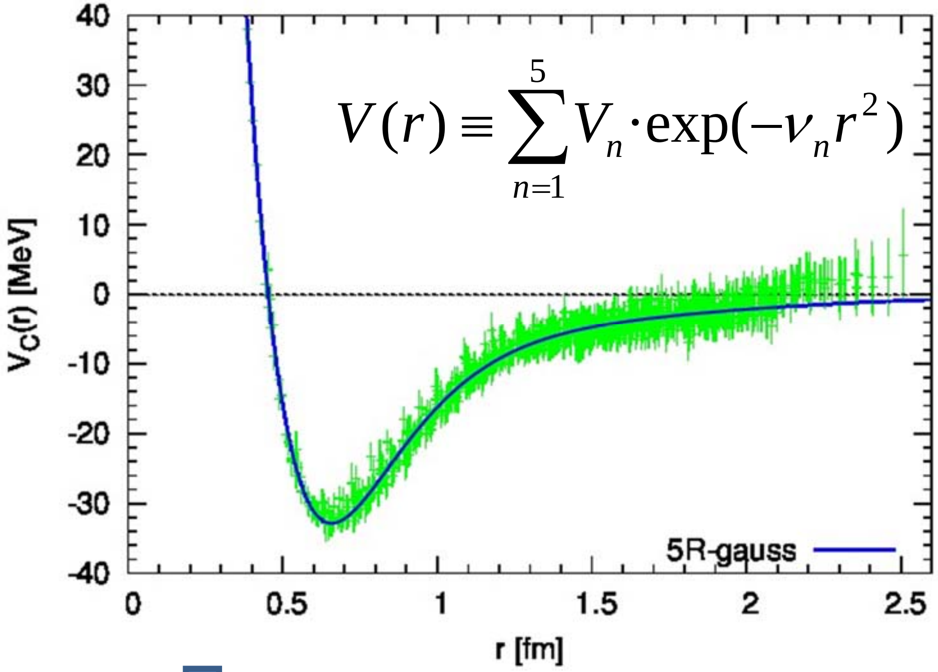
➔ time-dependent Schrodinger eq. in imaginary time

$$\left\{-\frac{\partial}{\partial t}-H_0\right\}R(t,\vec{x}) \simeq \int d^3x' U(\vec{x},\vec{x}')R(t,\vec{x}')$$

This approx. turns out to work well for NN system
(←→ The 3rd term gives negligible contribution)

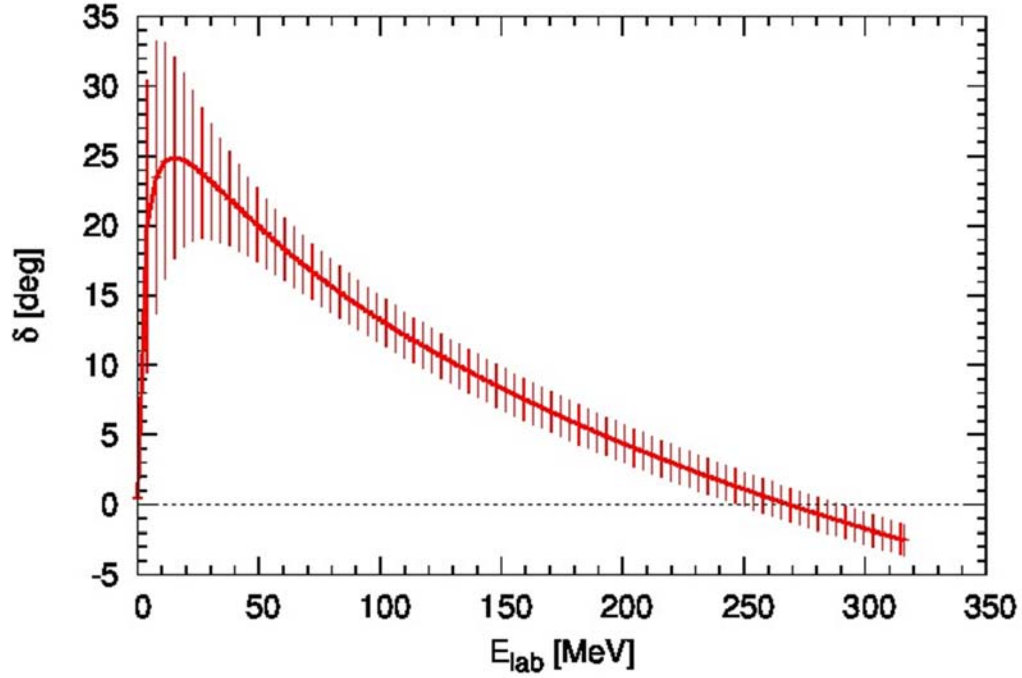
Significant enhancement in scat. length and scat. phase

5-range Gauss fit



$$V(r) \equiv \sum_{n=1}^{5} V_n \cdot \exp(-\nu_n r^2)$$

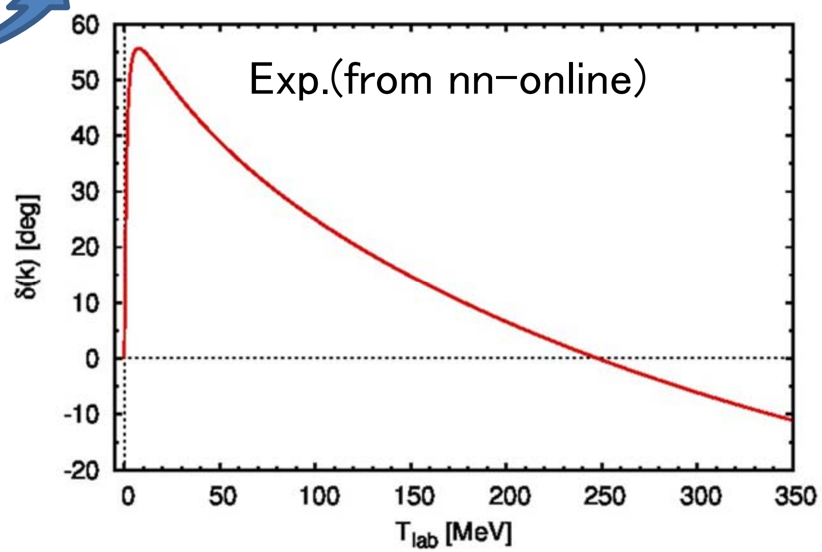
$\boldsymbol{\delta(1S0)}$



scattering length

$$a_0(^1S_0) = 1.6(11)\ \text{fm}$$

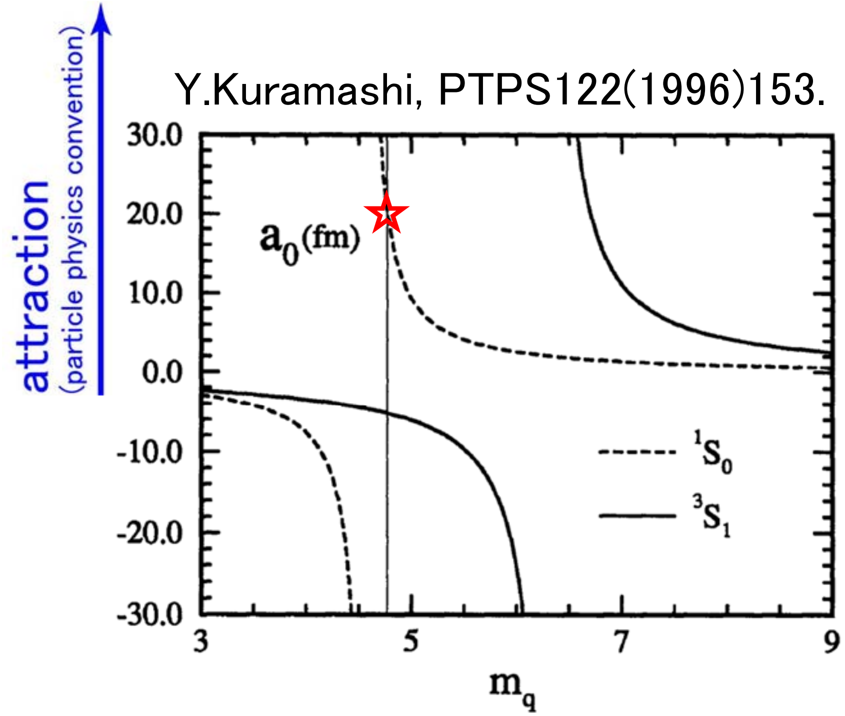
Exp : ～20 fm



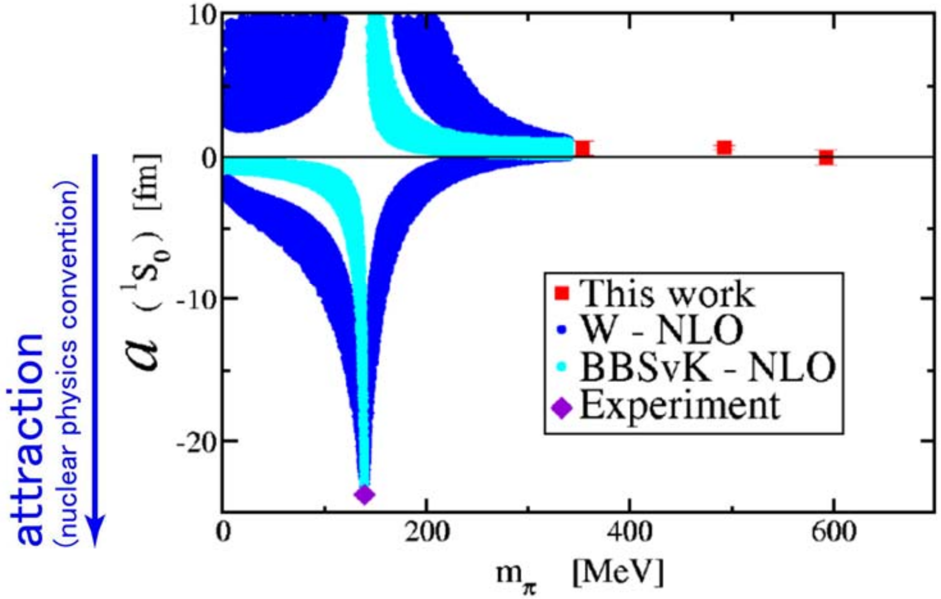
Remaining discrepancy is due to the heavy quark mass

- Several opinions against the quark mass dependence of scat. length

Y.Kuramashi, PTPS122(1996)153.



S.Beane et al., PRL97,012001(2006).



- They agree at the point:

Physical point is in the **unitary region.**

➔ **A rapid change of scat. length** due to a generation of new bound state

➔ **Direct calculations** in the **light quark mass region** desirable

Summary

- We have presented a new method for a lattice determination of nuclear force

"Time-dependent" effective Schrodinger eq.

$$\left\{ \frac{1}{4m_N} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right\} R(t,\vec{x}) = \int d^3x' U(\vec{x},\vec{x}') R(t,\vec{x}')$$

- Nuclear force is constructed based on time-evolution of 4-point correlator.

  ➔ We do not need

  ground state saturation

  - ☐ Potentials can be constructed within a limited range of t.
  - ☐ Powerful for large spatial volume:
    (the larger the volume ➔ the smaller the energy gap in the spectrum $\propto O(\frac{1}{L^2})$)

- Potential obtained by the new method:
  - ☐ Range gets longer. The attraction gets stronger.
  - ☐ Significant enhancement in the scattering length / phase.
    (Remaining discrepancy is due to the heavy quark mass)

backup slides

従来型 v.s. 新型

◆ 従来型の方法

- □ 定義 (effective Schrodinger eq.)

$$\left(k^2/m_N - H_0\right)\psi_{\vec{k}}(\vec{x}) = \int d^3x' U(\vec{x},\vec{x}')\psi_{\vec{k}}(\vec{x}') \quad \text{for} \quad E \equiv 2\sqrt{m_N^2 + \vec{k}^2} < 2m_N + m_\pi$$

- □ 構築法 (effective Schrodinger eq.)

$$\left(k^2/m_N - H_0\right)\psi_{\vec{k}}(\vec{x}) = \int d^3x' U(\vec{x},\vec{x}')\psi_{\vec{k}}(\vec{x}') \quad \text{for} \quad E \equiv 2\sqrt{m_N^2 + \vec{k}^2} < 2m_N + m_\pi$$

- □ 欠点

  single state saturation が必須

◆ 新型の方法

- □ 定義 (effective Schrodinger eq.)

$$\left(k^2/m_N - H_0\right)\psi_{\vec{k}}(\vec{x}) = \int d^3x' U(\vec{x},\vec{x}')\psi_{\vec{k}}(\vec{x}') \quad \text{for} \quad E \equiv 2\sqrt{m_N^2 + \vec{k}^2} < 2m_N + m_\pi$$

- □ 構築法 (**Time-dependent** effective Schrodinger eq.)

$$\left\{\frac{1}{4m_N}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right\} R(t,\vec{x}) = \int d^3x' U(\vec{x},\vec{x}')\, R(t,\vec{x}')$$

- □ 利点

  励起状態が混じっている状態でも平気。

An explicit construction of NN potential

- We assume linear independence of BS wave function below pion threshold.
  ➔ It has the dual basis (a left inverse)

$$\int d^3r\, \widetilde{\psi}_{\vec{k}'}(\vec{r})\psi_{\vec{k}}(\vec{r}) = (2\pi)^3\delta^3(\vec{k}'-\vec{k})$$

- More explicit construction of NN potential

$$K_{\vec{k}}(\vec{r}) \equiv \left(\Delta + k^2\right)\psi_{\vec{k}}(\vec{r})$$

$$= \int \frac{d^3k'}{(2\pi)^3} K_{\vec{k}'}(\vec{r}) \int d^3r'\, \widetilde{\psi}_{\vec{k}'}(\vec{r})\psi_{\vec{k}}(\vec{r})$$

$$= \int d^3r' \left\{ \int \frac{d^3k}{(2\pi)^3} K_{\vec{k}'}(\vec{r})\widetilde{\psi}_{\vec{k}'}(\vec{r}') \right\} \psi_{\vec{k}}(\vec{r}')$$

If we define

$$U(\vec{r},\vec{r}') \equiv \frac{1}{m_N}\int \frac{d^3k}{(2\pi)^3} K_{\vec{k}'}(\vec{r})\widetilde{\psi}_{\vec{k}'}(\vec{r})$$

then we have

$$\frac{1}{m_N}\left(\Delta + k^2\right)\psi_{\vec{k}}(\vec{r}) = \int d^3r' U(\vec{r},\vec{r}')\psi_{\vec{k}}(\vec{r})$$